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ASTRONOMY AND ASTROPHYSICS

Nonlinear self-consistent three-dimensional arcade-like solutions of the magnetohydrostatic equations

T. Neukirch

School of Mathematical and Computational Sciences, University of St. Andrews, St. Andrews, KY16 9SS, UK (thomas@dcs.st-and.ac.uk)

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Abstract. Exact self-consistent solutions of the magnetohydrostatic equations in three dimensions with a nonlinear relationship between the magnetic field and the current density are presented. The derivation is carried out in Cartesian coordinates including a constant gravitational force pointing into the negative z-direction. Out of several solution classes only one has an arcade-like magnetic field topology. This solution class is investigated further and two examples are shown for specific choices of the free functions and parameters which the solution class has.

Key words: magnetohydrodynamics – magnetic fields – plasmas – Sun: magnetic fields – Sun: corona – stars: magnetic fields

1. Introduction

Exact solutions of the magnetohydrostatic equations can only be found in special cases. The particularly well studied cases are those in which the solutions are invariant under either translations, rotations or combinations of the two, i.e. helically invariant (e.g. Lüst & Schlüter 1957; Grad & Rubin 1958; Shafranov 1958; Johnson et al. 1958), because in these and only in these cases the equations of magnetohydrostatics can be reduced to a single nonlinear elliptic equation (Solov'ev 1963; Edenstrasser 1980a, 1980b). Since in these three cases, the physical quantities depend only on two spatial coordinates whereas the third one is ignorable due to the symmetry, we refer to these cases as two-dimensional cases.[1] This equation still contains free functions resulting from the partial integration of the complete set of magnetohydrodynamic equations and this freedom may be exploited so that even in nonlinear cases many exact solutions are known. Equilibria of this type have been used to model space plasma systems like the magnetotail of the Earth (e.g. Schindler & Birn 1986), plasma structures in the solar atmosphere (e.g. Priest 1982) or astrophysical jets (e.g. Appl & Camenzind 1993). Furthermore, it is in principle straightforward to solve a quasi-linear elliptic equation with standard methods (e.g. Zwingmann 1983; Neukirch 1993a, 1993b). Though the study and application of symmetric solutions has been very helpful for a qualitative modelling of many plasma systems, for the majority of plasma systems three-dimensional solutions are crucial to obtain quantitative models. However, the step from the symmetric cases to cases with variations in all three spatial dimensions has proven to be very difficult. The reasons for this are the nonlinearity of the equations and that in 3D the problem can in principle only be reduced to a system of two coupled nonlinear partial differential equations of mixed type (if e.g. the magnetic field is represented by Euler potentials). Furthermore the very existence of three-dimensional equilibria has been questioned for toroidal geometries (Grad 1967) and recent work seems to corroborate this suggestion under certain conditions (Salat 1995). For astrophysical systems the non-existence of three-dimensional perturbative solutions near two-dimensional equilibria has been proved under certain assumptions (Parker 1979). If these assumptions are relaxed, it can be shown that three-dimensional equilibria may exist near two-dimensional equilibria and may be calculated by perturbation methods (Hu et al. 1983; Arendt & Schindler 1988).

Apart from the cases of potential and linear force-free fields only very few fully three-dimensional solutions of the MHS equations are known. The known examples comprise nonlinear force-free equilibria (Low 1988a, 1988b), equilibria with force balance between the pressure gradient and the $\boldsymbol{j} \times \boldsymbol{B}$-force (Woolley 1975, 1977; Shivamoggi 1986; Salat & Kaiser 1995; Kaiser & Salat 1996a, 1996b) and equilibria in which an additional external potential is present (Low 1982, 1984, 1985, 1991, 1992, 1993a, 1993b; Bogdan & Low 1986; Osherovich 1985a, 1985b; Neukirch 1995). Whereas in the first two classes of known solutions the magnetic field lines are restricted to lie in one coordinate plane (with exceptions in Kaiser and Salat 1996a), in the third case a larger variety of equilibria is allowed due to the inclusion of an additional degree of freedom by introducing an external force.

[1] If a non-vanishing magnetic field component in the invariant direction is allowed, some authors speak of the two-and-a-half dimensional cases. We will, however, not use this expression and will always speak of two-dimensional equilibria.

In the present contribution we follow a method developed by Low (1985, 1991, 1992). The idea behind Low's method is to introduce Euler potentials for the current density with one of the Euler potentials being identical with the potential of the external force (Low 1991). Exact solutions of the problem have so far been found if the other Euler potential was assumed to be a linear function of the magnetic field component parallel to the gradient of the external potential (Low 1985, 1991, 1992; Bogdan & Low 1986; Neukirch 1995). An additional linear force-free component of the current density may be included in the treatment (Low 1991, 1992; Neukirch 1995).

The objective of the present paper is to present for the first time self-consistent solutions of the MHS equations for which the relationship between the current density and the magnetic field component parallel to the gradient of the external potential is nonlinear. We treat the case in which the external force is gravitation and may be considered as being approximately constant.

The paper is organised as follows. In Sect. 2 we briefly recapitulate the general method of calculating three-dimensional equilibria. In Sect. 3 we calculate a class of non-linear self-consistent arcade-like equilibria. In Sect. 4 we present two examples out of this class of solutions and in Sect. 5 we conclude the paper. In addition, we present two other solution classes which do not lead to arcade-like magnetic topologies in two appendices.

