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Principles of statistical astrometry

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Abstract. We present a coherent scheme of 'statistical astrometry' for high-precision measurements. Statistical astrometry provides methods and tools for treating quantitatively the overall statistical effect of the presence of many individually unresolved or unmodelled astrometric binaries in a stellar ensemble. The non-linear motions of the photo-centers of these binaries give rise to deviations from long-term linear motions. These deviations may be called 'cosmic errors', since they represent a source of 'noise' (in addition to measuring errors) with respect to the assumed linear motions of the stars. For bright HIPPARCOS stars, the cosmic errors are on average larger than the measuring errors of the 'instantaneous' positions and proper motions of these stars. Basic tools of statistical astrometry are correlation functions between the orbital displacements in position and velocity relative to the mean motions of the stars. We present methods for calculating the mean errors of stellar positions predicted on the basis of measured instantaneous data or of mean data, including the cosmic errors. We discuss the comparison of astrometric catalogues, containing either instantaneous data or mean ones, the question of using acceleration terms, the treatment of 'averaged' observational data, and some problems connected with the determination and the behaviour of the correlation functions. Our general conclusion is that in high-precision astrometry, the effect of the cosmic errors is often dominant with respect to the measuring errors and should therefore be treated properly.

Key words: astrometry – catalogues – binaries: general

1. Introduction

With the advent of observing instruments like the ESA Astrometry Satellite HIPPARCOS, the measuring accuracy in astrometry has been dramatically increased. It is the purpose of the present paper to point out that this is not only a quantitative progress but that high-precision astrometry is opening also *qualitatively* a new area of astrometry. High-precision measurements of stellar positions and proper motions introduce a strong statistical element into astrometry. The statistical nature of high-precision astrometry is not due to the measuring errors, but is caused by the physical behaviour of the observed stars, many (if not most) of them being binaries, and by our incomplete knowledge of this behaviour. A quantitative treatment of high-precision astrometry requires new concepts and methods. I propose to call this new field 'statistical astrometry'.

In Sect. 2, the basic reason for statistical astrometry is outlined. The most important tools of statistical astrometry are correlation functions for the positions and velocities of the stars. These tools will be discussed in Sect. 3. In Sect. 4, methods of statistical astrometry are presented which make use of the correlation functions and of other concepts for predicting positions and their mean errors, for comparing proper motions measured at two different epochs, etc. As implied by the title, it is the aim of present paper to describe the basic principles of statistical astrometry. The applications of these principles to actual data will be presented in future papers. We can, however, already now assure the reader that the basic concepts of statistical astrometry are valuable and realistic, since we were able to make positive use of them during our tests of the reliability of the HIPPARCOS results by means of accurate ground-based data such as the FK5.

2. Fundamental reasons for statistical astrometry

In classical astrometry, the basic assumption is that a star moves linearily in time on a straight line in space:

$$\mathbf{x}(t) = \mathbf{x}_0(t_0) + \mathbf{v}_0(t - t_0) \ , \tag{1}$$

where $\mathbf{x}(t)$ is the position of the star at time t, and $\mathbf{v}_0$ is its constant velocity. Astrometric binaries, which do not follow Eq. (1), are treated in classical astrometry individually as rare exceptions (if at all).

The vector $\mathbf{x}(t)$ is meant to represent the three-dimensional position of the star in space. In astrometry, we usually consider only the projection of $\mathbf{x}(t)$ and of $\mathbf{v}(t)$ onto the sky, thereby introducing the angular position and the proper motion of a star. In this sense, one of the components of $\mathbf{x}$ and $\mathbf{v}$ may be e.g. the declination δ and the corresponding proper motion μ_δ of the star. In this paper, we shall in general not distinguish explicitly between v (in km/s) and μ (in mas/year), since the meaning of the quantity will be either clear from the context, or will be fixed only during actual applications. The constancy of $\mathbf{v}$

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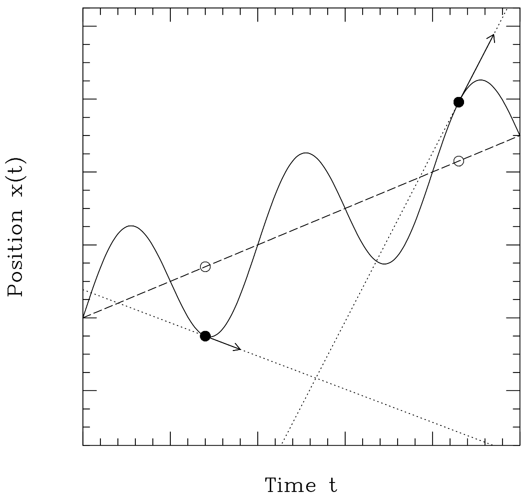

Fig. 1. Wavy motion of the photo-center of an astrometric binary (solid curve) around the linear motion of its center-of-mass (dashed line). Two instantaneous positions and proper motions are indicated (filled dots; arrows). The linear predictions based on the instantaneous values are shown as dotted lines.

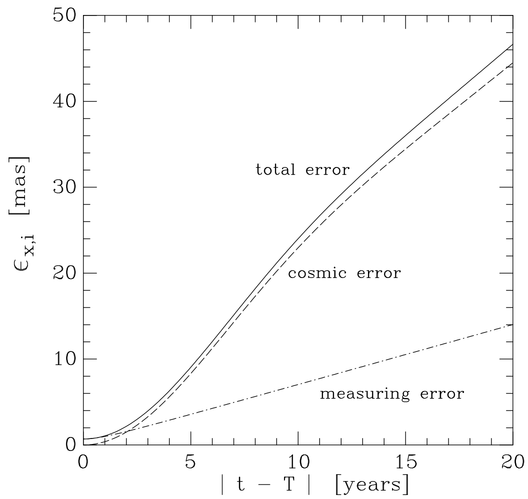

Fig. 2. Mean error $\varepsilon_{x,i}(t)$ of a predicted instantaneous position $x_p(t)$ at an epoch t, based on a linear extrapolation using an instantaneous position and an instantaneous proper motion observed at epoch T. The total mean error (solid curve) is build up by the cosmic error (dashed curve) and the measuring error (dash-dotted curve). The formulae and the numerical values used are given in Sects. 2, 3.6, and 4.2.1.

implies also constant components of the proper motion of the star if we neglect here for simplicity higher-order effects, such as the motion on a great circle instead of a linear motion in α and δ or the changing perspective due to the varying distance of the star (foreshortening). These higher-order effects have, of course, to be taken into account in real applications. The assumption of a constant proper motion of a star implies, for example, that we could predict the position of the star with infinite accuracy at any instant of time if we would have measured the position and proper motion of the star at one epoch t_0 with infinite accuracy. It will turn out that this is no longer true in the regime of statistical astrometry.

In the following, all the three directions of positional space and velocity space are in most cases statistically equivalent to each other. Hence, for writing economy, we will discuss only one component of **x** and **v**, which we denote by $x(t)$ and by $\dot{x}(t) = v(t)$. Usually, we envisage that x and v are one of the two tangential components in the plane of the sky. We may point out, however, that most of the results of statistical astrometry on v could equally well be used for a statistical treatment of high-precision radial-velocity measurements of stars.

We can now discuss the fundamental reason of statistical astrometry: *It is the astrometric-binary nature of many (or most) of the stars*. If the star is an astrometric binary (instead of being a single star), then its photo-center does not move according the Eq. (1). The photo-center of an (unresolved) astrometric binary moves on a Keplerian orbit around its center-of-mass. Hence the total motion of an astrometric binary in one component is composed of the still linear motion of the center-of-mass of the binary and of the wavy orbital motion of the photo-center around the center-of-mass. This is schematically illustrated in Fig. 1. It is the wavy orbital motion of the photo-center around the center-of-mass of astrometric binaries which is responsible for the statistical nature of high-precision astrometry: As soon as the measuring accuracy reaches the order-of-magnitude of the orbital displacements in position and in velocity, the linear model of Eq. (1) fails for the ensemble of observed stars. The consequences of this failure are treated statistically in statistical astrometry.

The failure of the linear model of Eq. (1) becomes evident if we try to use the instantaneously measured position and velocity of the star for predicting the position and proper motion of the star at another epoch. This is illustrated in Fig. 1. HIPPARCOS is measuring essentially such 'instantaneous' positions and proper motions. If we compare an instantaneous velocity $v_1(T_1)$, measured at one epoch T_1, with another instantaneous velocity $v_2(T_2)$ measured at T_2, or with the 'mean' velocity $v_{\rm cms}$, then the velocities will in general not agree. The differences between the velocities will not only reflect the measuring errors but also 'cosmic errors' caused by the binary motions of the stars under consideration.

How significant such 'cosmic errors' could be, is illustrated in Fig. 2. We investigate the mean error of a predicted position $x_p(t)$ which is based on instantaneous measurements of $x_{i,T}(T)$ and of $v_{i,T}(T)$ at an epoch T and on the linear model (Eq. (1)). We assume a measuring error in $x_{i,T}$ of $\epsilon_{x,i,T}$ = 0.7 mas and in $v_{i,T}$ of $\epsilon_{v,i,T}$ = 0.7 mas/year. The mean error in $x_p(t)$ due to the measuring errors alone is then

$$\epsilon_{x,\rm meas}(t) = (\epsilon^2_{x,i,T} + \epsilon^2_{v,i,T}(t - T)^2)^{1/2} \ . \quad (2)$$

The 'cosmic error' $\epsilon_{x,\rm cosm}$ in $x_p(t)$ is calculated according to Eq. (52) and using the correlation functions discussed in Sect. 3. While this example is rather arbitrarily chosen, it may reflect very roughly the situation for bright HIPPARCOS stars (such as the basic FK5 stars). Since the measuring errors and the cosmic errors are independent of each other, the total mean error in the predicted value of $x_p(t)$ is

$$\epsilon_{x,\mathrm{tot}}(t) = (\epsilon^2_{x,\mathrm{meas}}(t) + \epsilon^2_{x,\mathrm{cosm}}(t))^{1/2} \ . \qquad (3)$$

The example illustrated in Fig. 2 points out the general trend: The expected mean error of the predicted position $x_p(t)$ is governed by the measuring error in $x_{i,T}$ and $v_{i,T}$ for epochs close to T. The cosmic error $\epsilon_{x,\mathrm{cosm}}(t)$ is very small for small values of $|t - T|$, since close to T the linear model is a good approximation even for the stars which do show an orbital motion. For large values of $|t - T|$, however, the expected mean error of the predicted position $x_p(t)$ is mainly due to the cosmic error $\epsilon_{x,\mathrm{cosm}}(t)$, not due to the measuring error $\epsilon_{x,\mathrm{meas}}(t)$. This is only one striking example for the importance of the 'cosmic errors', or, more general, of applying methods of statistical astrometry instead of classical astrometry.

The 'cosmic errors' in the positions and velocities of stars are not real 'errors' but reflect only our limited knowledge of the actual motions of the stars. If we were able to determine for each astrometric binary its Keplerian orbit (and if no triple systems etc. would exist), then we could use the center-of-mass positions and proper motions and the orbital elements, and would not need any statistical astrometry. However, in reality, it is essentially impossible to determine the Keplerian orbit of each astrometric binary. In many cases, the measuring accuracy (with respect to the orbital displacements) and the available observational period of time (with respect to the orbital period) are not large enough for such an orbit determination. However, even if we are unable to determine *individual* orbits for most of the astrometric binaries, the *collective* effect of the unresolved astrometric binaries in a stellar sample can be quite significant. For example, if we compare the instantaneous proper motions of N stars (say N$\sim$100), measured at two different epochs of time, the *average* 'cosmic error' in these proper motions can be determined (and felt) even if this cosmic error is of the order or (slightly) smaller than the measuring errors of the proper motions, because of the $1/\sqrt{N}$ effect in the accuracy of the rms difference between the two proper motions. In such a case, only methods of statistical astrometry are able to handle the situation quantitatively.

