Astron. Astrophys. 325, 74–80 (1997)

ASTRONOMY AND ASTROPHYSICS

A deprojection method to obtain the spectral index distribution in diffuse radio halos of clusters of galaxies

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Received 2 January 1997 / Accepted 20 January 1997

Abstract. Diffuse radio halos of clusters of galaxies are generally classified as "steep-spectrum radio sources" according to the observed spectral index of the integrated flux density of $\alpha_S > 0.75$. It is shown that the spectral index of the underlying synchrotron emission coefficient can be considerably smaller than α_S due to a frequency-dependent effective emission volume observed in some radio halos. Deiss, Reich, Lesch & Wielebinski (1996 A&A, in press) have suggested an analytic deprojection method to derive the spatial distribution of the emissivity index from the observed spectral-index distribution of the surface brightness assuming a spherically symmetric gaussian radio halo source. The formalism also allows to check on the consistency of the observational data at different frequencies. The method is generalized to the case of any halo model that is symmetric about the center but otherwise arbitrary. It is shown that, as an example, in the central region of the extended radio halo of the Coma cluster, the spectral index of the emission coefficient may be as low as 0.43, which is smaller than the observed integrated-flux density index by about 0.8. Differences of this magnitude are significant, since α is an important discriminator of cosmic-ray acceleration models in the intracluster medium.

Key words: methods: data analysis – galaxies: clusters of – intergalactic medium – radio continuum: galaxies

1. Introduction

Some of the richest and most X-ray luminous clusters of galaxies show diffuse cluster-wide radio emission not associated with individual galaxies. These extended radio halos are usually classified as steep-spectrum radio sources due to the rather high spectral index ($\alpha > 0.75$) of their integrated diffuse radio flux densities. The diffuse radio emission is synchrotron radiation from relativistic electrons (e.g. Pacholczyk 1970) gyrating around the field lines of the magnetic field pervading the intracluster medium (ICM) (e.g. Kronberg 1994). In order to construct reliable models for halo formation (see Hanisch 1982, for a review; De Young 1992; Tribble 1993) one would like to know the spectral index of the synchrotron emission coefficient ϵ_ν and its spatial distribution in a given halo. However, the emissivity index is proportional to the observed integrated-flux index only in that case that the spatial distribution of ϵ_ν does not depend on frequency. Mathematically speaking, that means that then ϵ_ν may be written as a product of two functions, $\epsilon_\nu(\boldsymbol{r}) = f(\nu)g(\boldsymbol{r})$. On the other hand, if the full width half maximum (FWHM) inferred from the observed surface brightness distribution is different at different frequencies the extent of the effective emission volume depends on ν. Since the intracluster medium is optically thin for the emitted synchrotron radiation, the slope of the integrated-flux density spectrum is affected by such a variation of the effective emission volume, and it does not directly reflect the underlying emissivity index. For instance, for the Coma cluster the radio halo's FWHM tends to be smaller the higher the frequency (e.g., Deiss et al. 1996, and references therein); hence, the observed spectral index of the integrated diffuse radio flux density is necessarily higher than that of the synchrotron emission coefficient.

This has also implications for, e.g., the notion that if high-energy cosmic rays are present in the intracluster medium with an average intensity comparable to that observed in our Galaxy, they could produce the extragalactic diffuse gamma radiation by hadronic interaction with the thermal protons (Dar & Shaviv 1995). Recently, however, this model has been questioned by Chi et al. (1996) who argued that one would expect a spectral index of the synchrotron radiation emitted by the secondary electrons produced in these interactions of only 0.9, while the observed index were considerably steeper, namely about 1.34 (Kim et al. 1990). The latter number, however, refers to the integrated flux density spectrum; in order to obtain the spectral index of the underlying synchrotron emission, this figure has to be corrected for the variation of the effective emission volume.

Deiss et al. (1996) showed that, under some simplifying assumptions, it is possible to derive a halo's spectral-index distribution of the synchrotron emission coefficient using the combined data of integrated flux density, surface brightness distribu-

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tion and scale-size-vs.-frequency relation. Using available data for the Coma cluster, they demonstrated that the emission coefficient index can be smaller than the observed integrated-flux index by up to 1.1 at the cluster's center.

In addition, the formalism gives a rough estimate concerning the consistency of the measurements at different frequencies. Considering their new 1.4 GHz measurements and the 2.7 GHz measurements given by Schlickeiser et al. (1987) of the halo source Coma C they concluded that there are strong indications that the 2.7 GHz data are inconsistent in the sense that the given integrated diffuse flux at that frequency is probably too low. This implies that the strong spectral steepening at high frequencies claimed by Schlickeiser et al. appears to be unrealistic.

In their deprojection formalism, Deiss et al. considered a spherically symmetric gaussian halo model. However, real halos often appear to be elongated or distorted. For the halo source Coma C, the surface brightness distribution at various frequencies has been fitted by two-dimensional gaussians, the inferred scale sizes of which are: $\mathrm{FWHM}_{30.9\mathrm{MHz}} = 31' \pm 5' \times 15' \pm 5'$ (Henning 1989), $\mathrm{FWHM}_{326\mathrm{MHz}} = 28' \times 20'$ (Venturi et al. 1990) and $\mathrm{FWHM}_{1.4\mathrm{GHz}} = 18.'7 \times 13.'7$ (Kim et al. 1990).