2. Recapitulation of the method

The starting point of the investigation is the set of MHS equations including a pressure gradient and an external gravitational field, but without an energy equation or an equation of state:

$$\boldsymbol{j} \times \boldsymbol{B} - \nabla p - \rho \nabla \psi = 0, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3}$$

Here, p denotes the pressure, ρ the plasma density, and ψ the gravitational potential. Since the current density $\boldsymbol{j}$ is solenoidal it may represented by two Euler potentials F and G; as one of the Euler potentials, say G, we choose the external potential ψ:

$$\boldsymbol{j} = \nabla F \times \nabla \psi. \tag{4}$$

Low (1991) showed that the other Euler potential F may be a function of the potential ψ and of the component of the magnetic field parallel to the gradient of ψ

$$F = h(\psi, \boldsymbol{B} \cdot \nabla \psi). \tag{5}$$

So far, solutions have been found only for cases in which F depends linearly on $\boldsymbol{B} \cdot \nabla \psi$ (Low 1991, 1992; Neukirch 1995). In the present contribution, we will derive solutions for cases with a nonlinear dependence of F on $\boldsymbol{B} \cdot \nabla \psi$. In addition to the component of the current density described by Eq. (4) it is possible to include a component similar to a linear force-free current (Low 1991, 1992). If F is linear the problem can be reduced to a one dimensional Schrödinger problem if ψ is spherically symmetric or depends only on one Cartesian coordinate (Neukirch 1995).

We proceed as described in Neukirch (1995) for the linear case. For simplicity, we restrict the analysis here to the case of a constant gravitational force pointing in the negative z-direction. The gravitational potential is then $\psi = \mathrm{g}z$ and without loss of generality we may write

$$F = h(z, B_z) \quad . \tag{6}$$

We now insert (we absorb a factor g in F for convenience)

$$\boldsymbol{j} = \alpha \, \boldsymbol{B} + \nabla F \times \nabla z \tag{7}$$

into Ampère's law and apply the differential operator $\boldsymbol{e}_z \cdot (\nabla \times)$ to this equation. Introduction of the operator $\hat{\boldsymbol{L}} = -\mathrm{i}\boldsymbol{e}_z \times \nabla$ and some algebraic manipulations yield:

$$-\Delta B_z = \mu_0^2 \alpha^2 B_z + \mu_0 \hat{L}^2 F(z, B_z) \tag{8}$$

where $\hat{L}^2 = -\partial^2/\partial x^2 - \partial^2/\partial y^2$. Equation (8) is the basic equation we have to solve. F is still a free function of its arguments z and B_z and we will choose it in a convenient way. The other components of the magnetic field can be derived from Ampère's law and $\nabla \cdot \boldsymbol{B} = 0$. The pressure and the density can be obtained from the force balance. We will not give this derivation here, but discuss it at the appropriate place together with the solutions.

3. Nonlinear solutions

Since

$$\Delta B_z = \frac{\partial^2 B_z}{\partial z^2} - \hat{L}^2 B_z \tag{9}$$

we can rewrite Eq. (8) as

$$\frac{\partial^2 B_z}{\partial z^2} + \mu_0^2 \alpha^2 B_z - \hat{L}^2 (B_z - \mu_0 F) = 0 \tag{10}$$

Choosing $F = B_z/\mu_0 + \tilde{F}$, we finally arrive at

$$\frac{\partial^2 B_z}{\partial z^2} + \mu_0^2 \alpha^2 B_z + \mu_0 \hat{L}^2 \tilde{F} = 0. \tag{11}$$

To obtain explicit solutions of Eq. (11) we assume that $\tilde{F}$ as well as B_z have special forms. We assume that

$$\tilde{F} = \xi(z) B_z^{\kappa} \tag{12}$$

where $\xi(z)$ is left unspecified for the moment. Another choice for $\tilde{F}$ leading to exact solutions is discussed in Appendix A. We furthermore make the assumption that B_z is separable in its dependence on z and on x and y, respectively:

$$B_z = \Xi(z) \Upsilon(x, y). \tag{13}$$

Since $\hat{L}^2$ only operates on Υ Eq. (11) can be written as

$$\frac{1}{\xi \Xi^\kappa}\left(\frac{\mathrm{d}^2\Xi}{\mathrm{d}z^2} + \mu_0^2\alpha^2\Xi\right) = -\frac{\mu_0}{\Upsilon}\hat{L}^2\Upsilon^\kappa. \quad (14)$$

As the left hand side of Eq. (14) depends only on z whereas the right hand side depends only on x and y, they must both be constant. If we call this constant k, the two separate equations for Ξ and Υ are

$$\frac{\mathrm{d}^2\Xi}{\mathrm{d}z^2} + \mu_0^2\alpha^2\Xi - k\xi\Xi^\kappa = 0 \quad (15)$$

$$\hat{L}^2\Upsilon^\kappa + \frac{k}{\mu_0}\Upsilon = 0 \quad (16)$$

Since $\xi(z)$ is an arbitrary function Eq. (15) can in principle be solved by assuming a function $\Xi(z)$ and regard Eq. (15) as a function for $\xi(z)$. This leaves us the free choice concerning the z-dependence of the solutions.

A solution of Eq. (16) can be found by introducing a new function $\Gamma = \Upsilon^\kappa$ and polar coordinates $x - x_0 = r\cos\phi$, $y - y_0 = r\sin\phi$, where the point x_0, y_0 is some reference point outside the considered domain. Equation (16) is then transformed into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Gamma}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Gamma}{\partial\phi^2} + \frac{k}{\mu_0}\Gamma^{1/\kappa} = 0 \quad (17)$$

This equation is again solved by separation in r and ϕ. With

$$\Gamma = r^\delta\gamma(\phi) \quad (18)$$

we obtain

$$\gamma'' + \delta^2\gamma + \frac{k}{\mu_0}\gamma^{1/\kappa} = 0 \quad (19)$$

if $\delta - 2 = \delta/\kappa$. An equation similar to Eq. (19) has e.g. been encountered in models of two-dimensional sheared arcades (Priest & Milne 1980; Webb 1986, 1988) and closed solutions have been found for different values of κ. Equation (19) is analogous to the equation of motion in a potential in one dimension. It can be integrated once to give

$$\frac{1}{2}\gamma'^2 + V(\gamma) = E \quad (20)$$

where E is an integration constant and the potential $V(\gamma)$ is defined as

$$V(\gamma) = \frac{1}{2}\delta^2\gamma^2 + \frac{k\kappa}{\mu_0(\kappa+1)}\gamma^{(\kappa+1)/\kappa} \quad (21)$$

