Astron. Astrophys. 325, 1253–1258 (1997)

ASTRONOMY
AND
ASTROPHYSICS

Photometry with a periodic grid*

I. A new method to derive angular diameters and brightness distribution

D. Hestroffer[1,2] **and F. Mignard**[3]

[1] *Present address:* Astrophysics Division, ESTEC, Postbus 299, 2200 AG Noordwijk, The Netherlands
[2] Bureau Des Longitudes, URA CNRS 707, 77 Avenue Denfert Rochereau, F-75014 Paris, France
[3] Observatoire de la Côte d'Azur, Dept. CERGA, UMR CNRS 6527, Av. N. Copernic, F-06130 Grasse, France

Received 20 February 1997 / Accepted 15 April 1997

Abstract. The interest of observations of point sources with a periodic grid has been recognized for long in astrometry because of the direct relation existing between the position in the sky and the phase of the modulated signal. We investigate in this paper a complementary aspect by showing how observations of an extended stellar or planetary object carried out with a periodic grid may also yield valuable information on its size and on the light distribution over its surface. Several categories of objects may benefit from this kind of observation depending on the modulation period, the telescope focal length, and the signal–to–noise ratio that can be achieved. We discuss the overall principles of this method and derive an analytical formulation for the 'modulation function' defined in the text. An application based on Hipparcos observations of the largest minor planet (1) Ceres is provided as an illustration of the method.

Key words: photometry – scattering – minor planets, asteroids – stars: fundamental parameters of

1. Introduction

The list of numbered asteroids is increasing every year and there are now nearly ten thousand known small bodies orbiting mainly between the orbits of Mars and Jupiter. The dynamical and physical study of asteroids is an important tool for our understanding of the formation of the solar system and its evolution. During the past decade, photometric lightcurves of asteroids have been obtained on a large number of objects and these curves remain the primary data for the rotation rates, the shapes and the orientation of the axes of rotation. In addition, the change of the lightcurve parameters with the phase angle is linked to the scattering properties of the surface of the planet and its study provides an alternate and complementary means to spectrophotometry.

Send offprint requests to: D. Hestroffer, dhestrof@estec.esa.nl

* Based on data from the Hipparcos astrometry satellite.

Radiometry is the most widely used method for the assessment of asteroid diameters and of their albedo. The Asteroid and Comet Catalogue from the International Astronomical Satellite IRAS offers a large data base of such determinations. Radiometric diameters are however limited to a given wavelength band and are model dependent (Lebofsky 1989); their comparison or calibration from observations obtained in the visible remains an important goal for planetary scientists. It is true that observations of stellar occultations - performed simultaneously at different observatories - are the most accurate way to determine the diameter and the shape of the asteroids, they nevertheless remain difficult and costly to achieve and cannot be performed on a routine basis.

Before the advent of CCD chips, the use of periodic grids was of great interest for the positional measurements of celestial bodies. This was developed for instance at the photoelectric meridian circles of Bordeaux and La Palma or on many photoelectric or automatic astrolabes over the world. As the grid is coupled with a photometer, these observations can also yield valuable photometric information. The astrometric Hipparcos satellite of the European Space Agency was also equipped with a periodic grid leading in the end to astrometric and photometric measurements. The method proved very efficient despite the very modest aperture of the on-board telescope, less than 30 cm in diameter. In the case of Hipparcos it has been shown conclusively that the use of a particular combination of the amplitudes of the modulated signal as an estimator of the magnitude of point sources, was also a powerful means for detecting binary stars and determining their astrometric parameters (Mignard *et al.* 1995).

The modulated signal of an extended object seen through a periodic grid departs from the signal of a point source. This difference is essentially influenced by the apparent size of the object and its surface brightness distribution (Morando 1987; Lindegren 1987; Morando & Lindegren, 1989; Hestroffer *et al.* 1995). The first section of this paper deals with the fundamental equations that relate the modulated signal to the global properties of the light source while in the second section we investigate in more detail the influence of the light distribution on the amplitudes of the light curve (modulation function). Finally we present in the last section an application of this method to the observations of minor planet (1) Ceres made by Hipparcos.