To apply the method proposed by Deiss et al., one has to azimuthally average the observed surface brightness distribution in order to evaluate the averaged spatial scale size, i.e. the full-width-half-maximum, of the halo at a given frequency. The scale-size-vs.-frequency relation is the essential ingredient of the deprojection method. One might suspect that the azimuthal averaging and the assumption of a gaussian shape of the halo constraints the accuracy of the inferred deprojected spectral index as well as the applicability of the method. It is the aim of this paper to show that the applicability of the deprojection formalism suggested by Deiss et al. does not rely on the assumption of sphericity or gaussian shape of the halo's spatial profil, but that it can be applied to more general, non-spherical and non-gaussian halo models as well. Hence, in order to apply the deprojection method, it is not necessary to use a scale-size-vs.-frequency relation that is based on azimuthal averages.

2. Deprojection method

2.1. Basic assumptions

For the spatial distribution of the emission coeffient within a halo's volume, $\epsilon_\nu(\boldsymbol{r})$, Deiss et al. assumed a spherically symmetric gaussian. As a simple generalization of that, one may adopt a three-dimensional gaussian, i.e.,

$$\epsilon_\nu(\boldsymbol{r}) = \epsilon_\nu(0)\ \exp\left[-\frac{x_1^2}{2\Theta_{1,\nu}^2} - \frac{x_2^2}{2\Theta_{2,\nu}^2} - \frac{x_3^2}{2\Theta_{3,\nu}^2}\right] , \qquad (1)$$

where $\Theta_{1,\nu}$, $\Theta_{2,\nu}$ and $\Theta_{3,\nu}$ denote the scale sizes along the main axes of the ellipsoid at frequency ν. Or, if the observed surface brightness distribution has a power-law shape one may adopt a power-law distribution for $\epsilon_\nu()$ as well, e.g.

$$\epsilon_\nu(\boldsymbol{r}) = \epsilon_\nu(0)\ \left[1 + \frac{x_1^2}{2\Theta_{1,\nu}^2} + \frac{x_2^2}{2\Theta_{2,\nu}^2} + \frac{x_3^2}{2\Theta_{3,\nu}^2}\right]^{-\beta} . \qquad (2)$$

In the present work, we only retain two assumptions regarding the distribution function of the synchrotron emission coefficient: i) The $\epsilon-$distribution is symmetric about the origin; ii) the distribution function implies scale lengths along the (cartesian) coordinate axes x_i, i.e.,

$$\epsilon_\nu(\boldsymbol{r}) = \epsilon_\nu(0)\, f\left(\frac{x_\mathrm{i}}{\Theta_{i,\nu}}\right) = \epsilon_\nu(0)\, f\left(-\frac{x_\mathrm{i}}{\Theta_{i,\nu}}\right) , \qquad (3)$$

where the latter relation just represents the applied symmetry; function f is otherwise arbitrary.

In addition, we assume that, for any two different frequencies considered, say $\nu_1 = \mathrm{m}$ and $\nu_2 = \mathrm{n}$, the halo can be regarded as being self-similar in the sense that the ratios of the scale sizes are to be identical for each of the main axes. That means,

$$\frac{\Theta_{1,\mathrm{m}}}{\Theta_{1,\mathrm{n}}} = \frac{\Theta_{2,\mathrm{m}}}{\Theta_{2,\mathrm{n}}} = \frac{\Theta_{3,\mathrm{m}}}{\Theta_{3,\mathrm{n}}} \equiv \frac{\Theta_\mathrm{m}}{\Theta_\mathrm{n}} \qquad (4)$$

This assumption is also implicitely contained in the simple spherically symmetric case considered by Deiss et al..

For the sake of simplicity, we consider a locally isotropic synchrotron emission coefficient.

2.2. Consistency of observational data

The integrated diffuse flux density at frequency ν is given by

$$\begin{aligned} S_\nu &= (\mathrm{const})\,\epsilon_\nu(0) \int_{-\infty}^{\infty} \mathrm{f}\left(\frac{\mathrm{x_i}}{\Theta_{\mathrm{i},\nu}}\right) \mathrm{dx_1 dx_2 dx_3} \\ &= (\mathrm{const})\,\epsilon_\nu(0)\Theta_{1,\nu}\Theta_{2,\nu}\Theta_{3,\nu} \int_{-\infty}^{\infty} \mathrm{f(t_i)dt_1dt_2dt_3} , \end{aligned} \qquad (5)$$

where we applied the substitution $t_\mathrm{i} = x_\mathrm{i}/\Theta_{i,\nu}$. The remaining integral is just a number that does not depend on $\Theta_{i,\nu}$. For the ratio at two different frequencies, this number cancels out and one obtains

$$\frac{S_\mathrm{m}}{S_\mathrm{n}} = \frac{\epsilon_\mathrm{m}(0)}{\epsilon_\mathrm{n}(0)} \frac{\Theta_{1,\mathrm{m}}}{\Theta_{1,\mathrm{n}}} \frac{\Theta_{2,\mathrm{m}}}{\Theta_{2,\mathrm{n}}} \frac{\Theta_{3,\mathrm{m}}}{\Theta_{3,\mathrm{n}}} = \frac{\epsilon_\mathrm{m}(0)}{\epsilon_\mathrm{n}(0)} \left(\frac{\Theta_\mathrm{m}}{\Theta_\mathrm{n}}\right)^3 , \qquad (6)$$

where the latter relation follows from assumption (4).