Integrating once more, we obtain the implicit solution

$$\phi = \phi_0 + \int_0^\gamma \frac{\mathrm{d}\sigma}{\sqrt{2(E - V(\sigma))}} \quad (22)$$

To identify promising values of κ in order to get sensible solutions, we make the following considerations. The coronal magnetic field is structured into loops and arcade-like configurations

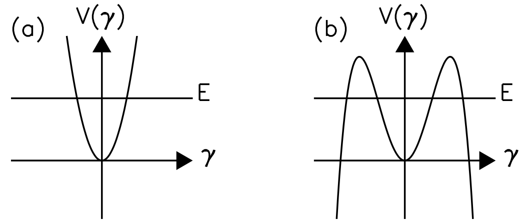

Fig. 1a and b. Sketch of the potential $V(\gamma)$ for $k > 0$ (**a**) and $k < 0$ (**b**).

and we therefore want to find solutions which are of this type. Arcade-like field lines connect regions of opposite polarity on the photosphere (which we assume to be identical to the plane $z = 0$) and thus we need to have regions with opposite sign of B_z on the photosphere. Since on the photosphere

$$B_z(x, y, 0) = \Xi(0)r^{\delta/\kappa}\gamma^{1/\kappa}(\phi) \quad (23)$$

only the last factor can cause a change of sign of B_z. This is only possible if κ is a fraction of odd integers and γ takes both positive and negative values.

Priest and Milne (1980) obtained solutions for $\kappa = -1/3$ which fulfils the first condition, but the corresponding potential $V(\gamma)$ has a singularity at $\gamma = 0$ and the solutions are either positive or negative definite. These solutions are not suitable for describing coronal arcades and are discussed in Appendix B for completeness.

Webb (1988) discusses the case $\kappa = 1/3$. In that case $\delta = -1$. The potential $V(\gamma)$ can then be written as

$$V(\gamma) = \frac{1}{2}\gamma^2 + \frac{k}{4\mu_0}\gamma^4. \quad (24)$$

A sketch of the potential $V(\gamma)$ for different parameter combinations is shown in Fig. 1. The solutions $\gamma(\phi)$ can be both positive and negative as required. Following Webb (1988) we write the solutions for $\gamma(\phi)$ as Jacobian elliptic functions (Abramowitz & Stegun 1965; Lawden 1989).

There are several different cases depending on the parameter values of the problem. In the case $k > 0$ the solution is

$$\gamma(\phi) = \frac{ab}{\sqrt{a^2+b^2}}\,\mathrm{sd}\left\{\left[\frac{k}{2\mu_0}(a^2+b^2)\right]^{1/2}(\phi - \phi_0)\,\middle|\,\frac{b^2}{a^2+b^2}\right\} \quad (25)$$

where

$$a^2 = \frac{\mu_0}{k}\left[\left(1 + 4E\frac{k}{\mu_0}\right)^{1/2} + 1\right] \quad (26)$$

$$b^2 = \frac{\mu_0}{k}\left[\left(1 + 4E\frac{k}{\mu_0}\right)^{1/2} - 1\right] \quad (27)$$

and $\mathrm{sd}(x|m)$ is one of the standard Jacobian elliptic functions. This solution is periodic in ϕ as can be seen from the analogy with particle motion in a potential well. However, as a detailed inspection of the complete magnetic field solution showed, this case is actually not of the form of a magnetic arch or arcade, because the magnetic field component across the polarity inversion line is giving field lines crossing the polarity inversion line a U-shape rather than an inverse U-shape necessary for an arch-like structure. Though recent observations show evidence for U-shaped magnetic field structures in the solar atmosphere (Lites et al. 1995), in the present paper we restrict our analysis to arcade-like structures and will therefore not consider this solution any further.

If $k < 0$ there are two separate cases depending on the value of E. If

$$0 < E < E_{\max} = \frac{\mu_0}{4|k|} \tag{28}$$

then

$$\gamma(\phi) = \pm g_1 \mathrm{sn}\left[\sqrt{\frac{|k|}{2\mu_0}} g_2(\phi - \phi_0) \left| \frac{g_1^2}{g_2^2} \right.\right] \tag{29}$$

with

$$g_1 = \sqrt{\frac{\mu_0}{|k|}} \left[1 - \left(1 - 4E\frac{|k|}{\mu_0}\right)^{1/2}\right]^{1/2} \tag{30}$$

$$g_2 = \sqrt{\frac{\mu_0}{|k|}} \left[1 + \left(1 - 4E\frac{|k|}{\mu_0}\right)^{1/2}\right]^{1/2}. \tag{31}$$

Again this solution is periodic in ϕ. For $E \to E_{\max}$ the period increases without bound and we obtain

$$g_1 = g_2 = -\sqrt{\frac{\mu_0}{|k|}} \tag{32}$$

and the solution reduces to

$$\gamma(\phi) = \sqrt{\frac{\mu_0}{|k|}} \tanh\left[\frac{1}{\sqrt{2}}(\phi - \phi_0)\right] \tag{33}$$

Finally, for $E > E_{\max}$ the solution has the form

$$\gamma(\phi) = \left(\frac{4\mu_0 E}{|k|}\right)^{1/4} \frac{\mathrm{sn}[\lambda(\phi-\phi_0)\,|\mu]\mathrm{dn}[\lambda(\phi-\phi_0)\,|\mu]}{\mathrm{cn}[\lambda(\phi-\phi_0)\,|\mu]} \tag{34}$$

with

$$\mu = \frac{1}{2}\left[1 + \sqrt{\frac{\mu_0}{4E|k|}}\right] \tag{35}$$

$$\lambda = \left(\frac{E\,|k|}{\mu_0}\right)^{1/4}. \tag{36}$$

This solution is non-periodic and has no lower or upper bound. However, this solution is singular at

$$\phi - \phi_0 = \frac{\mathrm{K}(\mu)}{\lambda} \tag{37}$$

where $\mathrm{K}(m)$ is the complete elliptic integral of the first kind (Abramowitz & Stegun 1965, Lawden 1989), and we will therefore not consider it any further. More details on the derivation of these solutions can be found in Webb (1988).