The importance of statistical astrometry depends, beside the measuring accuracy, on the number and 'strength' of unresolved astrometric binaries in the stellar sample under consideration. It is well-known (e.g. Duquennoy & Mayor 1991) that the majority of all the stars are members of double or multiple systems. Most of these binaries are difficult to detect individually as visual binaries, spectroscopic binaries, or astrometric binaries. Even if we exclude the obviously double stars from a stellar sample, a large fraction of the remaining, apparently 'single' stars will be in reality unresolved astrometric binaries. Hence as soon as the measuring accuracy reaches the level of the average orbital displacements of these individually undetected astrometric binaries, methods of statistical astrometry are required for a suitable handling of the results.

Statistical astrometry is mostly dealing with *ensembles* of stars. Methods of statistical astrometry allow, however, also predictions about the individual behaviour of stars in the sense of an expected mean behaviour of a typical member of the ensemble. For these individual predictions, we encounter a phenomenon which is unfamiliar in astrometry, but well-known e.g. in quantum mechanics or in other fields in which statistical effects play an important role: Our predictions do not depend on our actual astrometric measurements alone, but are strongly influenced by the additional knowledge of certain properties of the object, i.e. of the star under consideration. Take for example the problem of determining the mean error $\epsilon_{x,\mathrm{tot}}(t)$ of a predicted position $x_p(t)$, discussed already in relation with Fig. 2 (Eqs. (2) and (3)). If we do not know anything else than that the star belongs to the ensemble envisaged in Fig. 2 (i.e. bright HIPPARCOS stars), then our prediction by Eq. (3), including an average cosmic error, is the best one. However, if we would know from other, e.g. astrophysical considerations that the star is very probably single, then the (smaller) conventional error estimate by Eq. (2) would be appropriate, although nothing has changed in the astrometric observations itself. In general, our additional knowledge about the star will be less clear-cut and probably more subtle than in the simple example given above. Nevertheless, additional knowledge about e.g. the distance of the star, its spectral type etc. can be used to modify and hopefully to improve the predictions made.

Earlier studies which discussed the effect of unresolved astrometric binaries on high-precision astrometry have been presented by Lindegren (1979), Soederhjelm (1985), Tokovinin (1993), Brosche et al. (1995), and Wielen (1995a). These authors have concentrated on pointing-out the problem, but they have not provided a suitable mathematical frame-work for handling the statistical consequences of this effect on astrometry in general. Statistical astrometry is aiming at establishing this frame-work. A short discussion of the contributions of statistical astrometry to the planned astrometry mission GAIA (Lindegren & Perryman 1995) has been presented elsewhere (Wielen, 1995b).

While the orbital motions of unresolved astrometric binaries are the fundamental reason for statistical astrometry, other sources of 'noise' may also contribute to the statistical nature of high-precision astrometry. One example is the motion of the photo-center of unresolved double stars in which one component is a variable star. These 'variability-induced movers' or VIMs (Wielen 1996) can, similar to the orbital motions of astrometric binaries, be treated either individually (if the effect is large enough) or statistically (if the effect is only of the order of the measuring accuracy). In principle, also other reasons for 'cosmic errors' in very high-precision astrometry, such as irregular light distributions ('spots') in rotating stellar atmospheres, may exist. As far as the correlation functions etc., which are used in statistical astrometry, are determined *empirically*, all the different sources of 'cosmic errors' in the astrometric quantities are automatically included into statistical astrometry.

In classical astrometry, it is usually assumed that the position of a star is independent of the wavelength λ (or the photometric pass-band, such as B or V) at which the observations are carried out. For most astrometric binaries, this assumption is no longer correct. If the two stars of the binary system have different colours (due to different effective temperatures), then the position of the photo-center depends on λ (e.g. Christy et al. 1983, Sorokin & Tokovinin 1985, Wielen 1996). We have called such binaries with colour-induced displacements of the photo-centers 'CIDs' (Wielen 1996). The effect of CIDs is another source of 'cosmic noise' in high-precision astrometry and affects both positions and proper motions if the observations are carried out at more than one photometric pass-band (e.g. for TYCHO or GAIA). Since the colour-induced variations of the photo-center with λ are typically an order of magnitude smaller than the total orbital displacements of astrometric binaries, the 'cosmic errors' caused by CIDs can often be neglected with respect to the 'cosmic errors' due to the total orbital motions. This is in general the case for large epoch differences, while simultaneous observations at different wavelength λ (e.g. for GAIA) would reveal the statistical effects of CIDs most clearly.

Part X of the Double and Multiple Systems Annex of the HIPPARCOS Catalogue (ESA 1997) contains 'stochastic solutions' for objects for which no other reasonable solutions could be found in agreement with the standard errors of the HIPPARCOS observations. The excess scatter in the observations of such a star is also called its 'cosmic error' ϵ. In most cases, the source of the cosmic errors used in the stochastic solutions is probably the same as that of the cosmic errors used in the present paper, namely the astrometric-binary nature of the objects. In detail, however, the definitions of the cosmic errors in the stochastic solutions and in the present paper differ and should not be mixed up. In general, there is no simple quantitative relation between the two varieties of cosmic errors (Only if all the objects were short-period binaries, we had basically $< \epsilon^2 > = \xi(0)$, with ξ defined in Sect. 3.2). The basic concepts are, however, very similar. This justifies the use of the common term 'cosmic error' in both cases.

3. Correlation functions as basic tools of statistical astrometry

The basic tools of statistical astrometry are correlation functions between the orbital displacements of astrometric binaries. The usefullness of these tools will become evident in Sect. 4. Here we provide the basic definitions of the required correlation functions and discuss some of the properties of these functions.

3.1. Orbital displacements

We have first to define the orbital displacements in position, $\Delta x(t)$, and in velocity, $\Delta v(t)$, for astrometric binaries. For single stars, these displacements are simply zero, but have nevertheless to be included into our statistical treatment.

For the velocity displacement of a binary, $\Delta v(t)$, the definition is straight-forward:

$$\Delta v(t) = v(t) - v_{\rm cms} \ , \tag{4}$$

where $v(t) = \dot{x}(t)$ is the actual velocity (or proper motion) of the photo-center of the binary, and $v_{\rm cms}$ is the constant velocity of the center-of-mass of the system.

The definition of the orbital displacement in position, $\Delta x(t)$, is slightly more elaborate. Instead of using the position of the center-of-mass, $x_{\rm cms}(t)$, as the reference position, it is more appropriate to use a newly defined 'mean' position $x_m(t)$ for this purpose. We define first the displacement $\Delta\tilde{x}(t)$ with respect to the center-of-mass:

$$\Delta\tilde{x}(t) = x(t) - x_{\rm cms}(t) \ , \tag{5}$$

where $x(t)$ is the actual position of the photo-center of the binary, and

$$x_{\rm cms}(t) = x_{\rm cms}(t_0) + v_{\rm cms}(t - t_0) \tag{6}$$

is the instantaneous position of the center-of-mass of the system. We now define the long-term average of $\Delta\tilde{x}(t)$ by

$$< \Delta\tilde{x} >_t = \lim_{T\to\infty} \{ \frac{1}{T} \int\limits_{t_0}^{t_0+T} \Delta\tilde{x}(t')dt' \} \ , \tag{7}$$

which is a constant for each star (but different for each coordinate component). If the orbit of the binary is circular, i.e. if the eccentricity e is zero, then $\Delta\tilde{x}(t)$ is purely sinusoidal in t, and hence $< \Delta\tilde{x} >_t = 0$. However, if the orbit is elliptical ($e \neq 0$), which is in general the case, then $< \Delta\tilde{x} >_t$ is different from zero, except for some degenerated cases. We now define the 'mean' position of the photo-center, $x_m(t)$:

$$x_m(t) = x_{\rm cms}(t) + < \Delta\tilde{x} >_t \ . \tag{8}$$

The 'mean' position $x_m(t)$ moves with the same velocity ($v_{\rm cms}$) as the center-of-mass, but is offset from the center-of-mass by the constant amount $< \Delta\tilde{x} >_t$. We are now able to define $\Delta x(t)$ as

$$\Delta x(t) = x(t) - x_m(t) \ . \tag{9}$$

The situation is schematically illustrated in Fig. 3.

Why should we prefer $x_m(t)$ over $x_{\rm cms}(t)$ as the reference position ? The reason is that $x_m(t)$ is rather directly observable as the 'averaged' position of the photo-center if we have observations over a long period of time. In contrast, the center-of-mass is not observable if we are not able to determine an individual Keplerian orbit for the system. In addition, the properties of some correlation functions (e.g. their asymptotic behaviour for $\Delta t \to \infty$) become simpler if we use Δx instead of $\Delta\tilde{x}$.

As already mentioned in Sect. 2, the orbital displacements $\Delta x(t)$ and $\Delta v(t)$ depend in general on the photometric pass-band in which the observations are carried out. If we consider more than one pass-band, we should identify a pass-band symbolically by its central wave-length λ_I, λ_{II}, etc., and add this quantity λ as a parameter or variable to Δx and Δv, e.g. $\Delta x(t, \lambda)$, $\Delta v(t, \lambda)$.

What happens if the system is triple or multiple of higher order ? The basic concepts and definitions remain unchanged. There exists always a center-of-mass of the total system, which

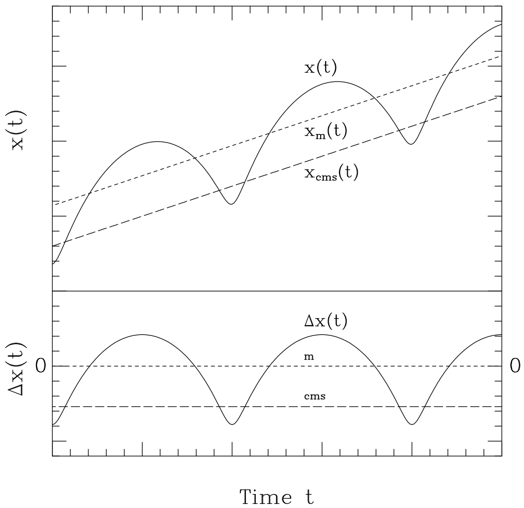

Fig. 3. Illustration of the instantaneous position $x(t)$ (solid curve), the mean position $x_m(t)$ (short-dashed line), and the center-of-mass position $x_{\rm cms}(t)$ (long-dashed line). In the lower panel, $\Delta x(t) = x(t) - x_m(t)$ is shown.

moves with a constant velocity, linearily in time. The only difference to the binary case is that the orbital displacements $\Delta x(t)$ and $\Delta v(t)$ are more complex and can no longer be described by a Keplerian two-body orbit.

The orbital displacements Δx and Δv defined above are 'instantaneous' values which would be measured if the observations were carried out during a time interval which is very short compared to the period of (all) the astrometric binaries. This is usually the case for measuring radial velocities. But it is not true for astrometric measurements which are usually summarized in the form of 'catalogues' and give 'averages' over many years of observations. We shall discuss the consequences of such an 'averaging' of astrometric observations later in Sect. 5.1. We think that it is better for the reader to get first acquainted with the much simpler results for 'instantaneous' observations. Real applications require, however, that the 'averaging' is taken into account.

3.2. Definition of correlation functions

We now define correlation functions between orbital displacements Δx and Δv at different epochs, e.g. at t and $t + \Delta t$. The operator $< q >$ means the average of a quantity q over the ensemble of stars under consideration and over the time t.

Correlation function $\xi(\Delta t)$ for positions:

$$\xi(\Delta t) = < \Delta x(t + \Delta t)\Delta x(t) > \quad . \tag{10}$$

Correlation function $\eta(\Delta t)$ for velocities:

$$\eta(\Delta t) = < \Delta v(t + \Delta t)\Delta v(t) > \quad . \tag{11}$$

Correlation function $\zeta(\Delta t)$ for positions and velocities:

$$\zeta(\Delta t) = < \Delta x(t + \Delta t)\Delta v(t) > \quad . \tag{12}$$

The functions $\xi(\Delta t)$ and $\eta(\Delta t)$ are in fact the auto-correlation functions of Δx and Δv. All the correlation functions depend, for a given ensemble of stars, on the epoch differences Δt only, not on the time t. Of course, the actual ensemble average of e.g. $\Delta x(t + \Delta t)\Delta x(t)$ at a given time t will also depend in general slightly on t, due to the time dependence of the orbital displacements and the finite number of the stars in the ensemble. However, since the various astrometric binaries are supposed to be statistically independent with respect to their orbital phases, the ensemble average at any time t should be rather close to the correlation function, defined above as the time average of the ensemble averages at various times t. In any case, the correlation functions are the 'expectation values' for the actual ensemble averages at any given time t. Hence we shall identify in actual applications the ensemble average of e.g. $\Delta x(t+\Delta t)\Delta x(t)$ with $\xi(\Delta t)$.