Next, we consider the surface brightness distribution which is, up to a constant, the two-dimensional projection of $\epsilon_\nu(\boldsymbol{r})$ onto the celestial sphere. For the further considerations, it is convenient to consider a cartesian coordinate system x'_i, where the x'_3-axis denotes the line of sight, while the x'_1 and x'_2-axis lie in the plane of the sky perpendicular to the l.o.s. The x_i-coordinate system is that system in which a given distribution function (3) may be conveniently represented; for instance, if one adopts a three-dimensional gaussian distribution (1) it is convenient to put the x_i-axes along the main axes of the ellipsoid. Hence, in general the x_i-system and the x'_i-system are arbitrarily rotated to each other while having the same origin. The rotation may be described by a rotation matrix with elements a_ik, i.e.,

$$x_\mathrm{i} = \sum_{\mathrm{k}=1}^{3} a_\mathrm{ik}\, x'_\mathrm{k} . \qquad (7)$$

The surface brightness distribution at the projected location (x_1', x_2') is then given by

$$I_\nu(x_1', x_2') = (\text{const})\, \epsilon_\nu(0) \int_{-\infty}^{\infty} \text{f}\left(\frac{1}{\Theta_{\text{i},\nu}} \sum_{\text{k}=1}^{3} \text{a}_{\text{ik}} \text{x}_\text{k}'\right) \text{dx}_3' \qquad (8)$$

This is again a symmetric distribution in the sense that $I_\nu(x_1', x_2') = I_\nu(-x_1', -x_2')$, which follows from assumption (3).

At a frequency ν = m the central surface brightness distribution is given by

$$\begin{aligned} I_\text{m}(0) &= (\text{const})\, \epsilon_\text{m}(0) \int_{-\infty}^{\infty} \text{f}\left(\frac{1}{\Theta_{\text{i,m}}} \sum_{\text{k}=3}^{3} \text{a}_{\text{ik}} \text{x}_\text{k}'\right) \text{dx}_3' \\ &= (\text{const})\, \epsilon_\text{m}(0) \int_{-\infty}^{\infty} \text{f}\left(\frac{1}{\Theta_{\text{i,m}}} \text{a}_{\text{i3}} \text{x}_3'\right) \text{dx}_3' \,. \end{aligned} \qquad (9)$$

At another frequency, say ν = n, it may be expressed by

$$\begin{aligned} I_\text{n}(0) &= (\text{const})\, \epsilon_\text{n}(0) \int_{-\infty}^{\infty} \text{f}\left(\frac{1}{\Theta_{\text{i,n}}} \text{a}_{\text{i3}} \text{x}_3'\right) \text{dx}_3' \\ &= (\text{const})\, \epsilon_\text{n}(0) \frac{\Theta_\text{n}}{\Theta_\text{m}} \int_{-\infty}^{\infty} \text{f}\left(\frac{1}{\Theta_{\text{i,m}}} \text{a}_{\text{i3}} \text{x}_3''\right) \text{dx}_3'' \,, \end{aligned} \qquad (10)$$

where we have used the definition

$$x_\text{k}'' = \left(\frac{\Theta_\text{m}}{\Theta_\text{n}}\right) x_\text{k}' \,. \qquad (11)$$

The remaining integrals in Eqs. (9) and (10) are identical; hence, we obtain for the ratio of the central surface brightness distribution at two different frequencies

$$\frac{I_\text{m}(0)}{I_\text{n}(0)} = \frac{\epsilon_\text{m}(0)}{\epsilon_\text{n}(0)} \frac{\Theta_\text{m}}{\Theta_\text{n}}, \qquad (12)$$

The ratio of the halo's internal scale sizes, $(\Theta_\text{m}/\Theta_\text{n})$, is identical to the ratio of the observed scale sizes of the surface brightness distribution, i.e. (see Appendix),

$$\frac{\Theta_\text{m}}{\Theta_\text{n}} = \frac{\text{FWHM}_{\text{x}_1',\text{m}}}{\text{FWHM}_{\text{x}_1',\text{n}}} = \frac{\text{FWHM}_{\text{x}_2',\text{m}}}{\text{FWHM}_{\text{x}_2',\text{n}}} \,. \qquad (13)$$