The other components of the magnetic field are determined as follows. Introducing the vector $\boldsymbol{B}_t = B_x\boldsymbol{e}_x + B_y\boldsymbol{e}_y$, we make the Ansatz:

$$\boldsymbol{B}_t = f(z)\nabla V(x,y) + g(z)\nabla W(x,y) \times \boldsymbol{e}_z \tag{38}$$

We insert Eq. (38) into Ampère's law and multiply the result with $\boldsymbol{e}_z$ and get

$$g(z)\hat{L}^2 W(x,y) = \mu_0\alpha\Xi(z)\Upsilon(x,y) \tag{39}$$

This equation can again be solved by separation leading to

$$g(z) = \frac{\mu_0\alpha}{c}\Xi(z) \tag{40}$$

and

$$\hat{L}^2 W(x,y) = c\Upsilon(x,y) \tag{41}$$

with c being the separation constant. Using Eq. (16), we may rewrite Eq. (41) as

$$\hat{L}^2(W + \frac{c\mu_0}{k}\Upsilon^\kappa) = 0 \tag{42}$$

with the solution

$$W = -\frac{c\mu_0}{k}\Upsilon^\kappa + \chi \tag{43}$$

where $\hat{L}^2\chi = 0$. By inserting Eqs. (13) and (38) into $\nabla \cdot \boldsymbol{B} = 0$, we can apply a similar method to obtain

$$f(z) = \frac{1}{d}\frac{\mathrm{d}\Xi}{\mathrm{d}z} \tag{44}$$

and

$$V(x,y) = -\frac{d\mu_0}{k}\Upsilon^\kappa + \zeta \tag{45}$$

with d being constant and $\hat{L}^2\zeta = 0$. The values of the constants c and d and the functions χ and ζ can be found by inserting the complete expression for $\boldsymbol{B}$ into Ampère's law. The result is that c and d are arbitrary, but have no influence on the solution because they drop out of the final expressions whereas χ and ζ have to be constant and may without loss of generality be set equal to zero. Putting everything together, we obtain for the components of the magnetic field

$$B_x = \frac{\mu_0}{k}\frac{1}{r^3}\left[\frac{\mathrm{d}\Xi}{\mathrm{d}z}\left((x-x_0)\,\gamma + (y-y_0)\,\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right) + \mu_0\alpha\,\Xi\left((y-y_0)\,\gamma - (x-x_0)\,\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right)\right] \tag{46}$$

$$B_y = \frac{\mu_0}{k}\frac{1}{r^3}\left[\frac{\mathrm{d}\Xi}{\mathrm{d}z}\left((y-y_0)\,\gamma-(x-x_0)\,\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right)-\mu_0\alpha\,\Xi\left((x-x_0)\,\gamma+(y-y_0)\,\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right)\right] \qquad (47)$$

$$B_z = \Xi\frac{1}{r^3}\gamma^3 \qquad (48)$$

The pressure and the density are more easily determined if they are formally regarded as functions of B_z and z rather than as functions of F and z. By integration of the force balance equation, we get

$$p = p_0(z) - \frac{1}{2\mu_0}B_z^2 - \frac{\kappa}{\kappa+1}\xi B_z^{\kappa+1} \qquad (49)$$

and

$$\rho = -\frac{1}{\mathrm{g}}\left[\frac{\mathrm{d}p_0}{\mathrm{d}z} - \left(\frac{1}{\mu_0}+\kappa\xi(z)B_z^{\kappa-1}\right)\boldsymbol{B}\cdot\nabla B_z - \frac{\kappa}{\kappa+1}B_z^{\kappa+1}\frac{\mathrm{d}\xi}{\mathrm{d}z}\right]. \qquad (50)$$

The temperature can formally be calculated by assuming that the plasma is an ideal gas:

$$T = \frac{mp}{\mathrm{R}\rho} \qquad (51)$$

where m is the mean molecular weight and R the universal gas constant.

4. Examples

In this section we show illustrative examples of the solution class found in Sect. 3. Since we still have the freedom to choose the function $\xi(z)$ it is impossible to give a complete discussion of all possibilities here. For convenience we choose $\xi(z)$ as

$$\xi(z) = \xi_0\exp(-z/L_\xi). \qquad (52)$$

Equation (15) is then solved for $\Xi(z)$ by

$$\Xi(z) = \Xi_0\exp(-z/L_\Xi) \qquad (53)$$

with $L_\Xi = 2L_\xi/3$ and $\Xi_0 = (k\xi_0 L_\Xi^2/(1+L_\Xi^2\mu_0^2\alpha^2))^{3/2}$. This solution for Ξ_0 implies that for negative k, ξ_0 does also have to be negative for Ξ_0 be real. As discussed in the Sect. 3, this is actually the case for the solutions found there. Finally, we choose the background pressure for convenience as

$$p_0(z) = \overline{p}_0\exp(-z/H) \qquad (54)$$

If we assume that the other two terms of the pressure have the same length scale in z as the background pressure, we obtain $L_\Xi = 2H$ and $L_\xi = 3H$. We make this assumption here because it simplifies the following analysis. In principle, H is not related to the other two length scales.