While the correlation functions $\xi(\Delta t)$, $\eta(\Delta t)$, and $\zeta(\Delta t)$ are certainly the most important ones, one can also define higher-order correlation functions. The HIPPARCOS catalogue will give for many stars not only positions and proper motions, but also 'acceleration terms', $\mathbf{g} = \ddot{\mathbf{x}} = d^2\mathbf{x}/dt^2$, or even $\dot{\mathbf{g}} = d\mathbf{g}/dt$. In such cases, correlation functions between two quantities $q_1(t + \Delta t)$ and $q_2(t)$ can be defined as

$$\gamma_{q_1 q_2}(\Delta t) = < q_1(t + \Delta t)q_2(t) > \ , \tag{13}$$

where q_1 and q_2 may be either Δx, Δv, g_x, or $\dot{g}_x$; for example:

$$\gamma_{xg}(\Delta t) = < \Delta x(t + \Delta t)g_x(t) > \quad . \tag{14}$$

There is no need to define correlation functions between various components of quantities, such as $< \Delta x(t+\Delta t)\Delta y(t) >$, since they vanish because of the random orientations of binary orbits in space.

3.3. Some basic properties of correlation functions

Due to the random orientation of binary orbits in space and due to the random distribution of the phases within the orbits at any given time, there are certain symmetries of the correlation functions with respect to Δt. $\xi(\Delta t)$ and $\eta(\Delta t)$ are symmetric (even) with respect to Δt:

$$\xi(\Delta t) \ = \ \xi(-\Delta t) \ , \tag{15}$$

$$\eta(\Delta t) \ = \ \eta(-\Delta t) \ . \tag{16}$$

$\zeta(\Delta t)$ is anti-symmetric (odd) in Δt:

$$\zeta(\Delta t) \ = \ -\zeta(-\Delta t) \ , \tag{17}$$

and hence

$$\zeta(0) \ = \ 0 \ . \tag{18}$$

The values of ξ and η for $\Delta t = 0$ are related to the root-mean-squared values of Δx and Δv, averaged over the ensemble and time:

$$\xi(0) \ = \ (\Delta x)^2_{\rm rms} \ , \tag{19}$$

$$\eta(0) \ = \ (\Delta v)^2_{\rm rms} \ . \tag{20}$$

We may call $\Delta x_{\rm rms}$ and $\Delta v_{\rm rms}$ the 'overall cosmic errors' in position and velocity.

The validity of Eqs. (15)-(18) can be shown rather explicitly for circular orbits. For a double star with a circular orbit (but any inclination i and any position angle of the node Ω), we have

$$\Delta x(t) = S_x \sin(\omega(t - t_0)) \ , \tag{21}$$

where S_x is the amplitude in x, $\omega = 2\pi/P$ the orbital frequency, P the period of the binary, and t_0 a time defining the phase in the orbit. We then have

$$\Delta x(t+\Delta t)\Delta x(t) = S_x^2 \Big(\sin^2\omega(t-t_0)\cos\omega\Delta t + \sin\omega(t-t_0)\cos\omega(t-t_0)\sin\omega\Delta t \Big) \ , \tag{22}$$

$$\Delta v(t+\Delta t)\Delta v(t) = \omega^2 S_x^2 \Big(\cos^2\omega(t-t_0)\cos\omega\Delta t - \sin\omega(t-t_0)\cos\omega(t-t_0)\sin\omega\Delta t \Big) \ , \tag{23}$$

$$\Delta x(t+\Delta t)\Delta v(t) = \omega S_x^2 \Big(\cos^2\omega(t-t_0)\sin\omega\Delta t + \sin\omega(t-t_0)\cos\omega(t-t_0)\cos\omega\Delta t \Big) \ . \tag{24}$$

If we average Eqs. (22)-(24) over time t, indicated by $< q >_t$, we obtain

$$< \Delta x(t+\Delta t)\Delta x(t) >_t = \frac{1}{2} S_x^2 \cos\omega\Delta t \ , \tag{25}$$

$$< \Delta v(t+\Delta t)\Delta v(t) >_t = \frac{1}{2} \omega^2 S_x^2 \cos\omega\Delta t \ , \tag{26}$$

$$< \Delta x(t+\Delta t)\Delta v(t) >_t = \frac{1}{2} \omega S_x^2 \sin\omega\Delta t \ , \tag{27}$$

since the time average of $\sin\omega(t-t_0)\cos\omega(t-t_0)$ vanishes, and the time averages of $\sin^2\omega(t-t_0)$ and of $\cos^2\omega(t-t_0)$ are both equal to 1/2. Since Eqs. (25) and (26) are symmetric with respect to Δt for all values of S_x, ω, and t_0, this property is also valid for the ensemble average, i.e. for $\xi(\Delta t)$ and $\eta(\Delta t)$. Eq. (27) is anti-symmetric in Δt, and hence $\zeta(\Delta t)$ is so by the same reasoning. A more general treatment shows also that eccentric orbits of binaries, and even the more complicated motions of the photo-centers of multiple systems, produce the same symmetry properties of ξ, η, ζ, etc. as obtained for circular orbits. The addition of single stars with $\Delta x = \Delta v = 0$ obviously does not change the symmetry properties of the correlation functions.

3.4. Some relations between correlation functions

It can be easily shown that the correlation functions for instantaneous orbital displacements satisfy certain relations:

$$\zeta(\Delta t) = -\frac{d\xi(\Delta t)}{d(\Delta t)} \ , \tag{28}$$

$$\eta(\Delta t) = +\frac{d\zeta(\Delta t)}{d(\Delta t)} = -\frac{d^2\xi(\Delta t)}{d(\Delta t)^2} \ . \tag{29}$$

The proof is similar to the corresponding proofs for correlation functions of 'stationary random functions' (see e.g. Yaglom 1987, especially Eqs. (1.50) and (1.51)). The relation

$$< \Delta x(t)\Delta v(t+\Delta t) > = - < \Delta x(t+\Delta t)\Delta v(t) > = -\zeta(\Delta t) \tag{30}$$

is a direct consequence of the anti-symmetry of $\zeta(\Delta t)$ and of the time-invariance of the correlation functions. The relation

$$\gamma_{xg}(\Delta t) = -\eta(\Delta t) \tag{31}$$

is a kind of generalized virial theorem. All these (and other) relations can usually be anticipated from results for circular orbits, which are easily obtainable (see Sect. 3.3).

It should be noted, however, that in principle the differential relations (Eqs. (28), (29), (31)) are valid for 'instantaneous' measurements only, but not for 'averaged' data, because the averaged results do not strictly fulfill the relations $v = \dot{x}$ or $g = \ddot{x}$. However, according to preliminary numerical results obtained by Biermann (1996, unpublished), the Eqs. (28) and (29) are quite accurate even for 'averaged' correlation functions.

3.5. Determination of correlation functions

The correlation functions required for statistical astrometry may be determined either 'theoretically' from binary statistics, or 'empirically' from calibrations using observed astrometric data, or by a combination of both methods.

The theoretical method uses essentially Eqs. (25)-(27), or more general expressions for non-circular orbits, and integrates these equations over 'all possible' orbits. For this, we need empirical information on the distribution functions of the important orbital elements, such as the period P, the semi-major axis a_p of the photo-center, and the eccentricity e. The distribution of the other elements, such as inclination i, position angle of the node Ω, longitude of periastron ω, and time of periastron passage T, is given by the (assumed or proven) random distributions of the orientation of orbits in space and of T in time. Hence the distribution function $f(P, a_p, e)$ and the fraction of binary stars in the ensemble of stars under consideration are actually required. While we often have good information on the distribution functions of P and e (e.g. for G dwarfs by Duquennoy & Mayor (1991)), the distribution of a_p is more difficult to obtain. a_p is given by

$$a_p[\mathrm{AU}] = (B-\beta)a[\mathrm{AU}] = (B-\beta)(P[\mathrm{years}])^{2/3}\big((\mathcal{M}_1+\mathcal{M}_2)[\mathcal{M}_\odot]\big)^{1/3} \ , \tag{32}$$

where a is the total semi-major axis, $B = \mathcal{M}_2/(\mathcal{M}_1+\mathcal{M}_2)$ the mass fraction, and $\beta = L_2/(L_1+L_2) = 1/(10^{0.4(m_2-m_1)}+1)$ the luminosity fraction of the fainter star. Hence we require, beside the distribution of P, also those of the stellar masses $\mathcal{M}_1$ and $\mathcal{M}_2$, and of the magnitude difference $\Delta m = m_2 - m_1$ in the pass-band of the astrometric observations. While in principle all this information may be available with different degrees of confidence, the overall accuracy of 'theoretically' determined correlation functions is difficult to assess.

The ‘empirical’ method of determining the correlation function is based on the comparison of the observed positions and proper motions (and perhaps accelerations) of the same stars at various epochs t of time. The required equations are derived in the following Sect. 4. However, in order to calibrate e.g. $\eta(\Delta t)$ as a function of Δt, we would need a comparison of proper motions for various different values of Δt. This is difficult in practice as long as we have only a few high-precision astrometric observations available. The situation gets even more complicated by the fact that the various astrometric catalogues ‘average’ the observations over different spans D of time (see Sect. 5.1).

Another method allows us also to determine at least $\eta(0)$ empirically: The observable dispersion among the instantaneously measured proper motions of the members of an open star cluster is often governed by the cosmic errors in these proper motions. If the dispersion (around the mean motion of the cluster) is corrected for the true velocity dispersion of the cluster (estimated by applying the virial theorem), for the measuring errors (complicated by the correlation between measurements of proper motions of stars which are located close together on the sky), and for the finite size of the cluster (especially in the radial direction), then the dispersion corrected in this way should be equal to $\Delta v_{\rm rms} = \sqrt{\eta(0)}$. We plan to apply this method to the HIPPARCOS observations of members of open star clusters.

Preliminary results for ‘theoretical’ correlation functions, based on binary statistics, have been obtained by Wielen (1995, unpublished) for circular orbits and extensively by Biermann (1996), who allowed for eccentric orbits of binaries. First indications on the amount of cosmic errors were obtained by Wielen (1994, unpublished) by using the ‘empirical’ method, when we compared preliminary HIPPARCOS data with the FK5 results in order to test the reliability of the HIPPARCOS data reduction procedures.