As it is usually done, we define power-law indices through the ratios

$$\frac{S_\text{m}}{S_\text{n}} = \left(\frac{m}{n}\right)^{-\alpha_{S,\text{mn}}}, \qquad (14a)$$

$$\frac{I_\text{m}(x_1', x_2')}{I_\text{n}(x_1', x_2')} = \left(\frac{m}{n}\right)^{-\alpha_{I,\text{mn}}(x_1', x_2')}, \qquad (14b)$$

$$\frac{\epsilon_\text{m}(\boldsymbol{r})}{\epsilon_\text{n}(\boldsymbol{r})} = \left(\frac{m}{n}\right)^{-\alpha_{\epsilon,\text{mn}}(\boldsymbol{r})}, \qquad (14c)$$

$$\frac{\text{FWHM}_\text{m}}{\text{FWHM}_\text{n}} = \left(\frac{m}{n}\right)^{-q_\text{mn}}. \qquad (14d)$$

From Eqs. (6) and (12) - (14) one obtains the relation

$$\alpha_{S,\text{mn}} = \alpha_{I,\text{mn}}(0) + 2\, q_\text{mn} \qquad (15)$$

which comprises observational quantities only: It relates to each other the measurements of the integrated flux, of the central surface brightness, and of the scale size of the halo at two different frequencies. Hence, expression (15) provides a check on the consistency of the observational data; it can give a hint to whether there are considerable systematic errors, e.g., in the measurement of the integrated flux densities.

Eq. (15) is identical to the respective expression given by Deiss et al. (1996) for a spherically symmetric gaussian halo. This shows that even with the less restrictiv assumptions (3) and (4) the consistency of the observational data can be checked by applying relation (15).

To illustrate the importance of relation (15), we consider radio observations of the Coma cluster: First, we consider measurements at frequencies m = 326 MHz (Venturi et al. 1990) and n = 1.38 GHz (Kim et al. 1990). The integrated fluxes are $S_{326\text{MHz}} = 3.18 \pm 0.03$ Jy and $S_{1.38\text{GHz}} = 0.53 \pm 0.05$ Jy, which yields a spectral index of $\alpha_{S,\text{mn}} = 1.24$. The observed scale sizes are $\text{FWHM}_{326\text{MHz}} = 28' \times 20'$ and $\text{FWHM}_{1.38\text{GHz}} = 18'.7 \times 13'.7$. The ratio of the scale sizes of the major axes is 0.668, while the respective ratio of the minor axes gives 0.685. Taking the mean of both values, we obtain a scale index of $q_\text{mn} = 0.27$. According to relation (15) the spectral index of the central surface brightness should be $\alpha_{I,\text{mn}}(0) = 0.70$. This is in very good agreement with the measurements of the spectral index distribution of the surface brightness by Giovannini et al. (1993), who inferred a value of 0.6 -0.8 for the central part of the halo. Hence, the observational data at 326 MHz and 1.38 GHz appear to be consistent to each other. For the spherically symmetric halo model, Deiss et al. derived $\alpha_{I,\text{mn}}(0) = 0.67$.

As a second example, we consider the measurements at 1.4 GHz (Deiss et al. 1996) and 2.7 GHz (Schlickeiser et al. 1987). The consistency of the data has already been discussed in the paper by Deiss et al. (1996). Since only azimuthally averaged FWHM-data exist, we just give the spectral indices inferred by Deiss et al.: $\alpha_{S,\text{mn}} = 3.37$, $\alpha_{I,\text{mn}}(0) = 1.40$ and $q_\text{mn} = 0.36$. Inserting these numbers in Eq. (15) shows that the data are obviously not consistent to each other. Since the measurements at 1.4 GHz appear to be in good agreement with the observations by Kim et al. (1990), this disagreement of the indices strongly indicates that the given value of the integrated diffuse flux at 2.7 GHz is too low. Hence, the sharp spectral break above 1.4 GHz claimed by Schlickeiser et al. appears to be questionable.

2.3. Spatial distribution of spectral indices

The power-law index of the synchrotron emission coefficient at the halo's center can be inferred from Eqs. (6) or (12). One obtains

$$\alpha_{\epsilon,\text{mn}}(0) = \alpha_{S,\text{mn}} - 3\, q_\text{mn}, \qquad (16)$$

and

$$\alpha_{\epsilon,\text{mn}}(0) = \alpha_{I,\text{mn}}(0) - q_\text{mn}\,, \qquad (17)$$

respectively. Eq. (16) shows that the emissivity index in the central region of a radio halo can be considerably smaller than the integrated-flux index. As an example, we consider the measurements of the Coma halo at m = 326 MHz and n = 1.38 GHz cited above. The integrated-flux index and the scale size index amount to $\alpha_{S,\mathrm{mn}} = 1.24$ and $q_{\mathrm{mn}} = 0.27$ respectively. Employing Eq. (16) we obtain $\alpha_{\epsilon,\mathrm{mn}}(0) = 0.43$. This rather small value implies an underlying energy spectrum index, x, of the relativistic electrons of $x = 1.86$ (see e.g. Pacholczyk 1970), i.e., a value smaller than 2. Particle acceleration in strong shocks produce a power law distribution with an index of at least $x = 2$ (e.g. Longair 1994). This is an indication for the existence of some very energetic processes operating in the intracluster medium at the center of the Coma cluster. Hence, without correcting for a possible variation of the effective emission volume, one may draw misleading conclusions concerning the physics of radio halo formation by just equating $\alpha_{\epsilon,\mathrm{mn}}$ and $\alpha_{S,\mathrm{mn}}$.