2. Modulated photometry

Consider an observation of a celestial body whose image is focussed on a grid of angular period s. The image moves on this grid perpendicularly to the slits at a speed V because of the motion of the platform (the Earth or a satellite) carrying the telescope. This produces a periodic output signal modulated by the slit system with a frequency $\omega = 2\pi\, V/s$. This periodic signal can be expanded in a Fourier series as:

$$S(t) = I + B + \sum_{m>0} I\, M_m^o \ \exp^{\, jm\,(\omega\, t + \varphi_m)} \qquad (1)$$

where I is the total intensity, B the unmodulated background signal and dark current, $j = \sqrt{-1}$, M_m^o and φ_m are respectively the modulation coefficients and the phase offset of the m^{th} harmonic. The fundamental signal being smooth, the expansion can be adequately limited to the first few harmonics. The intensity I depends on the magnitude of the source or that of its components in case of a multiple system; the modulation coefficients, for a point-like source, are linked to the instrument transfer function and thus can be calibrated with natural or artificial point sources. Let M_m^o be the reference modulation coefficients for an isolated point source. The diffraction pattern produced by an entrance pupil, real with no phase shift, is symmetric and the output signal after the grid can be made even with an appropriate origin. This implies that the phases φ_m do not depend on the harmonic rank and are all equal to $\varphi_1 = \varphi$ for a point source.

Observation of a binary with a linear detector yields the total signal:

$$\begin{aligned} S(t) &= S_1(t) + S_2(t) \\ &= B + \sum_{i=1,2} \left(I_i + \textstyle\sum_{m>0} I_i\, M_m^o \ \exp^{\, jm\,(\omega\, t + \phi_i)} \right) \\ &= I + B + \sum_{m>0} I\, M_m \ \exp^{\, jm\,(\omega\, t + \varphi_m)} \qquad (2) \end{aligned}$$

We introduce the intensity ratio $I_2/I_1 = 10^{-0.4\,\Delta m}$, where ($\Delta m > 0$) is the magnitude difference between the faint and the bright component of the binary, and $\phi = \phi_2 - \phi_1$ the phase difference, which is the same as the separation of the components projected on the scanning direction modulo the grid period. This leads to the relations (Mignard *et al.* 1989):

$$I = I_1 + I_2$$

$$\frac{M_m}{M_m^o} = \frac{\left[1 + (I_2/I_1)^2 + 2\, I_2/I_1 \cos(m\,\phi)\right]^{1/2}}{1 + I_2/I_1} \qquad (3)$$

There are also relations linking the phase offsets φ_m that are of interest only for the astrometric measurements and are not given here. Not surprisingly the modulation is smaller for a compound source than for a single star, which translates into the fact that for every harmonic m we have $M_m/M_m^o \leq 1$. From an astrometric point of view we may refer this fact to 'harmonic degradation' because this may hamper to considerable extent the determination of the position. Hereafter Eq. (3) is referred to as the modulation function.

Similarly, an extended source can be viewed as a continuum of closely packed sources and the resulting signal has amplitudes and phases depending on the source apparent size and its surface scattering properties or albedo distribution. More precisely we have:

$$S = B + \iint_{\mathscr{D}} S_\sigma(t)\ d\sigma \qquad (4)$$

where the integration holds over the illuminated part of the body visible from the telescope. The response function for the surface element $d\sigma$ is given by:

$$S_\sigma(t) = I_\sigma \left[1 + \sum_{m>0} M_m^o \ \exp^{\, jm\,(\omega\, t + \varphi_\sigma)} \right] \qquad (5)$$

expressing the fact that each surface element has the same behavior as a point source and thus the same modulation coefficients. The intensity I_σ is directly related to the light distribution over the surface and the phase offset φ_σ depends on the size of the object and the geometry of its projection in the scanning direction.