The mathematical structure of the resulting magnetic field is most easily comprehended if we write its components in cylindrical coordinates r, ϕ and z as defined above:[2]

$$B_r(r,\phi,z) = \frac{\mu_0\,\Xi_0}{k}\frac{1}{r^2}\left(\mu_0\alpha\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}-\frac{1}{2H}\gamma\right)\exp(-z/2H) \qquad (55)$$

$$B_\phi(r,\phi,z) = \frac{\mu_0\,\Xi_0}{k}\frac{1}{r^2}\left(\frac{1}{2H}\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}+\mu_0\alpha\gamma\right)\exp(-z/2H) \qquad (56)$$

$$B_z(r,\phi,z) = \frac{\Xi_0}{r^3}\gamma^3\exp(-z/2H) \qquad (57)$$

with the function γ defined in Eq. (29). We choose $\phi_0 = \pi/2$ to have the polarity inversion line lying along the y-axis. For this special choice of parameters, all three components of the magnetic field are products of functions depending on just one of the three cylindrical coordinates r, ϕ and z. For other parameter values this is not necessarily the case, but the basic properties of the magnetic field remain the same. The advantage of this special choice for the parameter values is the relative ease with which the properties of the solution can be analysed.

The total pressure can be written as

$$p = \exp(-z/H)\left(\overline{p}_0 - \frac{\Xi_0^2}{2\mu_0}\frac{1}{r^6}\gamma^6 - \frac{1}{4}\xi_0\Xi_0^{4/3}\frac{1}{r^4}\gamma^4\right) \qquad (58)$$

and the density as

$$\rho = \frac{1}{\mathrm{g}H}\exp(-z/H)\left\{\overline{p}_0 + \frac{\Xi_0^2}{2\mu_0}\frac{1}{r^6}\gamma^2\left[\frac{3\mu_0}{k}\left(\gamma^2+\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right)^2\right)-\gamma^4\right] + \frac{\xi_0\Xi_0^{4/3}}{6}\frac{1}{r^4}\left[\frac{3\mu_0}{k}\left(\gamma^2+\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\phi}\right)^2\right)-4\gamma^4\right]\right\} \qquad (59)$$

The temperature calculated from Eqs. (58) and (59) does not depend on z, because the z-dependent parts of p and ρ cancel in this case.

Though we are not aiming at modelling realistic solar structures, we nevertheless may take solar values as an orientation to choose reasonable parameter values. If we take a typical coronal value ($T = 2\cdot10^6$ K) for the background temperature, we get for the scale height of the background pressure $H = 6\cdot10^4$ km. We normalise the magnetic field with a typical field strength B_0 and the plasma pressure with $B_0^2/2\mu_0$ so that it corresponds to a plasma β_p.

The value of ξ_0 is constrained by the requirement that the pressure and the density have to be positive everywhere in the domain. We note that the last term contributing to the pressure in Eq. (49) is actually positive because ξ is negative. Once the parameters $\overline{p}_0$, k, E and α have been chosen, the absolute value

[2] Remember that the origin of this cylindrical coordinate system ($r = 0$) is outside the domain we are considering.

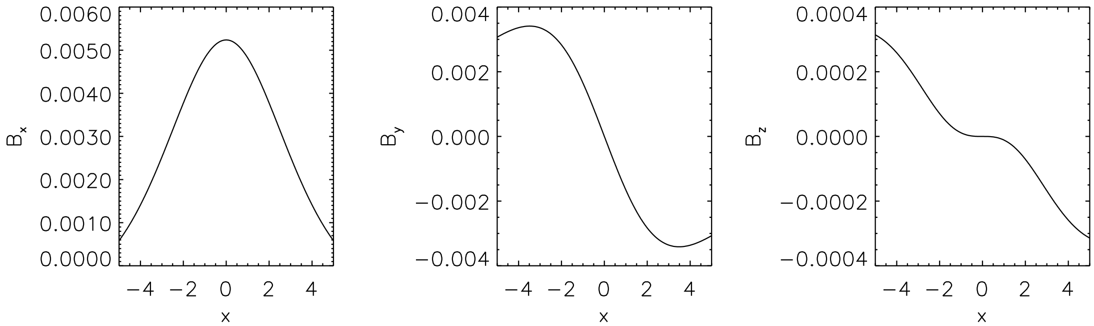

Fig. 2. Plot of the variation with x of all three magnetic field components along the middle of the lower boundary. We see that B_x is positive in the vicinity of the polarity inversion line ($x = 0$). The y-component of the magnetic field changes sign from positive to negative close to, but not exactly at the polarity inversion line. The B_z component is positive for negative x and vice versa. It has a saddle point at the polarity inversion line because it is proportional to the third power of γ which goes through zero at $x = 0$. For a detailed discussion of the implications of this structure of the magnetic fields see text.