From the ratio of the surface brightness distribution [Eq. (8)] at two different frequencies one obtains

$$\ln \frac{\int f\left(\Theta_{\mathrm{i,m}}^{-1}\sum_k^3 a_{\mathrm{ik}}x'_{\mathrm{k}}\right)\mathrm{d}x'_3}{\int \mathrm{f}\left(\Theta_{\mathrm{i,n}}^{-1}\sum_{\mathrm{k}}^3 \mathrm{a}_{\mathrm{ik}}\mathrm{x}'_{\mathrm{k}}\right)\mathrm{d}x'_3} = \ln\left(\frac{\mathrm{n}}{\mathrm{m}}\right)\left[\alpha_{I,\mathrm{mn}}(x'_1, x'_2) - \alpha_{I,\mathrm{mn}}(0) + q_{\mathrm{mn}}\right] \,. \tag{18}$$

The rhs of this equation comprises only observational quantities. Hence, this relation allows one, at least in principle, to place constraints on the profile function f. In general, this can be achieved only numerically. A feasible procedure could be to make an assumption on the shape of the profile function f and on the spatial orientation of the halo, to perform the integration along the l.o.s., and to compare the result with the rhs of Eq. (18). In that way, one could narrow the range of free parameters like the power law exponent β in Eq. (2). A further consideration of such a treatment of arbitrary distribution functions is, however, beyond the scope of this paper.

Here, we proceed by adopting a certain type of distribution function: We consider an axisymmetric gaussian halo model, which has the advantage that one can derive simple analytical relations from Eq. (18). The assumption of a gaussian halo model may be justified by the following argument: The scale size of the surface brightness distribution of a radio halo, i.e., its FWHM, is usually inferred from a gaussian fit to the observed two-dimensional spatial profil. Since the shape of the halos' surface brightness distribution is more or less gaussian, such a fit appears to be reasonable. The three-dimensional distribution of the emission coefficient underlying a projected two-dimensional gaussian distribution of the surface brightness distribution is, in general, a gaussian as well.

For the sake of a more convenient notation, we use (x, y, z) coordinates instead of (x_1, x_2, x_3) in the following. Without loss of generality, we assume

$$\Theta_{y,\nu} = \Theta_{z,\nu} \tag{19}$$

in expression (1); so the $x-$axis is the axis of symmetry. Let x' and y' denote the main axes of the two-dimensional surface brightness distribution in the plane of the sky, and the z'-axis be the direction of the l.o.s. The halo's orientation can then be characterized by a single angle γ between the $x-$axis and the $x'-$axis due to a rotation about the $y-$axis. The $y-$axis is assumed to be identical to the $y'-$axis.

Integrating (1) along the line of sight yields for the surface brightness distribution

$$I_\nu(x', y') = \sqrt{2\pi}\epsilon_\nu(0)\frac{\Theta_{x,\nu}\Theta_{y,\nu}}{\Theta_{x',\nu}} \times \exp\left[-\frac{x'^2}{2\Theta_{x',\nu}^2} - \frac{y'^2}{2\Theta_{y,\nu}^2}\right] \,, \tag{20a}$$

where

$$\Theta_{x',\nu}^2 = \Theta_{x,\nu}^2 \cos^2\gamma + \Theta_{z,\nu} \sin^2\gamma \,. \tag{20b}$$

Eq. (20) applies to a prolate as well as to an oblate radio halo. However, if one would like to compare an observed surface brightness distribution and the theoretical profil (20) one has to make an assumption on the three-dimensional shape of the halo: For an assumed prolate halo, the x' axis which corresponds to the halo's axis of symmetry has to be assigned to the major axis of the observed surface brightness distribution; for an assumed oblate halo, x' has to be assigned to the minor axis of the two-dimensional distribution. In both cases, the observed scale size along the $y'-$axis is a direct measure of $\Theta_{y,\nu}$. For $\gamma = 0$ the observed scale size along the x'-axis corresponds directly to $\Theta_{x,\nu}$ [Eq. (20b)].