If we neglect for the moment the unmodulated background B, the amplitude of each harmonic can be deduced from the Fourier transform in two dimensions:

$$S_m^*(x) = \iint_{\mathscr{D}} I_\sigma \ \exp^{\, -jm\, x\, \mathbf{u'n}}\ \mu\, d\sigma \ , \text{with} \quad x \equiv \frac{\pi\,\rho}{s} \qquad (6)$$

where $\mu\ d\sigma$ is the 3D surface element projected on the sky, ρ is the radial component of the polar coordinates of $d\sigma$ (i.e. the apparent diameter for a spherical object), s the grid step, $\mathbf{u}$ the scanning direction, $\mathbf{n}$ the normal to $d\sigma$ and $\varphi_\sigma = -x\,\mathbf{u'n}$. As a first step we consider spherical objects (ρ =constant) and simple functions for the intensity distribution I_σ, limiting the study to the case of geometrical scattering. Although geometrical scattering laws do not rest on serious physical grounds, nonetheless, they happen to give a satisfactory model of the photometric behavior of main-belt asteroids.

A normalized version of Eq. (6) leads to the modulation function defined as:

$$\frac{M_m}{M_m^o}(x) = \frac{|S_m^*(x)|}{\iint_{\mathscr{D}} I_\sigma\, \mu\ d\sigma} \qquad (7)$$

3. Light distribution model

For an object with azimuthal symmetry ($I_\sigma = I_r$ in polar coordinates where $r = \sin\theta$ with θ for the angle between the direction of the surface element $d\sigma$ and the direction from the centre of the planet to the observer), the integral in Eq. (7) is deduced from the Hankel transform of zero order of the function I_r:

$$\mathscr{H}_0[I_r;\, m\,x] = \int_0^\infty r\, I_r\, \mathrm{J}_0(m\,x\,r)\; dr$$

by:

$$\frac{M_m}{M_m^o}\,(x) = \frac{|\mathscr{H}_0[I_r;\, m\,x]|}{\int_0^\infty r\, I_r\; dr} \tag{8}$$

From now on, we consider either the case of uniform brightness:

$$\begin{array}{lll} I_r = I_o & ; & 0 \le r \le 1 \\ I_r = 0 & ; & r > 1 \end{array}$$

or the empirical Minnaert's law with the scalar parameter k (Minnaert 1961):

$$\begin{array}{lll} I_r = I_o\, \mu_o^k \mu^{k-1} & ; & 0 \le r \le 1 \\ I_r = 0 & ; & r > 1 \end{array}$$

where μ_o and μ are the cosines of the angles between the surface normal and respectively the incident and reflected ray.

The assumption of radial symmetry implies, for a solar system object, that the solar phase angle (the angular distance between the observer and the Sun as seen from the object) is zero. Under this hypothesis we have $\mu_o = \mu$, hence the intensity distribution may be written:

$$I_r = I_o\, \mu^{2\,(\nu-1)} \quad ; \quad \text{where we put } \nu \equiv k{+}1/2 \;\in \mathbb{R}$$

we see that the model of uniform brightness is now a particular case of the Minnaert scattering law with $k = 1/2$. Dark solar system bodies like asteroids are more likely uniformly bright objects, whereas icy satellites have a more pronounced centre–to–limb darkening. The frequently used Lambert's law corresponds here to $k = 1$.

Taking the geometrical law of Minnaert and noting that $\mu^2 = 1 - r^2$, we write from Eq. (7) and (3) the modulation function:

$$\frac{M_m}{M_m^o}\,(x) = \frac{\left|2\,\pi\, \int_0^1 r\,(1-r^2)^{\nu-1}\, \mathrm{J}_0(m\,x\,r)\; dr\right|}{\iint_{\mathscr{D}} \mu^{2\,(\nu-1)}\, \mu\; d\sigma} \tag{9}$$

where the denominator is equal to π/ν, and the numerator is a Hankel transform (Sneddon 1972). We finally deduce that:

$$\frac{M_m}{M_m^o}\,(x) = \Gamma(\nu{+}1)\; \frac{|\mathrm{J}_\nu(m\,x)|}{(m\,x/2)^\nu} \tag{10}$$

which is the norm of the hypergeometric function:

$$M_m/M_m^o = |{}_0F_1(;\, \nu+1;\, -(m\,x)^2/4)|$$

and yields for a uniformly bright object[1]:

$$\left(\frac{M_m}{M_m^o}(x)\right)_{|\,\nu=1} = 2\; \frac{|\mathrm{J}_1(m\,x)|}{m\,x} \tag{11}$$