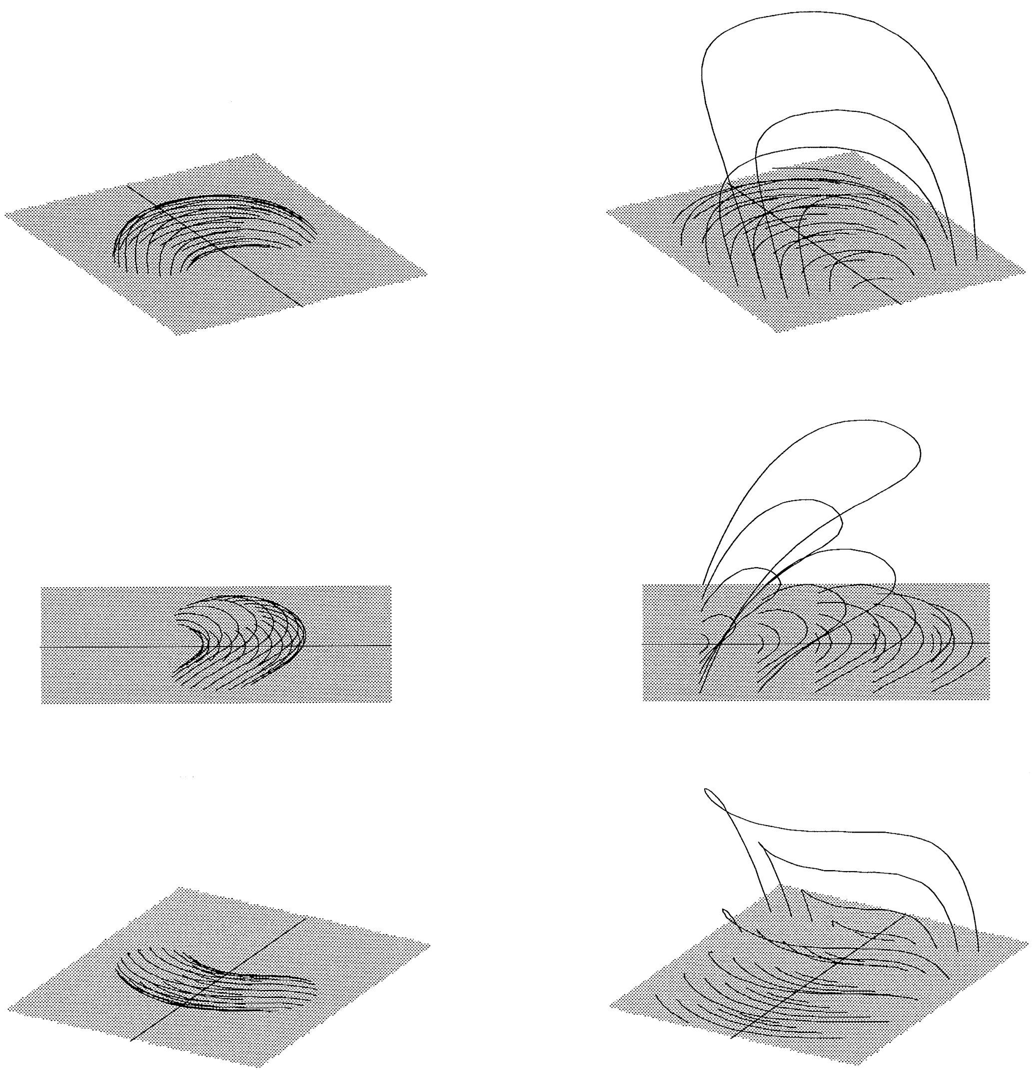

Fig. 3. Three-dimensional plot of magnetic field lines shown from three different perspectives. The parameter values used here are $\alpha = 0$, $k = -0.25$, $E = 0.5$ and $\xi_0 = -0.019$. On the left hand side a number of field lines emerging through the lower boundary on a circle are shown. On the right hand side a set of field lines with foot points on a regular grid in one half of the lower boundary are shown. The solid line on the lower boundary is the polarity inversion line. These plots show the basic structure of the field.

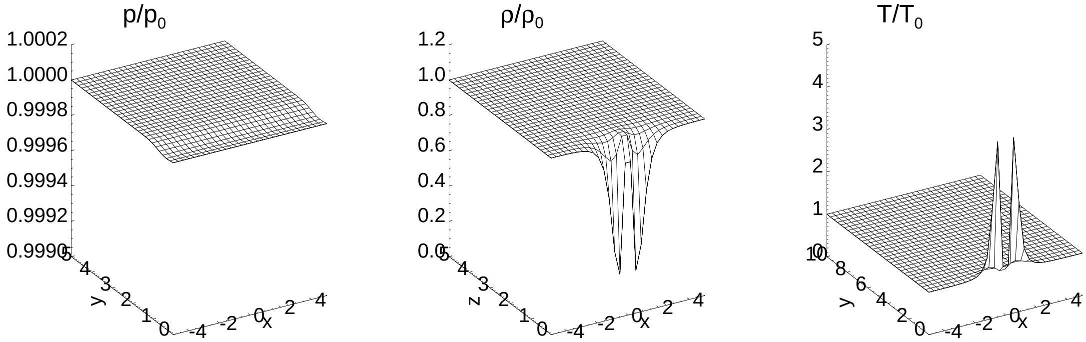

Fig. 4. Pressure, density and temperature in the plane z = 0.

of ξ_0 cannot be larger than a fixed maximum value. In the examples shown below we have always chosen sets of parameters which fulfill these constraints.

The actual values of k, E and ξ_0 do not influence the qualitative character of the solutions very much and we therefore only show solutions with a single representative choice of $k/\mu_0 = -0.25$, $E = 0.5$ and $\xi_0 = -0.019B_0^{4/3}/\mu_0$. The parameter α has a stronger influence on the solution as we will see below. The value of $\bar{p}_0$ has been set to $0.5B_0^2/2\mu_0$. For the following examples we chose $x_0 = 0.0$ and $y_0 = -1.0$ and restricted the solution to the half space $y \geq 0$. In Fig. 2 we show plots of all three magnetic field components in the plane z = 0 at y = 5.0. Since the z-dependence of the magnetic field components is given by a simple exponential factor, their qualitative behaviour stays the same for all values of z. We recognise that B_x is positive for all x-values shown, especially at the polarity inversion line ($x = 0$). Together with the fact that B_y goes through zero at the polarity inversion line, this means that the magnetic field points into the direction of the x axis when it crosses the polarity inversion line. Since B_z is positive for negative x and negative for positive x, we have the magnetic topology of an arcade-like magnetic field, with the field lines coming out of the lower boundary ("photosphere") for negative x, crossing the polarity inversion line and going down to the lower boundary again for positive x. For the solutions with $k > 0$ discussed in Sect. 3, we found that B_x always points into the opposite direction. Thus, these solutions do not show arcade-like structures. B_z has an inflection point at the polarity inversion line. This is due to its dependence on the third power of γ which has a simple zero there. In Fig. 3 we show several three-dimensional plots of the field lines. The field lines are shown from three different perspectives. On the left hand side we show a set of field lines emanating from the photosphere on a circle, giving the impression of a flux tube. On the right hand side, we show another set of field lines with foot points on a regular grid in the lower boundary region right of the polarity inversion line, which is shown as a line on the lower boundary. The plots clearly show the arcade-like structure of the magnetic field. The plots also show that the field lines are slightly bent which reflects the behaviour of B_y shown in Fig. 2.