Employing assumption (4) and Eqs. (12) - (14) and (20) one obtains from (18) a theoretical spectral-index distribution given by

$$\alpha_{I,\mathrm{mn}}(x', y') = \alpha_{I,\mathrm{mn}}(0) + \frac{x'^2}{r_{x',\mathrm{mn}}^2} + \frac{y'^2}{r_{y',\mathrm{mn}}^2} \,, \tag{21a}$$

where the characteristic radii $r_{x',\mathrm{mn}}$ and $r_{y',\mathrm{mn}}$ are defined through

$$r_{x',\mathrm{mn}} = (4\ln 2)^{-1/2}\mathrm{FWHM}_{\mathrm{x',m}}\left[\ln\left(\frac{\mathrm{n}}{\mathrm{m}}\right)\right]^{1/2} \times \left[\left(\frac{\mathrm{FWHM}_{\mathrm{x',m}}}{\mathrm{FWHM}_{\mathrm{x',n}}}\right)^2 - 1\right]^{-1/2} \tag{21b}$$

and

$$r_{y',\mathrm{mn}} = r_{x',\mathrm{mn}}\frac{\mathrm{FWHM}_{\mathrm{y',m}}}{\mathrm{FWHM}_{\mathrm{x',m}}} \tag{21c}$$

On the one hand, the characteristic radii $r_{x',\mathrm{mn}}$ and $r_{y',\mathrm{mn}}$ can be directly measured, at least in principle (e.g., Giovannini et al. 1993); on the other hand, they are determined by the relations (21b,c). Hence, the comparison of the theoretical profil (21) and the observed spectral-index distribution provides an additional check on the consistency of the observational data; and it gives a hint on the degree of reliability of the gaussian fits. As an example, we consider again the Coma observations at m = 326 MHz and n = 1.38 GHz. From Eqs. (21b,c) we infer characteristic radii of $r_{x',\mathrm{mn}} = 18\farcm1$ and $r_{y',\mathrm{mn}} = 13\farcm0$. Inserting these numbers in Eq. (21a) shows that the theoretical spectral index of the surface brightness distribution increases slowly from its central value of 0.70 to 0.90 within a region of diameter $16' \times 12'$. This corresponds to the central "plateau" according to the measurements of the spectral index distribution by Giovannini et al. (1993). At a projected radius of $18'$ along the main axis of the projected halo, and of $13'$ along the halo's minor axis, the theoretical spectral index reaches 1.7. This is also in accord with the observations given by Giovannini et al. This indicates that the theoretical spectral-index distribution of the Coma halo fits the main features of the observed distribution, and that the assumption of a gaussian profile function appears to be reasonable.

Taking advantage from the assumption of a gaussian distribution function, equation (20a) may be rewritten by

$$I_\nu(x', y') = \sqrt{2\pi}\epsilon_\nu(0)\frac{\Theta_{x,\nu}\Theta_{y,\nu}}{\Theta_{x',\nu}}\ \epsilon_\nu(x = x', \rho = y') \times \exp\left[\frac{x'^2}{2}\left(\frac{1}{\Theta^2_{x,\nu}} - \frac{1}{\Theta^2_{x',\nu}}\right)\right], \tag{22a}$$

where the azimuthal coordinate ρ of the axisymmetric halo is given by

$$\rho = (y^2 + z^2)^{1/2}. \tag{22b}$$

Using (4), (12), (13), (14), (20b) and (21) yields the desired expression for the spatial distribution of the spectral index of the synchrotron emission coefficient. It reads

$$\alpha_{\epsilon,\mathrm{mn}}(\boldsymbol{r})\big|_{(x=x',\rho=y')} = \alpha_{I,\mathrm{mn}}(x', y') - q_{\mathrm{mn}} - \frac{x'^2}{r^2_{x',\mathrm{mn}}}\sin^2\gamma\left(1 - \frac{\Theta^2_{y,\mathrm{m}}}{\Theta^2_{x,\mathrm{m}}}\right), \tag{23}$$

where the angle γ and the ratio of the halo's scale sizes $\Theta_{y,\mathrm{m}}/\Theta_{x,\mathrm{m}}$, i.e., its ellipticity, are free parameters.

For $\gamma = 0$, which implies $x = x'$, $y = y'$ and $z = z'$, the latter term of the sum in (23) vanishes, yielding

$$\alpha_{\epsilon,\mathrm{mn}}(\boldsymbol{r})\big|_{(x=x',\rho=y')} = \alpha_{I,\mathrm{mn}}(x', y') - q_{\mathrm{mn}}\,. \tag{24}$$

This shows that, for a decreasing scale-size-vs.-frequency-relation, the emissivity index is smaller than the spectral index of the surface brightness. Regardless of whether the halo has a prolate or an oblate shape, Eq. (24) gives directly the distribution of the emissivity index in the $(z = 0)-$plane, where $\rho = y = y'$. For the case of the Coma halo at 326 MHz and 1.38 Ghz, in that plane $\alpha_{\epsilon,\mathrm{mn}}$ increases from its central value of 0.43 to 0.75 within a region of diameter $20' \times 15'$. This demonstrates that the usual classification of the Coma halo as a "steep-spectrum radio source" does not apply to the local synchrotron emission coefficient in the halo's core region.

In order to construct the spatial distribution of $\alpha_{\epsilon,\mathrm{mn}}$ for $z \neq 0$, one has to make an assumption on the structural shape of the halo, i.e., whether it is prolate or oblate, as discussed above. For the case $\gamma = 0$, $\Theta_{x,\nu}$ and $\Theta_{y,\nu}$ are directly given by the observed scale lengths of the surface brightness distribution. Eq. (24) can then easily be applied to construct the spatial distribution of $\alpha_{\epsilon,\mathrm{mn}}(\boldsymbol{r})$ by applying index q_{mn} together with the theoretical distribution (21) or an observationally given distribution (e.g. Giovannini et al. 1993) of $\alpha_{I,\mathrm{mn}}$.