3.6. A simple example for correlation functions

In order to illustrate the results of methods of statistical astrometry, we shall use the following example of consistent correlation functions:

$$\xi(\Delta t) = \xi(0) e^{-\frac{(\Delta t)^2}{2S^2}} , \tag{33}$$

$$\zeta(\Delta t) = \eta(0) \Delta t e^{-\frac{(\Delta t)^2}{2S^2}} , \tag{34}$$

$$\eta(\Delta t) = \eta(0)(1 - (\frac{\Delta t}{S})^2) e^{-\frac{(\Delta t)^2}{2S^2}} , \tag{35}$$

with

$$\Delta x_{\rm rms} = \sqrt{\xi(0)} = 1\,{\rm AU} \;\widehat{=}\; 10\,{\rm mas} \quad {\rm at} \quad p = 10\,{\rm mas}, \tag{36}$$

$$\Delta v_{\rm rms} = \sqrt{\eta(0)} = 1\,{\rm km/s} \;\widehat{=}\; 2.11\,{\rm mas/yr} \quad {\rm at} \quad p = 10\,{\rm mas}, \tag{37}$$

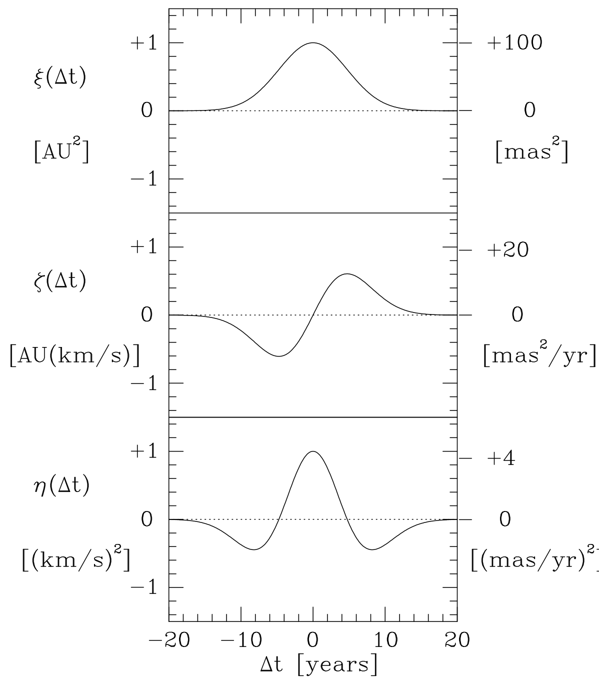

Fig. 4. Correlation functions $\xi(\Delta t)$, $\zeta(\Delta t)$, $\eta(\Delta t)$ in a simple numerical example. On the left-hand side, absolute units are used (AU^2 for ξ, AU km/s for ζ, (km/s)^2 for η). The right-hand side is valid for a star with a parallax p of 10 mas, and the astrometric units there are mas^2 for ξ, mas^2/yr for ζ, (mas/yr)^2 for η.

where p is the parallax of a star, and

$$S = \sqrt{\xi(0)/\eta(0)} = 4.74\,{\rm yr} \; . \tag{38}$$

The correlation functions are plotted in Fig. 4.

The rounded numerical values in Eqs. (36) and (37) are educated guesses, based both on statistics of binaries and on some experience with actual astrometric data. The use of these values in the graphical illustrations of the methods of statistical astrometry presented below, should produce results which have roughly the correct order of magnitude, but not more. A detailed quantitative determination of the correlation functions is beyond the scope of this paper, which tries to describe the *principles* of statistical astrometry.

The time-scale S of a few years is typical for $\eta(\Delta t)$ if we already include the ‘averaging’ discussed in Sect. 5.1 (for say $D_1 = D_2 = 3$ years, i.e. the HIPPARCOS observing period). Since $\xi(0)$ is mainly governed by long-period binaries, the actual time-scale of $\xi(\Delta t)$ should be much longer than the adopted 5 years. Since $\eta(\Delta t)$ is usually more important than $\xi(\Delta t)$ for the total result, we have preferred to use in our illustrative, simple example the time-scale of $\eta(\Delta t)$. In any case, $\xi(\Delta t)$ presents a special problem discussed in Sect. 5.2.

Since high-precision astrometry such as the HIPPARCOS mission provides, besides positions and proper motions, also accurate parallaxes for the measured stars, it is easy and appropriate to use basically correlation functions in ‘absolute units’, such as AU and km/s. With the help of the individual parallaxes

p of the stars, the correlation functions can then be transformed into 'astrometric units', such as mas or mas/yr, e.g. by

$$\eta_{\text{astrometric}} = \eta_{\text{absolute}}\, p^2. \quad (39)$$

For example, for a bright HIPPARCOS star or an FK5 star with a typical parallax of p = 10 mas, we obtain Δx_{rms} = 10 mas and Δv_{rms} = 2.11 mas/yr. In reality, even the correlation functions in 'absolute units' shall depend on the distance of the star, because of selection effects such as a catalogue limit in apparent magnitude, which produces an increase of the average stellar mass with increasing distance and hence an increase of the orbital displacements.

4. Methods of statistical astrometry

We shall now demonstrate some typical methods of statistical astrometry, in which we use the correlation functions described in Sect. 3.

For simplicity, we discuss here only the use of two limiting cases of astrometric measurements: (1) *instantaneous* results for positions and velocities, measured 'instantaneously' at a given epoch T of time, and (2) *mean* results for positions and velocities, obtained by averaging over an extremely long period of time. We denote *i*nstantaneous results by an index i, e.g. by $x_{i,2}(T_2)$ and $v_{i,2}(T_2)$, and *m*ean results by an index m, e.g. by $x_{m,1}(T_1)$ and $v_{m,1}$. To a first approximation, results of the HIPPARCOS mission (ESA 1997) may be treated as instantaneous data, while the positions and proper motions given in the FK5 catalogue (Fricke et al. 1988) may be considered as mean values. This is, of course, not strictly true: HIPPARCOS data are already averaged over about three years, while the FK5 has averaged the observations over (only) more than 200 years. Hence HIPPARCOS has already averaged-out the effect of short-period binaries, while about half of all existing binaries still have periods longer than the averaging time of the FK5.

In order to show the results of statistical astrometry most clearly, we usually give first the results without taking measuring errors into account. Later we include their effects as far as appropriate. In doing so, we always assume that the measuring errors are uncorrelated and independent of the 'cosmic errors'. This is probably a good approximation in most cases, although some part of the claimed 'measuring errors' may be in reality due to unmodelled 'cosmic errors'.

When we give numerical examples and graphical results for illustration, we use the correlation functions described in Sect. 3.6, and use for the mean measuring errors of positions and proper motions numbers which resemble the results of HIPPARCOS and of the basic FK5 for bright stars: $\epsilon_{x,i}$ = 0.7 milliarcseconds (mas), $\epsilon_{v,i}$ = 0.7 mas/yr, and $\epsilon_{x,m}$ = 20 mas, $\epsilon_{v,m}$ = 0.7 mas/yr. As a typical epoch difference $T_i - T_m$, we use 40 years, corresponding roughly to the difference between the central epochs of HIPPARCOS and of the FK5.

Furthermore we assume that the observational data have been 'prepared' in the following respects (see also Sect. 3 of Wielen (1995a)): (1) Systematic differences between catalogues must be removed before combining their data. For example, positions and proper motions given in the FK5 should be reduced to the HIPPARCOS system. Experience with actual data has already shown that the reduction of the FK5 onto the HIPPARCOS system is possible with an accuracy which is sufficient for studying cosmic errors. (2) The 'central' epoch T_c of a measurement of a position $x(T_c)$ and a proper motion $v(T_c)$ should be chosen such that $x(T_c)$ and $v(T_c)$ are not correlated. In the FK5, the central epoch T_c is given explicitly for each star and each coordinate. For HIPPARCOS, the central epoch T_c at which x and v are uncorrelated deviates for most of the stars by less than half a year from the overall catalogue epoch T_H = 1991.25 . The individual central epoch T_c can be obtained for each star and each coordinate by T_c = 1991.25 $-\big(\rho_{xv}(T_H)\epsilon_{x,i}(T_H)/\epsilon_{v,i}(T_H)\big)$, where ρ_{xv} is the correlation coefficient between x and v. If we change the epoch from T_H to T_c, then, of course, we have also to compute $x_i(T_c)$ and $\epsilon_{x,i}(T_c)$ and to use these quantities instead of $x_i(T_H)$ and $\epsilon_{x,i}(T_H)$. If we would keep the epoch T_H, we had to add terms with ρ_{xv} to many of our equations, e.g. $2\rho_{xv}(T_H)\epsilon_{x,i}(T_H)\epsilon_{v,i}(T_H)(t-T_H)$ to $\epsilon^2_{x,\text{meas}}$ in Eq. (2).

4.1. Comparison of proper motions

The comparison of proper motions given in two catalogues for a common subset of stars, is one of the simplest applications of statistical astrometry. Similar to the procedures described by Wielen (1995a), we derive for each star the difference $v_2 - v_1$ of the proper motions v_1 and v_2, given in the two catalogues, and form the root-mean-squared (rms) value D_{21} of these differences:

$$D^2_{21} = <(v_2 - v_1)^2> \quad . \quad (40)$$

The operator $<\ >$ denotes the averaging over the common set of stars.

4.1.1. Two instantaneous proper motions

Let us assume that both catalogues contain instantaneous proper motions, $v_{i,1}(T_1)$ and $v_{i,2}(T_2)$. In order to apply the correlation function $\eta(\Delta t)$ defined for Δv, we have to write v in terms of Δv and v_m, which is easily done here:

$$\begin{aligned} D^2_{21} &= <\big(v_{i,2}(T_2) - v_{i,1}(T_1)\big)^2> \\ &= <\big((v_{i,2} - v_m) - (v_{i,1} - v_m)\big)^2> \\ &= <\big(\Delta v_{i,2} - \Delta v_{i,1}\big)^2> \\ &= <\big(\Delta v_{i,1}\big)^2> + <\big(\Delta v_{i,2}\big)^2> \\ &\quad -2 <\Delta v_{i,1}\Delta v_{i,2}> \quad . \end{aligned} \quad (41)$$

We now assume in accordance with our assumptions in Sect. 3 that

$$<\big(\Delta v_{i,1}(T_1)\big)^2> = <\big(\Delta v_{i,2}(T_2)\big)^2> = \eta(0)\ , \quad (42)$$

and

$$<\Delta v_{i,1}(T_1)\Delta v_{i,2}(T_2)> = \eta(T_2 - T_1)\ . \quad (43)$$

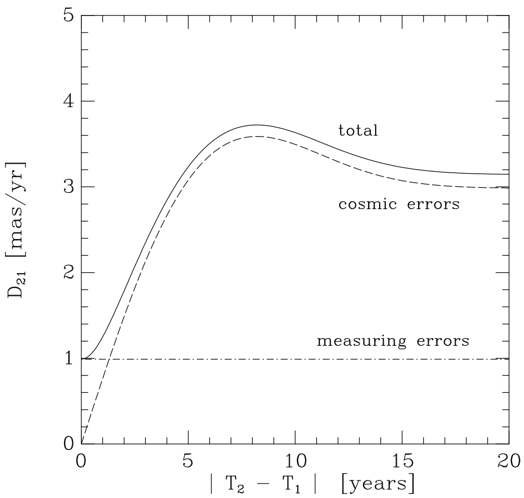

Fig. 5. The rms difference D_{21} between two instantaneous proper motions $v_{i,1}(T_1)$ and $v_{i,2}(T_2)$, measured at the epochs T_1 and T_2, as a function of the epoch difference $T_2 - T_1$. The total rms difference (solid curve) is build up by the cosmic errors (dashed curve) and the measuring errors (dash-dotted line). In our numerical example, the curves remain approximately constant for epoch differences of more than 20 years.

Hence we obtain finally:

$$D_{21}^2 = < \big(v_{i,2} - v_{i,1}\big)^2 > = 2\big(\eta(0) - \eta(T_2 - T_1)\big) \ . \tag{44}$$

This result is pleasing because it includes automatically the two limiting cases of $T_2 - T_1 = 0$ and $|T_2 - T_1| \to \infty$. For $T_2 = T_1$, the two instantaneous proper motions should be identical (except for measuring errors), and hence $D_{21} = 0$. This is also correctly obtained from the right-hand side of Eq. (44), since $\eta\,(T_2 - T_1 = 0) = \eta\,(0)$. For $|T_2 - T_1| \to \infty$, the correlation $\eta\,(T_2 - T_1)$ between $\Delta v_{i,1}$ and $\Delta v_{i,2}$ vanishes, and $D_{21}^2 = 2\,\eta\,(0) = 2(\Delta v)^2_{\mathrm{rms}}$, as expected for two uncorrelated sets of Δv_i. Between these two limiting cases, Eq. (44) gives a smooth transition, as illustrated in Fig. 5.