In Fig. 4 we show pressure (left), density (middle) and temperature (right) normalised to their respective background values as surface plots over the z = 0 plane. Since the dependence on z is purely multiplicative, the basic structure of the three quantities does not change with height. The pressure shows only very minor changes compared with the background pressure (note the different scaling of the axis for the pressure plot). The density has two sharp minima on the boundary $y = 0$, which show up as maxima in the temperature.

To illustrate the influence of α on the magnetic field structure we show in Fig. 5 the same plot as in Fig. 2 but for α = 0.05. The finite value of α introduces an asymmetry into B_x and B_y. Especially important here is that B_x changes its sign in the region of positive B_z. This corresponds to an angle ϕ for which B_ϕ (see Eq. (56)) vanishes for all r and z. Since B_ϕ is zero, no field line can cross this surface and it is therefore a separatrix surface between different field line topologies. In Fig. 6 we show again the field line plots. These plots show the additional twist introduced by a non-vanishing α and the effect of the separatrix surface forcing the field lines to bend back to the photosphere closer to the polarity inversion line than in the α = 0 case. In Fig. 7 the pressure, density and temperature are shown for α = 0.05. The minima of the density are not as small as for the case α = 0 and thus also the temperature maxima are lower than previously. The general structure of the density and the temperature has not changed very much.

5. Summary and discussion

In this paper, we have derived analytical three-dimensional self-consistent solutions of the MHS equation which are not laminated and have a nonlinear relation between the current density and the magnetic field. This has been achieved by assuming a special relationship between the current density and the magnetic field. We have exploited these assumptions by seeking separable solutions. We have focussed on solutions which are arcade-like and found a solution class of this type for which the basic nonlinear equation can be solved in terms of Jacobian elliptic functions. Other solutions classes which do not have an arcade-like behaviour have been discovered as well and were briefly discussed in two appendices. The z-dependence of

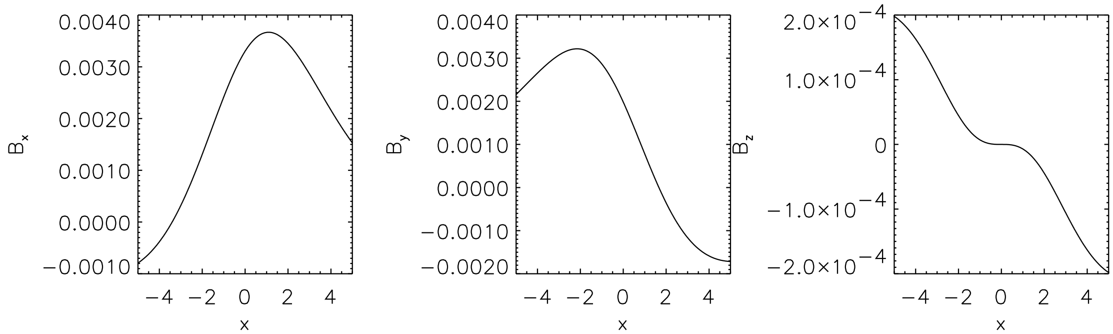

Fig. 5. Plot of the magnetic field components across the middle of the lower boundary for α = 0.05. The finite value of α introduces an asymmetry in B_x and B_y. Of particular importance is the zero of B_x at about $x = -3.5$, which introduces a separatrix surface into the field structure.