For $\gamma \neq 0$ ("oblique halo"), the ratio $\Theta_{y,\mathrm{m}}/\Theta_{x,\mathrm{m}}$ is an additional free parameter in Eq. (23). However, once that this ratio and the angle γ are fixed, the structural shape as well as the orientation of the halo are completely determined: i) $\Theta_{y,\mathrm{m}}/\Theta_{x,\mathrm{m}} < 1$ implies a prolate halo, and the $x'-$axis has to be assigned to the major axis of the observed surface brightness distribution; ii) $\Theta_{y,\mathrm{m}}/\Theta_{x,\mathrm{m}} > 1$ implies an oblate halo, and the $x'-$axis has to be assigned to the minor axis. Eq. (23) then gives the spatial distribution of the emissivity index, where one may use the theoretical profile (21) or an observed distribution of $\alpha_{I,\mathrm{mn}}$. For an oblique prolate halo, one infers an even smaller emissivity index than for the case $\gamma = 0$; for an oblique oblate halo, the reverse is true. Unfortunately, the structural shape and the orientation of the halo generally cannot be resolved from observations of the diffuse radio halo itself; however, theoretical considerations on halo formation as well as observations of the galaxy distribution and/or X-ray observations may place some constraints on that.

For an assumed spherically symmetric halo, Eq. (23) reduces to (Deiss et al. 1996)

$$\alpha_{\epsilon,\mathrm{mn}}(r)\big|_{(r=x')} = \alpha_{I,\mathrm{mn}}(x') - q_{\mathrm{mn}}\,. \tag{25}$$

In that case, $\alpha_{\epsilon,\mathrm{mn}}(r)$ is completely determined by (25) without any further assumptions.

3. Distribution of the emission coefficient $\epsilon_\nu(\boldsymbol{r})$

From a theoretical point of view, one would like to know not only the spatial distribution of a few spectral-indices of the emission coefficient but the complete spatial and spectral distribution of $\epsilon_\nu(\boldsymbol{r})$ itself. In principle, the distribution of $\epsilon_\nu(\boldsymbol{r})$ can be constructed by successively applying Eqs. (16) or (17) and (23) for different pairs of frequencies. This implies, however, that the radio images of a halo have a sufficiently high spatial resolution to allow for a reliable evaluation of its scale size, and that these images are available at as many as possible frequencies over a broad spectral range - a condition that is hardly fullfilled considering the presently available data.

4. Summary and conclusions

Theories on the formation of diffuse radio halos of clusters of galaxies have to rely on the observational determination of the spectral as well as of the spatial distribution of the synchrotron emission coefficient $\epsilon_\nu(\boldsymbol{r})$. Usually the observed integrated flux density spectrum is regarded as being representative for the spectral distribution of ϵ_ν. However, as has been observed, e.g., in the halo source Coma C, the scale size of a halo may be different at different frequencies. Hence, the integrated-flux spectrum as well as the spectral-index distribution of the surface brightness is necessarily affected by that frequency-dependend variation of the effective emission volume.

Deiss et al. (1996) have shown that, using the combined data of integrated flux density and scale-size-vs.-frequency relation, an analytic deprojection of the spectral-index distribution of the surface brightness distribution can be achieved, which allows one to derive the spatial distribution of the spectral index of ϵ_ν. The authors considered a spherically symmetric gaussian radio halo. A crucial parameter of the deprojection method is the spectral index of the scale size, which is a measure for the variation of the effective emission volume at different frequencies. The consideration of a spherically symmetric halo model implies that this index has to be evaluated from azimuthally averaged radio halo images. Since real halos often appear to be elongated or distorted, the averaging procedure constraints the accuracy of the scale-size index.

In the present work, the deprojection formalism is generalized to the case of any halo model that is symmetric about the halo's center but otherwise arbitrary. This allows to use a scale-size index that is not necessarily based on azimuthally averaged data.

The main results are: i) For arbitrary (but symmetric) halo models, we obtained analytical expressions which allow to check the consistency of the observational data at different frequencies and to infer the spectral index of the synchrotron emission coefficient in the central region of a halo. ii) We derived an equation that allows one to place constraints on the distribution function of the emission coefficient; in general, this equation has to be solved numerically. iii) For an axisymmetric gaussian halo model, we derived a theoretical spectral index distribution of the surface brightness that can be compared with an observed distribution, as well as an analytical expression that relates the spatial distribution of the emission coefficient to observationally given quantities.

As an example, we considered measurements of the diffuse radio halo of the Coma cluster given in the literature. We showed that the deprojected, "real" spectral index of the underlying synchrotron emission coefficient may be smaller than the observed integrated-flux index by about 0.8. This has considerable implications for radio halo formation theories as well as for extragalactic cosmic-ray models. Since deprojection has such a drastic effect on the spectral index of the synchrotron emission, the reliability of spectral fits to the integrated flux density using a spatially averaged halo model, as it was done by Schlickeiser et al. 1987, is rather limited.