[1] This particular result was erroneously given in Hestroffer *et al.* (1995) with omitted absolute value.

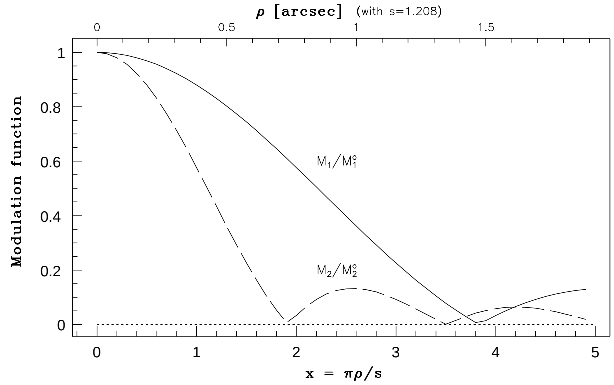

Fig. 1. Modulation function for the first two harmonics of a uniformly bright disc as a function of the spatial frequency x. The abscissa scale is also given in terms of apparent diameter for the Hipparcos main grid of period $s = 1\farcs208$

As expected one finds the limiting cases of objects of apparent diameter much smaller than the grid step, and for any harmonic and parameter ν:

$$\lim_{x \to 0}\; \frac{M_m}{M_m^o}\,(x) = 1$$

and the expansions:

$$\begin{aligned} \frac{M_m}{M_m^o}\,(x) &= \varepsilon_x \sum_{k \ge 0} (m\,\pi)^{2\,k}\; \frac{(-1/4)^k}{k!\;(\nu+1)_k}\; (\rho/s)^{2\,k} \\ &= \varepsilon_x \left[1 - \frac{1}{4}\, \frac{(m\pi)^2}{\nu+1}\; (\rho/s)^2 \right. \\ &\quad \left. + \frac{1}{16}\, \frac{(m\pi)^4}{2\,(\nu+2)\,(\nu+1)}\; (\rho/s)^4 \right] + o\,(\rho/s)^6 \end{aligned} \tag{12}$$

where $\varepsilon_x = \mathrm{sgn}\,[\mathrm{J}_\nu(m\,x)]$ and $(\nu+1)_k$ is the Pochhammer symbol, $(x_k = x(x+1)\ldots(x+k-1))$.

Fig. 1 shows the attenuation of the first and second harmonics for a uniformly bright disk. In the general case of Minnaert's law and for an object seen close to its opposition to the Sun, one sees from Eq. (10) that the harmonics disappear at spatial frequencies related to the zeros of the Bessel function J_ν. The modulation function for any harmonic, and any power law for the light distribution, is derived from the function $M_1/M_1^o\,(x)$ of the first harmonic by a scaling of the spatial frequency x:

$$\frac{M_m}{M_m^o}\,(x) = \frac{M_1}{M_1^o}\,(m\,x)$$

so that we can hereafter focus on this particular function.