Taking the measuring errors $\epsilon_{v,i,1}$ and $\epsilon_{v,i,2}$ of $v_{i,1}$ and $v_{i,2}$ into account, we generalize Eq. (44) to:

$$\begin{aligned} D_{21}^2 &= < \big(v_{i,2} - v_{i,1}\big)^2 > \\ &= 2\big(\eta(0) - \eta(T_2 - T_1)\big) + \epsilon_{v,i,1}^2 + \epsilon_{v,i,2}^2 \ . \end{aligned} \tag{45}$$

In our numerical example (Fig. 5), the cosmic-error term dominates D_{21} for all epoch differences larger than a few years, thereby demonstrating the importance of the cosmic errors for high-precision astrometry.

4.1.2. Difference between an instantaneous proper motion and a mean one

We now study the case where Catalogue 1 contains mean proper motions, v_m, and Catalogue 2 provides instantaneous proper motions, $v_{i,2}(T_2)$. For the rms difference D_{21}, we obtain

$$D_{21}^2 = < \big(v_{i,2} - v_m\big)^2 > = \eta(0) + \epsilon_{v,m}^2 + \epsilon_{v,i,2}^2 \ . \tag{46}$$

The epoch T_2 does not matter, since v_m is supposed to be independent of time. Hence D_{21} does not change with time T_2. Eq. (46) indicates that the cosmic error in the HIPPARCOS proper motions, $\sqrt{\eta(0)}$, can be determined from a comparison of HIPPARCOS proper motions with FK5 proper motions, since the measuring errors of these two sets of proper motions are known.

4.2. Prediction of positions

We now study a standard problem of astrometry, namely the prediction of stellar positions at an arbitrary epoch t if we know the positions and proper motions of the stars at some other epochs. We are especially interested in the expected accuracy of these predicted positions and in the question whether or not we could improve the predictions (on average) by allowing deviations from a purely linear extrapolation of the stellar motions.

4.2.1. Prediction based on an instantaneous catalogue

This case corresponds roughly to a prediction of stellar positions from the HIPPARCOS catalogue. We assume that the catalogue gives instantaneous positions, $x_{i,T}(T)$, and instantaneous proper motions, $v_{i,T}(T)$, for an epoch T. We would like to predict the instantaneous positions $x_i(t)$ at another epoch t. Having no other information than $x_{i,T}$ and $v_{i,T}$ at hand, we can only make a linear extrapolation in order to derive the predicted value (expectation value) $x_p(t)$ for the true value $x_i(t)$:

$$x_p(t) = x_{i,T}(T) + v_{i,T}(T)(t - T) \ . \tag{47}$$

We now ask for the mean error $\epsilon_{x,i}(t)$ of the predicted position $x_p(t)$:

$$\epsilon_{x,i}^2(t) = < (x_p(t) - x_i(t))^2 > \ . \tag{48}$$

As in Sect. 4.1.1, we have to write $x(t)$ in terms of $\Delta x(t)$ and the unknown function $x_m(t)$ in order to apply the concept of the correlation functions of Sect. 3:

$$\epsilon_{x,i}^2(t) = < \Big(\big(x_p(t) - x_m(t)\big) - \big(x_i(t) - x_m(t)\big)\Big)^2 > \ . \tag{49}$$

We have

$$x_i(t) - x_m(t) = \Delta x(t) \ , \tag{50}$$

and

$$\begin{aligned} x_p(t) - x_m(t) & \\ &= x_{i,T}(T) + v_{i,T}(T)(t - T) - x_m(T) - v_m(t - T) \\ &= \big(x_{i,T}(T) - x_m(T)\big) + \big(v_{i,T}(T) - v_m\big)(t - T) \\ &= \Delta x(T) + \Delta v(T)(t - T) \ . \end{aligned} \tag{51}$$

Inserting Eqs. (50) and (51) into (49), we obtain:

$$\epsilon_{x,i}^2(t) \ = \ < \Big(\Delta x(T) + \Delta v(T)(t - T) - \Delta x(t)\Big)^2 >$$

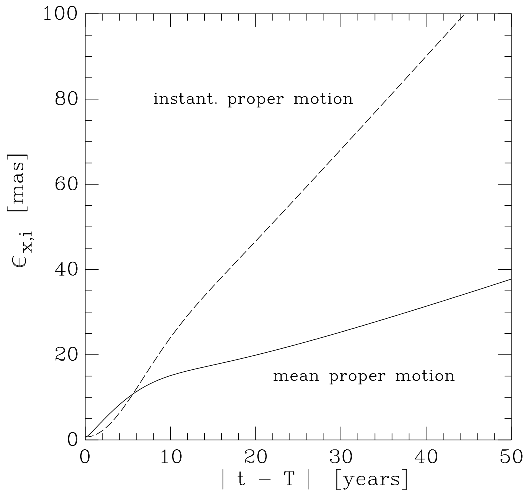

Fig. 6. Mean error $\varepsilon_{x,i}(t)$ of a predicted instantaneous position $x_p(t)$ at an epoch t, based on a linear extrapolation using an instantaneous position observed at epoch T and a mean proper motion (solid curve). The dashed curve gives the total mean error shown in Fig. 2, using an instantaneous proper motion observed at epoch T instead of a mean one.

$$\begin{aligned} &= < \big(\Delta x(T)\big)^2 + \big(\Delta x(t)\big)^2 - 2\Delta x(t)\Delta x(T) \\ &\quad +2\Delta x(T)\Delta v(T)(t-T) - 2\Delta x(t)\Delta v(T)(t-T) \\ &\quad +\big(\Delta v(T)\big)^2(t-T)^2 > \\ &= 2\big(\xi(0) - \xi(t-T)\big) \\ &\quad -2\zeta(t-T)(t-T) + \eta(0)(t-T)^2 \ , \end{aligned} \tag{52}$$

since $< (\Delta x(T))^2 > = < (\Delta x(t))^2 > = \xi(0)$ and $\zeta(0) = 0$. The result is again pleasing for both limiting cases of $\Delta t = t - T$. For $\Delta t \to \infty$, $\xi(\Delta t)$ and $\zeta(\Delta t)$ vanish, and we obtain:

$$\epsilon^2_{x,i}(t) = 2\xi(0) + \eta(0)(t-T)^2 \ , \tag{53}$$

as to be expected.

For studying the limit $\Delta t \to 0$, we have to expand the correlation functions into Taylor series in Δt, remembering Eqs. (28-29):

$$\xi(\Delta t) = \xi(0) - \frac{1}{2}\eta(0)(\Delta t)^2 + \frac{1}{24}\xi_0^{(IV)}(\Delta t)^4 \ , \tag{54}$$

$$\zeta(\Delta t) = \eta(0)\Delta t - \frac{1}{6}\xi_0^{(IV)}(\Delta t)^3 \ , \tag{55}$$

$$\eta(\Delta t) = \eta(0) - \frac{1}{2}\xi_0^{(IV)}(\Delta t)^2 \ . \tag{56}$$

Inserting Eqs. (54-56) into Eq. (52), we obtain:

$$\epsilon^2_{x,i}(t) = \frac{1}{4}\xi_0^{(IV)}(\Delta t)^4 \ . \tag{57}$$

This result shows that our prediction $x_p(t)$ deviates from the true value $x_i(t)$ for small values of Δt by a quadratic term only, because we are using the correct instantaneous values of $x_{i,T}(T)$ and $v_{i,T}(T)$ for $x_p(t)$. The transition between the quadratic increase of $\epsilon_{x,i}$ with Δt for small values of Δt and the linear increase for large values of Δt is shown in Fig. 2 for our numerical example.

Taking the measuring errors $\epsilon_{x,i,T}$ and $\epsilon_{v,i,T}$ of $x_{i,T}(T)$ and $v_{i,T}(T)$ into account, we get the complete expression for the mean error of the predicted instantaneous position at epoch t (see also Fig. 2):

$$\begin{aligned} \epsilon^2_{x,i}(t) &= 2\big(\xi(0) - \xi(t-T)\big) - 2\zeta(t-T)(t-T) + \epsilon^2_{x,i,T} \\ &\quad +\big(\eta(0) + \epsilon^2_{v,i,T}\big)(t-T)^2 \ . \end{aligned} \tag{58}$$

For an application, let us assume that a mission like GAIA is reobserving the bright HIPPARCOS stars. What is the accuracy with which GAIA can predict backwards the positions of these stars at the observing epoch of HIPPARCOS? We neglect the small measuring errors of GAIA, adopt an epoch difference between the two missions of Δt = 25 years, and use the correlation functions of Sect. 3.6. From Eq. (52) or Fig. 2, we obtain then a mean error of the GAIA prediction at the HIPPARCOS epoch of about 50 mas. This is obviously much larger than the measuring accuracy of HIPPARCOS itself. Our example indicates that most positional results of a high-precision astrometric mission are of 'unique' value and cannot be 'recovered' by later measurements.

4.2.2. Prediction based on a mean catalogue

This case reflects roughly the prediction of positions using the FK5. We assume that the catalogue gives mean positions at the epoch T, $x_{m,T}(T)$, and mean proper motions, v_m. The instantaneous position $x_i(t)$ at another epoch t is predicted by

$$x_p(t) = x_{m,T}(T) + v_m(t-T) \ . \tag{59}$$

Similarly to Sect. 4.2.1, the mean error $\epsilon_{x,i}(t)$ of $x_p(t)$ is obtained by

$$\epsilon^2_{x,i}(t) = \xi(0) + \epsilon^2_{x,m,T} + \epsilon^2_{v,m}(t-T)^2 \ . \tag{60}$$

If we neglect the measuring errors and compare Eqs. (57) and (60), we see that the prediction from a mean catalogue is more accurate than that of an instantaneous catalogue for approximately

$$|t - T_i| \ > \ \big(4\xi(0)/\xi_0^{(IV)}\big)^{1/4} \ , \tag{61}$$

i.e. a few years (about 5 years in our numerical example) away from the epoch T_i of the instantaneous catalogue. The measuring errors, however, can modify this result quite significantly. In our numerical example, the prediction of the mean catalogue is better for $t > T_i + 21$ yr and for $t < T_i - 12$ yr.

4.2.3. Prediction based on an instantaneous position and a mean proper motion

A simple, but still rather accurate prediction can be based on an instantaneous position, $x_{i,T}(T)$, and a mean proper motion, v_m:

$$x_p(t) = x_{i,T}(T) + v_m(t-T) \ . \tag{62}$$

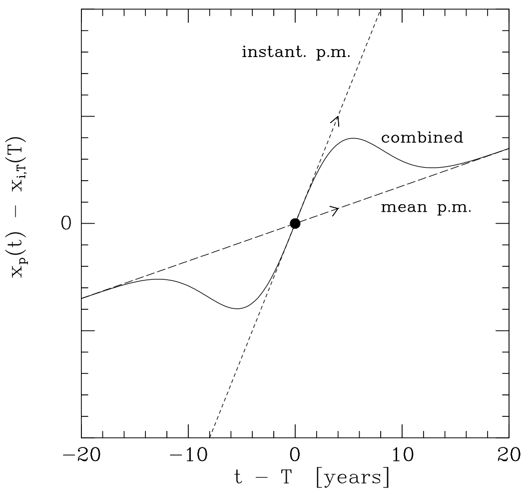

Fig. 7. Transition (solid curve) from the short-term prediction $x_{p,2}(t)$ (short-dashed line), which uses the instantaneous proper motion, to the long-term prediction $x_{p,1}(t)$ (long-dashed line), which uses the mean proper motion. The filled dot represents the instantaneous position used for both predictions. Measuring errors are neglected here to show the basic principle more clearly.

The mean error $\epsilon_{x,i}(t)$ of $x_p(t)$ follow as

$$\epsilon^2_{x,i}(t) \;=\; 2\big(\xi(0) - \xi(t-T)\big) + \epsilon^2_{x,i,T} + \epsilon^2_{v,m}(t-T)^2 \,. \quad (63)$$

For very small values of $|t-T|$, the prediction by Eq. (47), which uses instantaneous values only, is certainly the best linear extrapolation. For larger values, however, the 'mixed' prediction by Eq. (62) is better than that by Eq. (47), since the mean proper motion has no cosmic error (see Fig. 6).