Acknowledgements. I would like to acknowledge the kind hospitality of the Department of Astronomy, Univ. of Toronto, Canada, where portions of this work were done. And I thank P.P. Kronberg for helpful discussions.

Appendix A

The projected coordinates (x_1', x_2) may be scaled by some characteristic scale lengths, $\Theta'_{1,\nu}$ and $\Theta'_{2,\nu}$, which can be determined by observations. These characteristic scale lengths may be, for instance, the FWHM of a two-dimensional gaussian surface brightness distribution. At frequencies $\nu_1 = \mathrm{m}$ and $\nu_2 = \mathrm{n}$, the surface brightness distribution (8) may be written

$$I_\mathrm{m}\left(\frac{x_1'}{\Theta'_{1,\mathrm{m}}}, \frac{x_2'}{\Theta'_{2,\mathrm{m}}}\right) = \mathrm{(const)}\,\epsilon_\mathrm{m}(0) \int_{-\infty}^{\infty} \mathrm{f}\left(\frac{1}{\Theta_{\mathrm{i,m}}} \sum_{\mathrm{k}=1}^{3} \mathrm{a_{ik}}\, \Theta'_{\mathrm{k,m}} \frac{\mathrm{x}_\mathrm{k}'}{\Theta'_{\mathrm{k,m}}}\right) \mathrm{dx}_3' \quad \text{(A1)}$$

and

$$I_\mathrm{n}\left(\frac{x_1'}{\Theta'_{1,\mathrm{n}}}, \frac{x_2'}{\Theta'_{2,\mathrm{n}}}\right) = \mathrm{(const)}\,\epsilon_\mathrm{n}(0) \int_{-\infty}^{\infty} \mathrm{f}\left(\frac{1}{\Theta_{\mathrm{i,n}}} \sum_{\mathrm{k}=1}^{3} \mathrm{a_{ik}}\, \Theta'_{\mathrm{k,n}} \frac{\mathrm{x}_\mathrm{k}'}{\Theta'_{\mathrm{k,n}}}\right) \mathrm{dx}_3' \,, \quad \text{(A2)}$$

where, for the sake of simple notation, we also formally introduced a scale length $\Theta'_{3\mathrm{m}}$ which, however, is equal to unity. Using (11), (12) and (A1), the rhs of Eq. (A2) can be rewritten by

$$\begin{aligned} \mathrm{rhs} &= \mathrm{(const)}\,\epsilon_\mathrm{n}(0) \frac{\Theta_\mathrm{n}}{\Theta_\mathrm{m}} \\ &\times \int_{-\infty}^{\infty} f\left(\frac{1}{\Theta_{\mathrm{i,m}}} \sum_{k=1}^{3} a_{\mathrm{ik}}\, \Theta'_{\mathrm{k,m}} \frac{x_\mathrm{k}''}{\Theta'_{\mathrm{k,m}}}\right) \mathrm{dx}_3'' \\ &= \frac{I_\mathrm{n}(0)}{I_\mathrm{m}(0)} I_\mathrm{m}\left(\frac{x_1''}{\Theta'_{1,\mathrm{m}}}, \frac{x_2''}{\Theta'_{2,\mathrm{m}}}\right) . \end{aligned} \quad \text{(A3)}$$

From that we obtain the relation

$$\frac{I_\mathrm{m}\left(\frac{x_1'}{\Theta'_{1,\mathrm{n}}} \frac{\Theta'_{1,\mathrm{n}}}{\Theta'_{1,\mathrm{m}}} \frac{\Theta_\mathrm{m}}{\Theta_\mathrm{n}}, \frac{x_2'}{\Theta'_{2,\mathrm{n}}} \frac{\Theta'_{2,\mathrm{n}}}{\Theta'_{2,\mathrm{m}}} \frac{\Theta_\mathrm{m}}{\Theta_\mathrm{n}},\right)}{I_\mathrm{n}\left(\frac{x_1'}{\Theta'_{1,\mathrm{n}}}, \frac{x_2'}{\Theta'_{2,\mathrm{n}}}\right)} = \frac{I_\mathrm{m}(0)}{I_\mathrm{n}(0)} \,, \quad \text{(A4)}$$

where the rhs is a constant. Since we assumed the three-dimensional halo to be self-similar at any two different frequencies, the resulting projected profils are also self-similar. The ratio at the lhs of Eq. (A4) is a constant if the coordinates (x_1', x_2') are measured by the same scale. This implies

$$\frac{\Theta_\mathrm{m}}{\Theta_\mathrm{n}} = \frac{\Theta'_{1,\mathrm{m}}}{\Theta'_{1,\mathrm{n}}} = \frac{\Theta'_{2,\mathrm{m}}}{\Theta'_{2,\mathrm{n}}} \,, \quad \text{(A5)}$$

which is equivalent to relation (13).

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