Fig. 2 gives this modulation function for different theoretical brightness distributions - ranging from a uniformly bright disk to a fully darkened limb as a function of the spatial frequency. The main free parameters are the exponent in the Minnaert's law, the period of the grid and the apparent diameter

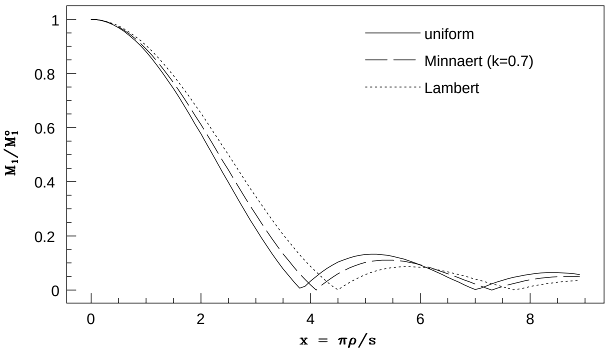

Fig. 2. Modulation function of the first harmonic as a function of the spatial frequency x, for a uniformly bright disk (solid curve), a Lambertian disk (dotted) and distribution following Minnaert's law with $k = 0.7$ (dashed)

of the source. Thus one can derive different applications from this modulation function depending on the free parameters and on the signal to noise ratio. We can however distinguish the following main cases:

i. A body of constant apparent diameter observed with a slit system composed of different spacings between the slits. For a given brightness distribution it is possible to derive from the observed signals a - model dependent - apparent diameter ;

ii. A solar system object with significantly varying geocentric distance. Knowing the apparent diameter it is possible to extract information on the light distribution of the resolved disk.

These two situations are closely related to the interpretation of visibility in the observations of giant stars with an interferometer, and provide a new method to derive the diameter and the scattering properties of the light sources.

Indeed, the ratio obtained for the first harmonic attenuation in Eq. (11) is, under the same assumptions of uniformly bright circular disk, strictly identical to the visibility function of the fringes with a stellar interferometer (Hanbury *et al.* 1974). The typical scales for the spatial frequencies between the two kinds of instruments are however different, so that we have different classes of interesting objects - for each harmonic - with the modulated photometry. Although observations spread over a large range of spatial frequencies are preferable, data acquired around the zeros of the modulation function are best suited to determine both the diameter and the properties of the light distribution.

By analogy with the observations with a stellar interferometer, the match to changing the baseline length (when observing an object of constant apparent diameter) would be a change in the grid step, so that one could in theory sample a larger part of the modulation function and get a better knowledge of the body size and of the light scattering.

With a slit of period p, a telescope focal length of f, and the first zero of the modulation function x_o, we have the corresponding angular diameters for each harmonic:

$$\rho_m = \frac{p\, x_o}{m\, \pi\, f}$$

For the Hipparcos observations with a grid step of $s = p/f = 8.2\,10^{-6}/1.4 \sim 1''\!.208$, this yield $\rho_1 \sim 1''\!.49$, $\rho_2 \sim 0''\!.74$. There were very few objects in this range of size among the solar system objects observed during the mission : (1) Ceres as the one minor planet, and the planetary satellites J2 Europa, S6 Titan and S8 Iapetus. Unfortunately, because of the residual light due to the nearby planet, there is no unmodulated photometry available for these satellites, thus the analysis of the photometry departs slightly from the present study (Hestroffer & Mignard, in prep.).

4. Application to asteroid (1) Ceres

The observed modulation curve in Eq. (1) was expanded for Hipparcos up to the second harmonic. A second magnitude estimator $H_{\rm ac}$, in the Hipparcos system, was constructed from a combination of the two amplitudes:

$$H_{\rm ac} = -2.5 \log \frac{I}{I_{\rm ref}} \; \frac{M_1\, M_1^o + M_2\, M_2^o}{(M_1^o)^2 + (M_2^o)^2}$$

where $I_{\rm ref}$ is the reference intensity of the magnitude scale. The apparent magnitude is given by:

$$H_{\rm dc} = -2.5 \log \frac{I}{I_{\rm ref}}$$

For a point source there is no attenuation, so that $H_{\rm ac} = H_{\rm dc}$. For a extended object, the difference $\Delta H = H_{\rm ac} - H_{\rm dc}$ is a weighted mean of the harmonics attenuation:

$$\Delta H(x) = -2.5 \log \frac{M_1\, M_1^o + M_2\, M_2^o}{(M_1^o)^2 + (M_2^o)^2} \,. \quad (13)$$

Fig. 3 shows the plot of this difference for the Hipparcos observations of (1) Ceres together with the theoretical functions obtained with two parameters of the Minnaert law. The adopted diameter of Ceres is $D = 913$ km. It is stressed that, given the values of the reference modulation coefficients in the Hipparcos detection chain ($M_1^o = 0.72$ and $M_2^o = 0.25$), $\Delta H(x)$ is (1) primarily function of the attenuation of the first harmonic ; and (2) depends heavily on the apparent diameter.