4.2.4. Prediction based on an instantaneous position, an instantaneous proper motion, and a mean proper motion.

This case provides not only a very accurate prediction in practice, but is also interesting from a methodological point of view: We encounter here for the first time a non-linear prediction.

We assume that we have instantaneous data for an epoch T, i.e. $x_{i,T}(T)$ and $v_{i,T}(T)$, and in addition a mean proper motion v_m. For the prediction of the instantaneous position $x_i(t)$, we use as an 'Ansatz' a linear combination of a long-term prediction, $x_{p,1}(t)$, and a short-term prediction, $x_{p,2}(t)$:

$$x_p(t) = \big(1-\beta(t)\big)x_{p,1}(t) + \beta(t)x_{p,2}(t) \;, \quad (64)$$

with

$$x_{p,1}(t) \;=\; x_{i,T}(T) + v_m(t-T) \;, \quad (65)$$
$$x_{p,2}(t) \;=\; x_{i,T}(T) + v_{i,T}(T)(t-T) \;. \quad (66)$$

$\beta(t)$ is a function of time to be determined. Inserting Eqs. (65) and (66) into Eq. (64) leads to

$$x_p(t) = x_{i,T} + \Big((1-\beta)v_m + \beta v_{i,T}\Big)(t-T) \;. \quad (67)$$

It is intuitively clear that β should be 1 for small values of $|t-T|$, since then the instantaneous proper motion gives the best results. On the other hand, for large values of $|t-T|$, the long-term prediction using v_m is better, and hence β should approach 0 in this limit.

In order to determine $\beta(t)$, we ask for that function $\beta(t)$ which minimizes the mean error $\epsilon_{x,i}(t)$ of $x_p(t)$ for every value of t. For $\epsilon^2_{x,i}(t)$ we have

$$\begin{aligned}\epsilon^2_{x,i}(t) \;&=\; <\big(x_p(t) - x_i(t)\big)^2> \\ &=\; <\Big(\Delta x(T) + \beta\Delta v(T)(t-T) - \Delta x(t)\Big)^2> \\ &=\; 2\big(\xi(0)-\xi(t-T)\big) - 2\beta\zeta(t-T)(t-T) \\ &\quad +\beta^2\eta(0)(t-T)^2 \;. \end{aligned} \quad (68)$$

We now differentiate $\epsilon^2_{x,i}(t)$ with respect to β, and find the minimum of $\epsilon_{x,i}(t)$ with respect to β by setting the result equal to zero:

$$\frac{d\epsilon^2_{x,i}}{d\beta} = -2\zeta(t-T)(t-T) + 2\beta\eta(0)(t-T)^2 = 0 \;. \quad (69)$$

The value of β which minimizes $\epsilon_{x,i}(t)$ follows as

$$\beta(t) = \frac{\zeta(t-T)}{\eta(0)(t-T)} \;. \quad (70)$$

This function has the correct limits of 1 for $t-T \to 0$, because of Eq. (55), and of 0 for $|t-T| \to \infty$, because of $t-T$ in the denominator and since $\zeta \to 0$. For values of $\Delta t = t-T$ inbetween 0 and ∞, $\beta(t)$ varies smoothly from 1 to 0, proportional to $\zeta(\Delta t)/\Delta t$. In summary, the prediction $x_p(t)$ changes in a steady way from the short-term prediction to the long-term one as $|\Delta t|$ increases (see Figs. 7 and 8). Inserting Eq. (70) for β into Eq. (68) leads to

$$\epsilon^2_{x,i}(t) = 2\big(\xi(0)-\xi(t-T)\big) - \frac{\big(\zeta(t-T)\big)^2}{\eta(0)} \;. \quad (71)$$

In the limit of $|t-T| \to \infty$, we find

$$\epsilon^2_{x,i}(t) = 2\xi(0) \;, \quad (72)$$

as in Sect. 4.2.3. For $|t-T| \to 0$, $\epsilon_{x,i}(t)$ follows Eq. (57) of Sect. 4.2.1.

Let us now take the measuring errors into account. In this case, they do not simply produce additional terms, but change the function $\beta(t)$. We have

$$\begin{aligned}\epsilon^2_{x,i}(t) \;&=\; 2(\xi(0)-\xi(t-T)) \\ &\quad -2\beta\zeta(t-T)(t-T) \\ &\quad +\beta^2\eta(0)(t-T)^2 \\ &\quad +\epsilon^2_{x,i,T} + \Big(\beta^2\epsilon^2_{v,i,T} + (1-\beta)^2\epsilon^2_{v,m}\Big)(t-T)^2, \end{aligned} \quad (73)$$

and therefore

$$\begin{aligned}\frac{d\epsilon^2_{x,i}}{d\beta} \;&=\; -2\zeta(t-T)(t-T) + 2\beta\eta(0)(t-T)^2 \\ &\quad +\Big(2\beta\epsilon^2_{v,i,T} - 2(1-\beta)\epsilon^2_{v,m}\Big)(t-T)^2 = 0 \;, \end{aligned} \quad (74)$$

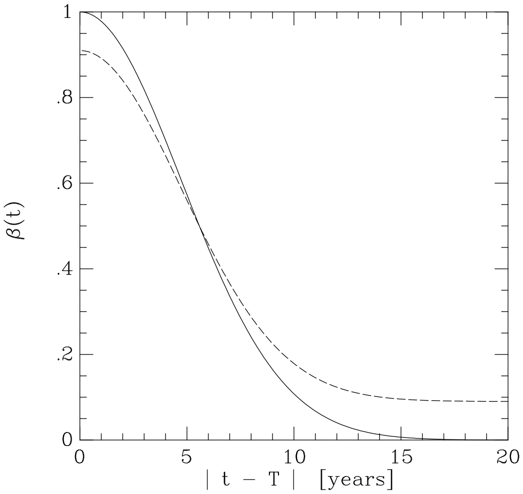

Fig. 8. The transition function $\beta(t)$ between the short-term prediction and the long-term prediction. Solid curve: using cosmic errors only (Eq. (70)). Dashed curve: measuring errors included (Eq. (75)).

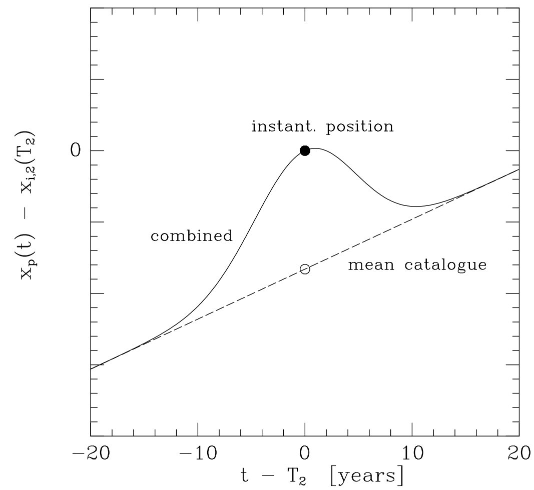

Fig. 9. Prediction $x_p(t)$ of an instantaneous position at an epoch t (solid curve), based on a mean catalogue (dashed line) and an instantaneous position observed at epoch T_2 (filled dot). Measuring errors are neglected here.

which leads to

$$\beta(t) = \frac{\zeta(t-T) + \epsilon^2_{v,m}(t-T)}{\left(\eta(0) + \epsilon^2_{v,i,T} + \epsilon^2_{v,m}\right)(t-T)} \ . \tag{75}$$

The function $\beta(t)$ varies now between $(\eta(0) + \epsilon^2_{v,m})/(\eta(0) + \epsilon^2_{v,i,T} + \epsilon^2_{v,m})$ for $\Delta t \to 0$ and $\epsilon^2_{v,m}/(\eta(0) + \epsilon^2_{v,i,T} + \epsilon^2_{v,m})$ for $\Delta t \to \infty$, instead of 1 and 0 without measuring errors (Fig. 8). This is in accordance with the expectation from a usual weighting of the proper motions $v_{i,T}(T)$ and v_m, including the cosmic error $\sqrt{\eta(0)}$ properly in both limits. Inserting Eq. (75) into Eq. (73) gives:

$$\begin{aligned} \epsilon^2_{x,i}(t) \ = \ & 2\left(\xi(0) - \xi(t-T)\right) \\ & + \epsilon^2_{x,i,T} + \epsilon^2_{v,m}(t-T)^2 \\ & - \frac{\left(\zeta(t-T) + \epsilon^2_{v,m}(t-T)\right)^2}{\eta(0) + \epsilon^2_{v,i,T} + \epsilon^2_{v,m}} \ . \end{aligned} \tag{76}$$

In this subsection, we encounter the problem of 'conditioned' correlation functions. In deriving $\beta(t)$, we have implicitely assumed that the orbital displacements $\Delta x(t)$ and $\Delta v(t)$ behave completely randomly as in a stochastic process. This is not true in reality, of course. For example, a large difference $v_{i,T} - v_m$ (in km/s) is an indication of a short-period binary, since a long-period binary produces typically only a slow orbital motion. Hence we should use here 'conditioned' correlation functions which are based only on such orbital configurations which produce (within the measurement errors) the observed difference $|v_{i,T} - v_m|$. The use of such conditioned correlation functions would not only change the transition function $\beta(t)$, but would also affect our estimates of the cosmic errors and their effects for individual stars directly. We shall neglect this complication here and in the following sections of the present paper. It can only be solved properly if the correlation functions and their conditioned counterparts are derived from binary statistics, as discussed in Sect. 3.5.

4.2.5. Predictions based on other combinations of observed data.

If we have available a mean catalogue, i.e. $x_{m,1}(T_1)$ and v_m, and an instantaneous position, $x_{i,2}(T_2)$, then we may use as a prediction for $x_i(t)$:

$$\begin{aligned} x_p(t) \ = \ & x_{m,1}(T_1) + v_m(t - T_1) \\ & + \gamma(t)\left(x_{i,2}(T_2) - x_{m,1}(T_1) - v_m(T_2 - T_1)\right) \ , \end{aligned} \tag{77}$$

and find that the minimum of $\epsilon_{x,i}(t)$ is obtained for

$$\gamma(t) = \frac{\xi(t - T_2)}{\xi(0)} \ . \tag{78}$$

The prediction $x_p(t)$ makes an 'excursion' from $x_m(t)$ to go through $x_{i,2}(T_2)$ (see Fig. 9). The form of this excursion as a function of t is determined by the 'scaled' correlation function $\xi(t - T_2)$. The excursion is symmetric around T_2 and reaches its maximum at $t = T_2$. If we take the measuring errors into account, we get

$$\gamma(t) = \frac{\xi(t - T_2) + \epsilon^2_{x,m} + \epsilon^2_{v,m}(t - T_1)(T_2 - T_1)}{\xi(0) + \epsilon^2_{x,i,2} + \epsilon^2_{x,m,1} + \epsilon^2_{v,m}(T_2 - T_1)^2} \ . \tag{79}$$

It is easy to show that we obtain in the important limiting cases, $\epsilon_{x,i,2} \to \infty$, or $\epsilon_{x,m,1} \to \infty$, or $\epsilon_{v,m} \to \infty$, 'automatically' the correct predictions $x_p(t)$. This is especially interesting for the case $\epsilon_{v,m} \to \infty$, which gives correctly

$$\gamma(t) = \frac{t - T_1}{T_2 - T_1} \ , \tag{80}$$

and hence

$$x_p(t) = x_{m,1}(T_1) + \frac{x_{i,2}(T_2) - x_{m,1}(T_1)}{T_2 - T_1}(t - T_1) \ , \tag{81}$$

although this is not obviously built-in into Eq. (77).