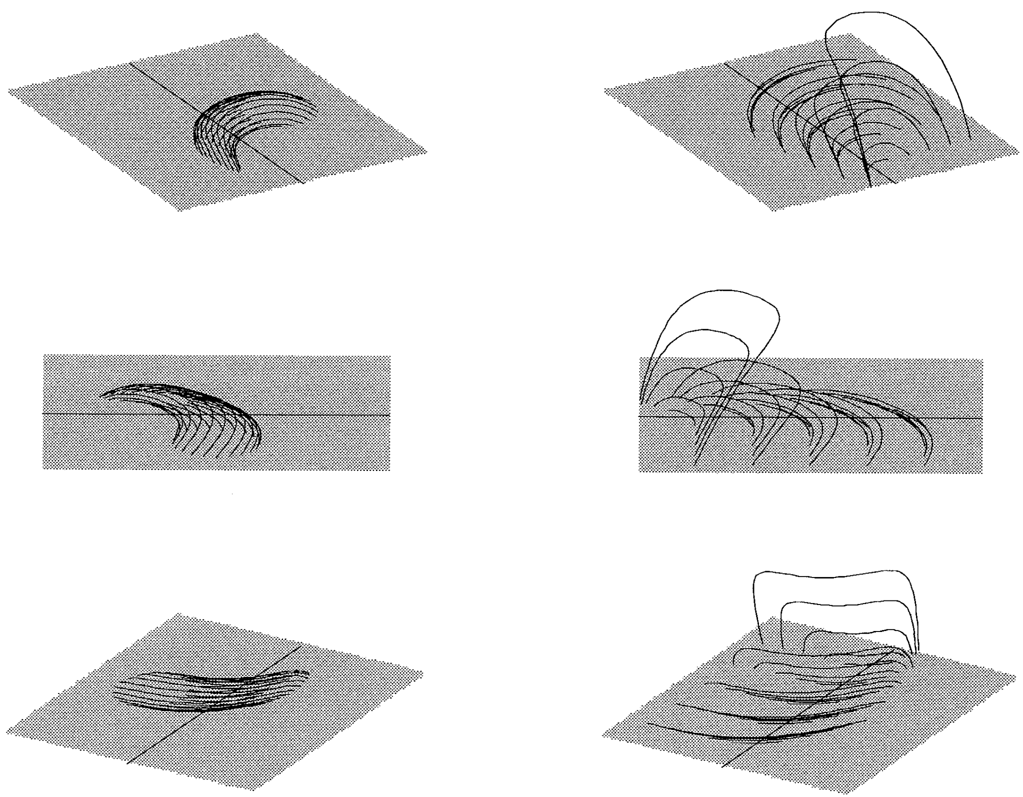

Fig. 6. Three-dimensional plot of magnetic field lines shown from three different perspectives. The parameter values used here are α = 0.05, $k = -0.25$, E = 0.5 and $\xi_0 = -0.019$. The additional field line twist introduced by the finite value of α is clearly discernible. Furthermore it can be seen that all field lines end on the photosphere to the right of the separatrix surface.

the solutions is still free and could be used to model different behaviour of the arcades with height. We have shown two illustrative examples of the solution class for two different values of the parameter α, corresponding to the linear field-aligned component of the current density.

The price one has to pay for getting analytical solutions of the nonlinear MHS equations in three dimensions is that the solutions have a fairly simple structure and that they contain an isolated singular line. The second shortcoming can be bypassed by considering the solution only in a restricted domain and placing the singularity outside that domain. Concerning the simple structure of the solutions we remark that the solution class still contains two free functions of z and at least four free parameters. In the present contribution we have by no means been able to investigate the space of possible solutions completely and that there may still be interesting properties of the solutions to be discovered.

The aim of the present contribution was to demonstrate the possibility of calculating self-consistent three-dimensional MHS equilibria beyond the linear regime. We consider this as a first step towards more general investigations of three-dimensional MHS equilibria. As the experience with two-

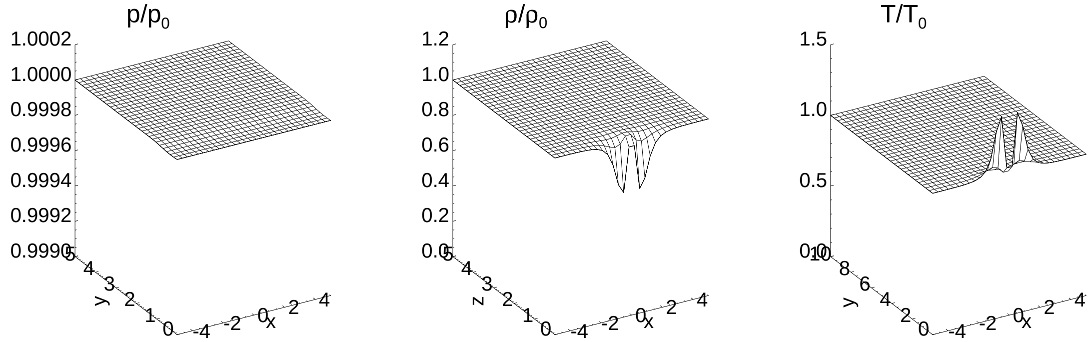

Fig. 7. Pressure, density and temperature for $\alpha = 0.05$. Though the extrema of the density and the temperature are smaller than in the previous case, the overall structure of these quantities has not changed.

dimensional equilibria shows, the existence of analytical solutions can be of great value for early modelling efforts and for testing numerical codes. Since it is quite certain that most of the work on three-dimensional equilibria has to carried out numerically, analytical equilibria, especially nonlinear ones, become even more important.

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Appendix A: the case $\tilde{F} = \xi(z) \ln B_z$

Another class of solutions may be obtained by assuming that

$$\tilde{F} = \xi(z) \ln B_z \tag{A1}$$

and that B_z may be written in the form of Eq. (13). Again, this leads to a form of Eq. (11) which is separable and we get

$$\frac{\mathrm{d}^2\Xi}{\mathrm{d}z^2} + \mu_0^2\alpha^2\Xi = k\xi \tag{A2}$$

and

$$\hat{L}^2 \ln \Upsilon = k\Upsilon \tag{A3}$$

Equation (A2) is a linear inhomogeneous ordinary second order differential equation which may either be solved by standard methods if $\xi(z)$ is given or which may again be regarded as an equation for determining $\xi(z)$ if $\Xi(z)$ is given.

With the definition $\Gamma(x, y) = \ln \Upsilon$, Eq. (A3) can be transformed into Liouville's equation (Liouville 1853). The general solution for this equation is:

$$\Upsilon = \frac{\left|\frac{\mathrm{d}\Psi}{\mathrm{d}u}\right|^2}{\left(1 + \frac{k}{8}|\Psi|^2\right)^2}. \tag{A4}$$

Here, $u = x + \mathrm{i}y$ and Ψ is an analytic function. However, once again this class of solutions allows only one polarity for $z = 0$ and is therefore not discussed any further.

Appendix B: solution with $\kappa = -1/3$

An exact solution is also possible for $\kappa = -1/3$ ($\delta = -2$). In that case Eq. (22) leads to

$$\gamma(\phi) = \pm\frac{1}{2}\left\{2E + \sqrt{4E^2 + 4\frac{k}{\mu_0}\sin(4(\phi - \bar{\phi}_0))}\right\}^{\frac{1}{2}} \tag{B1}$$

with

$$\bar{\phi}_0 = \phi_0 - \frac{1}{2\sqrt{2}} \arcsin\left(\frac{2E}{\sqrt{4E^2 + 4\frac{k}{\mu_0}}}\right) \tag{B2}$$

However, as already discussed in Sect. 3, this solution does not change sign with ϕ and therefore does not give rise to arcade-like solutions.

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