In the previous sections we have assumed that the planet was observed at the opposition which was not the case for the Hipparcos observations, which occurred preferentially around the quadratures. However the modulation function depends essentially on the projection along the scanning direction, so that increasing slightly the solar phase angle is equivalent to a small drop of the apparent diameter.Thus we can still attempt to interpret the observed attenuation with the simplified model for various subsets of solar phase angle. Taking the set of magnitude differences ΔH of Eq. (13), we have constructed the residuals

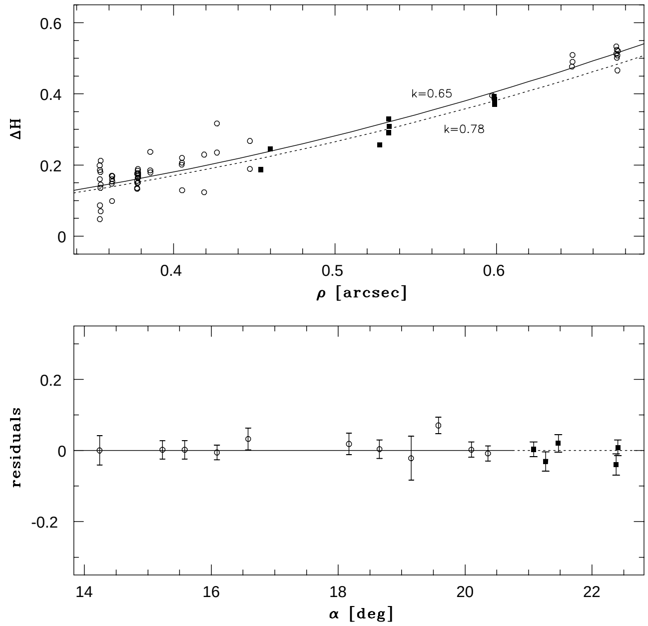

Fig. 3. Harmonics attenuation expressed on logarithmic scale (magnitude bias ΔH) for Hipparcos observations of minor planet (1) Ceres (top). Each point correspond to a transit of the planet across the field of view. The residuals (bottom) are calculated from a light scattering following Minnaert law, for two subsets in solar phase angle: circles, $\alpha < 21^\circ$ and squares, $\alpha > 21^\circ$. Each point is a mean of successive transits

shown in Fig. 3. They are calculated for two subsets of solar phase angle: $\alpha < 21^\circ (k = 0.65)$ and $\alpha > 21^\circ (k = 0.78)$. The precision of the measurements depended essentially on the apparent magnitude of the minor planet, and was of the order of 0.03 mag for Ceres, whose average magnitude is 9.2. Neglecting the dark region, constrains the model to adjust for a pronounced center to limb darkening, and the numerical results for the parameter k are overestimated. On the other hand, this provides a qualitative test for the value of the method. A future step will be to correct the magnitude offset for the known solar phase. Apart from this effect, the values are in global agreement with the expected lunar-like behaviour of a C class asteroid (Lumme & Bowell 1981b).

This application also highlights what could be achieved in the case of stellar observations since the Hipparcos main grid was not optimised for stellar angular diameters measurements. From the approximation:

$$\Delta H(x) \sim 1.2\,\rho^2$$

we can see from Figs. 1 and 3, that the attenuation is still significant for $x \sim 0.5$. So the analysis of the second harmonic of a modulation performed with a grid step five times smaller, could provide the diameter of a star of angular size $\phi \approx 20$ mas.