We assume now that we have two catalogues: a mean catalogue at epoch T_1, giving $x_{m,1}(T_1)$ and v_m, and an instantaneous catalogue at epoch T_2, giving $x_{i,2}(T_2)$ and $v_{i,2}(T_2)$. We adopt here the following prediction for $x_i(t)$:

$$\begin{aligned} x_p(t) \; = \; & x_{m,1}(T_1) + v_m(t - T_1) \\ & + \tilde{\gamma}(t)\big(x_{i,2}(T_2) - x_{m,1}(T_1) - v_m(T_2 - T_1)\big) \\ & + \tilde{\beta}(t)(v_{i,2}(T_2) - v_m)(t - T_2) \ . \end{aligned} \tag{82}$$

The minimum of $\epsilon_{x,i}(t)$ is obtained for

$$\tilde{\gamma}(t) \; = \; \frac{\xi(t - T_2)}{\xi(0)} \ , \tag{83}$$

$$\tilde{\beta}(t) \; = \; \frac{\zeta(t - T_2)}{\eta(0)(t - T_2)} \ , \tag{84}$$

as to be expected from Eqs. (78) and (70). A possible prediction in which we force $\tilde{\beta}(t) = \tilde{\gamma}(t)$ in Eq. (82), gives results with larger mean errors $\epsilon_{x,i}(t)$. The prediction $x_p(t)$ now makes an 'excursion' from $x_m(t)$ which is no longer symmetric around T_2 (see Fig. 10). The prediction satisfies, of course, the conditions $x_p(T_2) = x_{i,2}(T_2)$ and $\dot{x}_p(T_2) = v_{i,2}(T_2)$. For the apparent 'acceleration' at $t = T_2$, we find

$$\ddot{x}_p(T_2) = -\frac{\eta(0)}{\xi(0)}\big(x_{i,2}(T_2) - x_{m,1}(T_1) - v_m(T_2 - T_1)\big) \ , \tag{85}$$

independent of $v_{i,2} - v_m$. This is, by the way, the same 'acceleration' $\ddot{x}_p$ as implied by Eqs. (77) and (78) for $t = T_2$, and by Eqs. (67) and (70) for $t = T$.

If we take the measuring errors of the two catalogues into account, we find two linear equations of condition for $\tilde{\gamma}(t)$ and $\tilde{\beta}(t)$:

$$\begin{aligned} & \big(\xi(0) + \epsilon^2_{x,m,1} + \epsilon^2_{x,i,2} + \epsilon^2_{v,m}(T_2 - T_1)^2\big)\tilde{\gamma}(t) \\ & \quad + \epsilon^2_{v,m}(t - T_2)(T_2 - T_1)\tilde{\beta}(t) \\ & \quad = \xi(t - T_2) + \epsilon^2_{x,m,1} + \epsilon^2_{v,m}(t - T_1)(T_2 - T_1) \ , \end{aligned} \tag{86}$$

$$\begin{aligned} & \epsilon^2_{v,m}(T_2 - T_1)\tilde{\gamma}(t) + \big(\eta(0) + \epsilon^2_{v,m} + \epsilon^2_{v,i,2}\big)(t - T_2)\tilde{\beta}(t) \\ & \quad = \zeta(t - T_2) + \epsilon^2_{v,m}(t - T_1) \ . \end{aligned} \tag{87}$$

The explicit solutions of Eqs. (86) and (87) for $\tilde{\gamma}$ and $\tilde{\beta}$ are somewhat lengthy and therefore not given here.

We could continue to present results for other combinations of observed data, e.g. for two instantaneous catalogues. The results, however, are rather bulky, and do not add much insight into the principles of statistical astrometry in general. They are, nevertheless, important for actual applications, and should therefore be discussed in such a context.

4.2.6. Should acceleration terms be used?

The HIPPARCOS catalogue gives for many stars 'acceleration terms' $g_{i,T}(T) = \ddot{x}_i(T)$, or even $\dot{g}_{i,T}(T)$. Should we use such terms for the prediction of the instantaneous positions at other epochs t? It is intuitively clear that the use of $g_{i,T}$ or $\dot{g}_{i,T}$ will improve, at least in principle, the accuracy of the prediction for epochs t near to the HIPPARCOS central epoch T. On the other hand, we will show that a long-term prediction should not make use of these acceleration terms, as already recommended in the HIPPARCOS Catalogue (ESA 1997). We plan to discuss in a subsequent paper the implications of the statistical significance of the terms $g_{i,T}$ or $\dot{g}_{i,T}$ given in the HIPPARCOS Catalogue.

We investigate the following prediction for the instantaneous position $x_i(t)$:

$$x_p(t) = x_{i,T}(T) + v_{i,T}(T)(t - T) + \frac{1}{2} g_{i,T}(T)(t - T)^2 \ . \tag{88}$$

The square of the expected mean error $\epsilon_{x,i}(t)$ of $x_p(t)$, averaged over the whole ensemble of stars, is given by

$$\begin{aligned} \epsilon^2_{x,i} \; = \; & 2(\xi(0) - \xi(t - T)) - 2\zeta(t - T)(t - T) \\ & + \eta(t - T)(t - T)^2 + \frac{1}{4}\gamma_{gg}(0)(t - T)^4 \ , \end{aligned} \tag{89}$$

where $\gamma_{gg}(\Delta t)$ is the correlation function between $g_i(t)$ and $g_i(t + \Delta t)$, according to Eq. (13).

For studying the short-term behaviour of $\epsilon_{x,i}(t)$, we have to add to Eqs. (54-56) the terms with $\xi_0^{(VI)} = (d^6\xi(\Delta t)/d(\Delta t)^6)(\Delta t = 0)$. We obtain for $\Delta t = t - T \rightarrow 0$:

$$\epsilon^2_{x,i}(t) = \frac{1}{36}\left|\xi_0^{(VI)}\right|(\Delta t)^6 \ . \tag{90}$$

Hence for small values of Δt, the prediction by Eq. (88), which uses $g_{i,T}$, is better than the linear prediction by Eq. (47), because of $(\Delta t)^6$ in Eq. (90) instead of $(\Delta t)^4$ in Eq. (57).

For $\Delta t \rightarrow \infty$, we find

$$\epsilon^2_{x,i}(t) = 2\xi(0) + \frac{1}{4}\gamma_{gg}(0)(\Delta t)^4 \ . \tag{91}$$

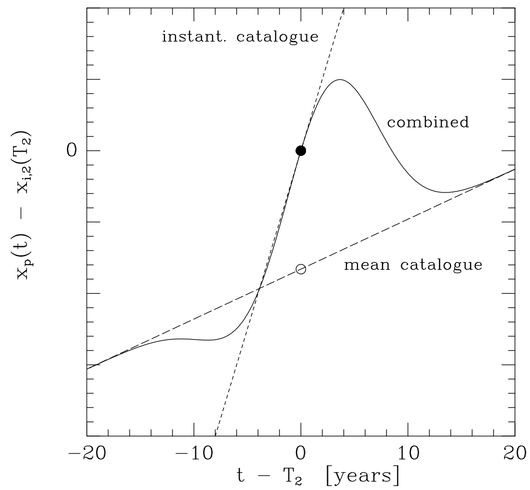

Fig. 10. Prediction $x_p(t)$ of an instantaneous position at an epoch t (solid curve), based on a mean catalogue (long-dashed line) and on an instantaneous catalogue observed at epoch T_2 (filled dot, short-dashed line). Measuring errors are neglected here.

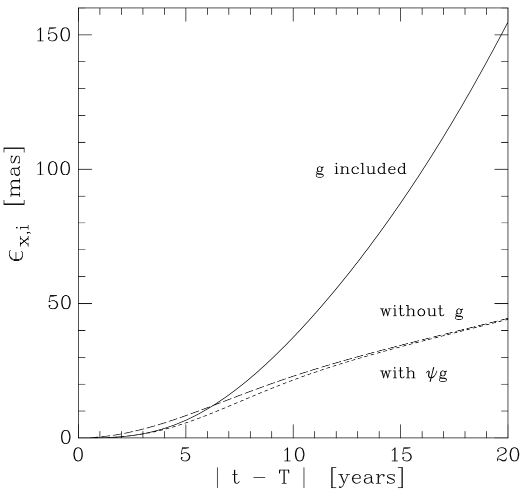

Fig. 11. Comparison of the mean errors $\epsilon_{x,i}(t)$ of a predicted instantaneous position, using either a quadratic prediction (Eq. (88)) with an acceleration term $g_{i,T}$ (solid curve), or a linear prediction (Eq. (47)) without such a term (long-dashed curve). The short-dashed curve is valid for a modified quadratic prediction in which $g_{i,T}$ is multiplied by $\psi(t-T)$ (Eq. (93)). The values of $\epsilon_{x,i}$ given are rms averages over the whole ensemble of stars. Measuring errors are neglected here.

Hence the long-term prediction using $g_{i,T}$ produces always larger errors than the linear prediction, because of $(\Delta t)^4$ in Eq. (91) instead of $(\Delta t)^2$ in Eq. (53).

Comparing Eqs. (89) and (52), we see that the quadratic prediction is less accurate than the linear prediction if $|\Delta t|$ is larger than

$$\Delta t_{\rm lim} = 2\sqrt{\big(\eta(0) - \eta(\Delta t_{\rm lim})\big)/\gamma_{gg}(0)}\ . \tag{92}$$

which is an implicit equation for $\Delta t_{\rm lim}$. In our numerical example, we find $\Delta t_{\rm lim}$= 6.3 years. Hence in most cases, the linear prediction, neglecting $g_{i,T}$, is better than the quadratic prediction, using $g_{i,T}$. This is illustrated in Fig. 11. The use of $\dot{g}_{i,T}$ in a cubic prediction would produce even less accurate results for large values of Δt.

For HIPPARCOS, the measuring error of $g_{i,T}$ is typically between 1 and 2 mas/yr^2 for bright stars. This is larger than the overall cosmic error of $g_{i,T}$ in our numerical example, namely $\sqrt{\gamma_{gg}(0)} = \sqrt{3\eta(0)}/S = 0.77$ mas/yr^2. Hence for large values of $|\Delta t|$, the total mean error of a quadratic prediction would be governed by the measuring error in $g_{i,T}$, not by the cosmic errors.

A smooth transition between the two limiting cases (with or without g) can be obtained by multiplying $g_{i,T}(T)$ in Eq. (88) with a function $\psi(t-T)$. The function ψ which minimizes $\epsilon_{x,i}(t)$ (rms averaged over the ensemble) is given by

$$\psi(t-T) = \frac{2\big(\eta(0) - \eta(t-T)\big)}{\gamma_{gg}(0)(t-T)^2}\ , \tag{93}$$

and varies between 1 for $\Delta t = 0$ and 0 for $|\Delta t| \rightarrow \infty$. For large values of $|\Delta t|$, the modified last term in Eq. (88), $\frac{1}{2}\psi(\Delta t)g_{i,T}(T)(\Delta t)^2$, tends towards a constant value Δx_g, which is the expectation value of the difference between $x_{m,T}$ and $x_{i,T}$. The result that Δx_g has the same sign as $g_{i,T}$, is to be expected from orbital mechanics: For circular orbits, the vector $\mathbf{g}_{i,T}$ points always from the instantaneous position $\mathbf{x}_{i,T}$ towards the mean position $\mathbf{x}_{m,T}$, and it is better for a long-term prediction to use $x_{m,T}$ instead of $x_{i,T}$. The rms average of Δx_g over the whole ensemble is given by

$$< (\Delta x_g)^2 >^{1/2} = \eta(0)/\sqrt{\gamma_{gg}(0)}\ , \tag{94}$$

or 5.77 mas in our numerical example. The individual values of Δx_g can be accurately determined from conditioned correlation functions only. The use of the overall correlation functions, valid for the whole ensemble, would be rather misleading in deriving individual values of Δx_g for stars with significant acceleration terms $g_{i,T}$.

4.3. Comparison of catalogues

In an earlier paper (Wielen 1995a), we have developed a method for determining the individual accuracy of astrometric catalogues. A variation of this method allows us to obtain empirical information about the cosmic errors.

We assume that we have available a mean catalogue, such as the FK5, at epoch T_1, i.e. $x_{m,1}(T_1)$ and v_m, and an instantaneous catalogue, such as the HIPPARCOS catalogue, at epoch T_2, i.e. $x_{i,2}(T_2)$ and $v_{i,2}(T_2)$. From the two positions, we derive (Wielen 1988) a third proper motion v_0:

$$v_0 = \frac{x_{i,2}(T_2) - x_{m,1}(T_1)}{T_2 - T_1}\ , \tag{95}$$

and compare v_0 with both v_m and $v_{i,2}(T_2)$:

$$\begin{aligned} D_{10}^2 &= < (v_m - v_0)^2 > \\ &= \frac{\xi(0)}{(T_2-T_1)^2} + \frac{\epsilon_{x,m,1}^2 + \epsilon_{x,i,2}^2}{(T_2-T_1)^2} + \epsilon_{v,m}^2\ , \end{aligned} \tag{96}$$

$$\begin{aligned} D_{20}^2 &= < (v_{i,2}(T_2) - v_0)^2 > \\ &= \eta(0) + \frac{\xi(0)}{(T_2-T_1)^2} + \frac{\epsilon_{x,m,1}^2 + \epsilon_{x,i,2}^2}{(T_2-T_1)^2} + \epsilon_{v,i,2}^2\ . \end{aligned} \tag{97}$$

From Sect. 4.1.2, we know:

$$D_{21}^2 =< (v_{i,2}(T_2) - v_m)^2 >= \eta(0) + \epsilon_{v,m}^2 + \epsilon_{v,i,2}^2\ . \tag{98}$$

From the three empirically available quantities D_{10}, D_{20}, and D_{21}, we can obviously determine $\xi(0)$ and $\eta(0)$ if all the measuring errors are known. This provides the most direct and powerful method to determine empirically the overall cosmic errors.

If the measuring errors are unknown, we can only determine $\epsilon_{v,m}$ and the two combinations $\xi(0) + \epsilon_{x,m,1}^2 + \epsilon_{x,i,2}^2$ and $\eta(0) + \epsilon_{v,i,2}^2$:

$$\begin{aligned} \xi(0) + \epsilon_{x,m,1}^2 &+ \epsilon_{x,i,2}^2 \\ &= \frac{1}{2}(T_2-T_1)^2(D_{20}^2 + D_{10}^2 - D_{21}^2)\ , \end{aligned} \tag{99}$$

$$\eta(0) + \epsilon_{v,i,2}^2 = \frac{1}{2}(D_{21}^2 + D_{20}^2 - D_{10}^2) \ , \tag{100}$$

$$\epsilon_{v,m}^2 = \frac{1}{2}(D_{21}^2 + D_{10}^2 - D_{20}^2) \ . \tag{101}$$

Most catalogues give the results for two coordinates, say right ascension α and declination δ. On the other hand, $\xi(0)$ and $\eta(0)$ should not depend on these coordinates. This, however, does not help to determine $\xi(0)$ or $\eta(0)$ if the measuring errors are unknown. We conclude that it is very important to produce reliable estimates of the measuring errors already in the course of the catalogue construction.

We now compare two instantaneous catalogues which provide data for two epochs T_1 and T_2. Forming a third proper motion $\tilde{v}_0$ in analogy to Eq. (95),

$$\tilde{v}_0 = \frac{x_{i,2}(T_2) - x_{i,1}(T_1)}{T_2 - T_1} \ , \tag{102}$$

we obtain

$$\begin{aligned} \tilde{D}_{10}^2 &= < (v_{i,1}(T_1) - \tilde{v}_0)^2 > \\ &= \frac{2\big(\xi(0) - \xi(T_2 - T_1)\big)}{(T_2 - T_1)^2} - \frac{2\zeta(T_2 - T_1)}{T_2 - T_1} + \eta(0) \\ &\quad + \frac{\epsilon_{x,i,1}^2 + \epsilon_{x,i,2}^2}{(T_2 - T_1)^2} + \epsilon_{v,i,1}^2 \ , \end{aligned} \tag{103}$$

$$\tilde{D}_{20}^2 = < (v_{i,2}(T_2) - \tilde{v}_0)^2 > = \tilde{D}_{10}^2 - \epsilon_{v,i,1}^2 + \epsilon_{v,i,2}^2 \ , \tag{104}$$

$$\begin{aligned} \tilde{D}_{21}^2 &= < (v_{i,2}(T_2) - v_{i,1}(T_1))^2 > \\ &= 2\big(\eta(0) - \eta(T_2 - T_1)\big) + \epsilon_{v,i,1}^2 + \epsilon_{v,i,2}^2 \ . \end{aligned} \tag{105}$$

If all the measuring errors are known, we can determine the combinations $\eta(0) - \eta(T_2 - T_1)$ and $\eta(0)(T_2 - T_1)^2 - 2\zeta(T_2 - T_1)(T_2 - T_1) + 2(\xi(0) - \xi(T_2 - T_1))$. The most promising possibility of using Eqs. (103-105) is probably to assume an appropriate analytic approximation for $\xi(\Delta t)$, and hence also for $\zeta(\Delta t)$ and $\eta(\Delta t)$ by using Eqs. (28) and (29), and to determine then up to two free parameters from the empirical quantities $\tilde{D}_{10}$, $\tilde{D}_{20}$, and $\tilde{D}_{21}$ in the case of two catalogues. More (say n) instantaneous catalogues would allow to determine more (up to $n(n-1)$) free parameters. The comparison of instantaneous catalogues observed at different epochs thus provides a rather direct method to determine empirically the dependence of the correlation functions on the epoch difference Δt.

Obviously the number of available high-precision catalogues and their epoch differences affect strongly the quality of the empirically determined correlation functions. At present, only the HIPPARCOS Catalogue and the basic FK5 are clearly qualified for being used. In the future, a great improvement in the empirical determination of the correlation functions should occur if the planned high-precision astrometric missions like GAIA and DIVA would be realized.

5. Complications

5.1. 'Averaged' observations

In Sects. 3 and 4, we have assumed for simplicity that 'instantaneous' observations are obtained during an infinitesimally short time interval. This is, of course, not true in reality. For example, the data given in the HIPPARCOS catalogue are already averaged over about three years of time, essentially by fitting a linear least-squares solution to a few dozens of individual positions of each star.

Let us assume that we observe continuously in time the actual positions $x_t(t)$ over the interval of time $T - (D/2) < t < T + (D/2)$, i.e. over a time span of D, centered at T. We neglect the parallax and fit a straight line to these observations by the least-squares method. If the star is an astrometric binary with a circular orbit and an orbital frequency ω, then the resulting 'averaged' orbital displacements in position, $\Delta x_D(T)$, and in proper motion, $\Delta v_D(T)$, are given by

$$\Delta x_D(T) = \Delta x_t(T)\varphi_x(\omega D/2), \tag{106}$$

$$\Delta v_D(T) = \Delta v_t(T)\varphi_v(\omega D/2), \tag{107}$$

where $\Delta x_t(T)$ and $\Delta v_t(T)$ are the actual orbital displacements in position and proper motion at $t = T$, and φ_x and φ_v are 'damping functions', due to the finite (non-zero) length D of observation:

$$\varphi_x(\tau) = \frac{1}{\tau}\sin\tau, \tag{108}$$

$$\varphi_v(\tau) = \frac{3}{\tau^3}(\sin\tau - \tau\cos\tau). \tag{109}$$

If the number of actual observations is finite and/or the observations are not spread homogeneously over time in the interval D, then the situation is more complex, but Eqs. (106-109) should still give a first impression of the effect of the averaging.

The correlation functions between averaged observations, such as

$$\bar{\xi}(\Delta t; D_1, D_2) = < \Delta x_{D2}(t + \Delta t)\Delta x_{D1}(t) >, \tag{110}$$

will differ in general from the corresponding instantaneous functions, such as $\xi(\Delta t) = \bar{\xi}(\Delta t; 0, 0)$. If we compare observations, obtained at the epochs $t = T_1$ and $t + \Delta t = T_2$, and averaged over intervals of time D_1 and D_2, then we have to replace e.g. Eq. (26) by

$$\begin{aligned} &< \Delta v_{D2}(t + \Delta t)\Delta v_{D1}(t) >_t \\ &\quad = \frac{1}{2}\omega^2 S_x^2 \cos\omega\Delta t\, \varphi_v(\omega D_1/2)\, \varphi_v(\omega D_2/2). \end{aligned} \tag{111}$$

Since we have $|\varphi_x(\tau)| < \varphi_x(0) = 1$ and $|\varphi_v(\tau)| < \varphi_v(0) = 1$ for all values of $\tau = \omega D/2$, the averaged term in Eq. (111) is always smaller in absolute value than the instantaneous term in Eq. (26). Hence for $\Delta t = 0$, all the averaged correlation functions are smaller than the corresponding instantaneous functions. For most other (but not all) values of Δt, these inequalities (e.g. $|\bar{\eta}(\Delta t; D_1, D_2)| \leq |\eta(\Delta t)|$) are also satisfied.

The damping functions φ_x and φ_v can easily be used in the 'theoretical' derivation of correlation functions, based on binary statistics. Preliminary results have been obtained by Wielen (1995, unpublished) and especially by Biermann (1996). The effect of the damping is dramatic for $\eta(\Delta t)$ if $|\Delta t|$ is small, but is minor for ξ and ζ.

5.2. *The problem of* $\eta(0)$ *and* $\xi(0)$

Many analytic functions which are used to fit the observed statistics of binaries (e.g. a gaussian distribution of $\log P$, as proposed by Duquennoy & Mayor (1991)), lead to diverging values of $\eta(0)$ and $\xi(0)$. While this is partially an artifact of the fitting functions, the problem is real in the sense that $\eta(0)$ and $\xi(0)$ tend to high values: $\eta(0)$ is governed by a few, very short-period binaries with high orbital velocities; $\xi(0)$ is mainly caused by some very long-period binaries with large values of their semi-major axis.

The problem of $\eta(0)$ is solved for typical astrometric measurements by the averaging process discussed in the former section, since $\overline{\eta}(0; D_1, D_2)$ behaves well if D_1 or D_2 are larger than about a year. This condition is satisfied in most astrometric applications. It is, however, violated for spectroscopic measurements of the orbital velocities of close binaries, where v is directly observed and where D is typically of the order of a few minutes.

The large, or even diverging, values of $\xi(\Delta t)$ are not changed by the averaging process, since here long-period binaries are responsible for the problem. In many applications, the problem of the large values of ξ or $\bar{\xi}$ disappears 'automatically': For example, Eq. (63) for the expected mean error $\epsilon_{x,i}(t)$ of $x_p(t)$ contains the difference $\xi(0) - \xi(\Delta t)$ only. This difference is 'well-behaved' and finite. The situation is somewhat similar to the 'renormalization' in quantum electrodynamics, where also differences between formally infinite quantities are used. Nevertheless, the problem remains disturbing: it indicates that even catalogues like the FK5, which averages over about two centuries, cannot be treated fully as a 'mean' catalogue as far as positions, and hence ξ and ζ, are concerned. In practice, it may still be a fair approximation to classify such catalogues like the FK5 as a mean one: The photo-centers of very long-period binaries move for most epoch differences of interest nearly on straight lines, and behave therefore like single stars, except for an off-set in position.

5.3. *Distribution functions of* Δx *and* Δv

The correlation functions describe the second-order moments of the distribution functions of the orbital displacements Δx and Δv. We may also study the distribution functions of Δx and Δv directly. This would be especially important if a few, very large values of Δx or Δv would govern the correlation functions. The tools for handling distribution functions, namely conditioned transition probabilities, are more complicated than the correlation functions, and are not discussed further in this paper.

6. Conclusions

In high-precision astrometry, such as provided by HIPPARCOS, the non-linear orbital motions of many individually undetected or unmodelled astrometric binaries act on average as 'cosmic errors' in the assumed linear motions. These cosmic errors and their effects on e.g. predicted positions at other epochs are often larger than those of the measuring errors. It is therefore important to take the cosmic errors and their consequences properly into account if one deals with high-precision astrometric data. The general methods and tools of statistical astrometry presented in this paper allow us to do so consistently and quantitatively. A first practical application of statistical astrometry will be made in the construction of the FK6 (Wielen et al. 1997, in preparation) in which we shall combine the FK5 data of the basic fundamental stars with those of HIPPARCOS, taking cosmic errors and concepts of statistical astrometry into account.

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