5. Discussion

This study is the first development of a new method to infer diameters and surface brightness distributions of celestial bodies. Some limitations of the actual model are that objects a few hundredth of arcsec in diameter — like asteroids — are not spherical but better approximated by triaxial ellipsoids. Another limitation in this model arises from the solar phase angle or the hypothesis of radial symmetry of the intensity distribution. An improved analytical formulation can be derived for the simplest uniform brightness distribution only; other more realistic scattering functions, deduced from radiative transfer theory, could be adopted such those from Hapke (1981) or Lumme & Bowell (1981a). These laws introduce many free parameters and are still subject to debates; a simpler analytical formulation for the scattering function has also been derived by Buratti & Veverka (1983). There are however no strong limitations because calculations for more complex scattering functions, convex shapes including oblique illumination by the Sun can be performed by numerical methods. The analytical solution nevertheless yields tractable expressions useful for the comparison and the assessment of the method; moreover it is completely justified for many celestial bodies other than asteroids like planetary satellites and stars.

For a typical apparent diameter, there is a particular grid period which is the most efficient to retrieve the information from the first two harmonics. Analyzing higher harmonics could in theory yield information on smaller objects for a given grid step. But the remaining signal in these higher harmonics depends now on the relative size of the grid step and that of the diffraction pattern, that is to say of the telescope aperture. A compromise has thus to be found between the science objectives, the manufacture of the instrument and the costs.

Finally, as pointed out in Sect. 4 there exists, for a given size of an object, a particular grid step for which the influence of the light scattering can be neglected on the modulation function of the first harmonic (see Fig. 3); this allows to determine in a first step the apparent diameter. In a second step, the modulation function of higher harmonics and/or obtained with smaller grid step will be more sensitive to the brightness distribution, and would enable to estimate the parameters of the light scattering.

6. Conclusion

Photometric measurements carried out with a periodic grid at the focal plane of a telescope, allows to determine either the angular size of the source or the brightness properties of its surface, or both of them, in a range of diameters connected to the diffraction pattern of the telescope and to the period of the grid. We have given a first analytical expression of the modulation function for a spherical body as a function of the spatial frequency $x = \pi\rho/s$ where ρ is the apparent diameter of the celestial body and s the angular period of the grid. Application to Hipparcos observations ($s = 1\farcs208$) of the minor planet (1) Ceres, whose apparent diameter varies between 0.35 and 0.7 arcsec, illustrates the value of this method. It is found that the surface optical properties depart only slightly from a uniformly bright disk with a moderate limb-darkening effect.

References

Buratti B., Veverka J., 1983, Icarus 55, 93.
Hanbury-Brown R., Davis J., Lake R.J.W., Thompson R.J., 1974, MNRAS 167, 475.
Hapke B., 1981, JGR 86, 3039.
Hestroffer D, Morando B., Mignard F., Bec-Borsenberger A., 1995, A&A 304, 168.
Lebofsky L.A., 1989, Icarus 78, 355.
Lindegren L., 1987, *In* Third FAST thinkshop, Bernacca and Kovalevsky (eds.), p. 285.
Lumme K., Bowell E., 1981a, AJ, 86, 1694
Lumme K. Bowell E., 1981b, AJ, 86, 1705.
Mignard F., Borriello L., Kovalevsky J., Prezioso N., Bernacca P.L., 1989, *In* The Hipparcos Mission. ESA SP-1111, Vol.3, p. 243.
Mignard F., Söderhjelm S., Bernstein H., *et al.*, 1995, A&A 304, 94.
Minnaert M., 1961, *In* Planets and Satellites, Vol 3, The Solar System. Kuiper and Middlehurst (eds.), p. 213.
Morando B., 1987, *In* Third FAST thinkshop, Bernacca and Kovalevsky (eds.), p. 277.
Morando B., Lindegren L., 1989, *In* The Hipparcos Mission. ESA SP-1111, Vol 3, p. 269.
Sneddon I.A., 1972, The use of integral transform, p. 313. McGraw-Hill Book Company, New York.

This article was processed by the author using Springer-Verlag LaTeX A&A style file *L-AA* version 3.