Astron. Astrophys. 326, 417–418 (1997)

ASTRONOMY
AND
ASTROPHYSICS

Research Note

On the use of nonrelativistic bremsstrahlung cross sections in astrophysics

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Received 27 February 1997 / Accepted 28 February 1997

Abstract. The nonrelativistic Bethe-Heitler cross section for bremsstrahlung often used in astrophysical applications is a poor approximation at initial electron energies above 30 keV. Therefore a simple formula is given which is accurate up to semirelativistic energies. An extension of this expression can be applied in the whole energy range.

Key words: radiation mechanisms: non-thermal

Energetic electrons are present in a great variety of cosmic plasmas throughout the universe. These electrons can be observed via the radiation they are emitting. One of the most important emission processes is bremsstrahlung production. In order to evaluate the radiation flux the cross section is required. It is given in first Born approximation by the well known Bethe-Heitler formula (Heitler, 1954) which is sufficiently accurate in astrophysical applications since the target nuclei — essentially hydrogen and helium — have low atomic numbers Z. It is, however, a common practice to use the nonrelativistic limit of the Bethe-Heitler cross section for problems involving electrons with energies of up to a few hundred keV. This expression has the form

$$\frac{d\sigma_B}{dk} \approx \frac{16\alpha Z^2 r_0^2}{3kp_1^2} \ln \frac{p_1 + p_2}{p_1 - p_2} \tag{1}$$

where $\alpha \approx 1/137$ is the fine structure constant, r_0 is the classical electron radius, k is the photon energy in units of the electron rest energy, mc^2, and p_1, p_2 are the momenta (in units of mc) of the incident and scattered electrons, respectively.

Due to its simplicity the cross section (1) has the advantage that calculations may be performed analytically, e.g., to derive parent electron spectra (Brown, 1971). But the accuracy of (1) falls off rapidly for increasing electron energies E_1 as is shown in

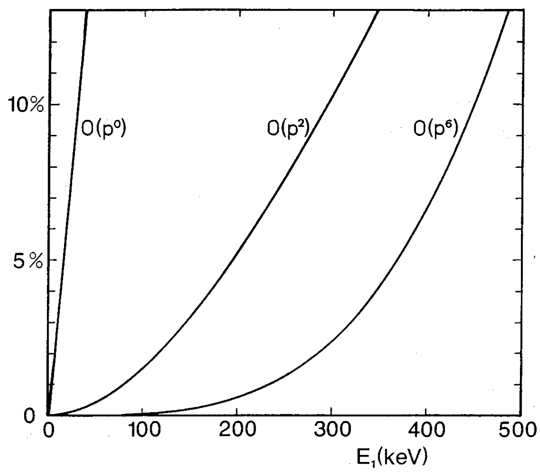

Fig. 1. Maximum relative error of the nonrelativistic approximations to the Bethe-Heitler cross section (expansions up to orders p^0, p^2 and p^6) as a function of the incident electron energy.

Fig. 1. For $E_1 > 30\,\mathrm{keV}$ the maximum relative error is more than 10%. It is by no means sufficient to enter just the relativistically correct values of the electron momenta in (1) to improve the results. In problems dealing with mildly relativistic electrons where one does not want to employ the fully relativistic cross section, it is rather expedient to use an expansion in powers of p_i^2. Up to order p_i^6 it is given by

$$\begin{aligned}\frac{d\sigma_B}{dk} \approx \frac{8\alpha Z^2 r_0^2}{3kp_1^2} \Bigg\{ &\Big[2 + p_1^2 + p_2^2 - \tfrac{1}{20}\left(p_1^2 - p_2^2\right)^2 \\ &+ \tfrac{3}{56}\left(p_1^2 - p_2^2\right)^2 \left(p_1^2 + p_2^2\right)\Big] \ln \frac{p_1 + p_2}{p_1 - p_2} \\ &- p_1 p_2 \Big[1 - \tfrac{2}{5}\left(p_1^2 + p_2^2\right) + \tfrac{87}{560}\left(p_1^4 + p_2^4\right) + \tfrac{1}{120} p_1^2 p_2^2 \Big] \Bigg\} .\end{aligned} \tag{2}$$

In applying this formula it is important to employ the relativistically correct values of the momenta, $p_i =$

$\sqrt{(E_i/mc^2)(2+E_i/mc^2)}$, where E_i (i = 1, 2) are the kinetic electron energies.

The maximum relative error of the expansions in p_i^2 up to orders p_i^0 (Eq. 1), p_i^2 and p_i^6 (Eq. 2) is shown in Fig. 1 as a function of the incident electron energy E_1. One can see the significant improvement attained even by the correction of order p_i^2. The full cross section (2) represents an excellent approximation for electron energies below E_1 = 236 keV. At this energy the error is less than 1% throughout the photon spectrum and below 0.5% for photon energies $h\nu > 160$ keV. The high accuracy of formula (2) is surprising since the expansion parameter p_1 is larger than unity for $E_1 > 212$ keV. Of course, the errors grow for further increasing E_1 and formula (2) gets inappropriate due to the high powers of p_i.

In order to obtain an approximate cross section which is more accurate at high energies one can combine Eq. (2) with the extreme-relativistic limit of the Bethe-Heitler formula,

$$\frac{d\sigma_B}{dk} \approx \frac{2\alpha Z^2 r_0^2}{kp_1^2}\left(\tfrac{4}{3}\epsilon_1\epsilon_2 + k^2\right)\left(2\ln\frac{2\epsilon_1\epsilon_2}{k} - 1\right), \tag{3}$$

where ϵ_1 and ϵ_2 are the total energies of the initial and final electrons in units of mc^2. This results in

$$\begin{aligned}\frac{d\sigma}{dk} \approx\ & \frac{2\alpha Z^2 r_0^2}{kp_1^2}\left\{\tfrac{4}{3}\epsilon_1\epsilon_2 + k^2 - \frac{7}{15}\frac{k^2}{\epsilon_1\epsilon_2} - \frac{11}{70}\frac{k^2(p_1^2+p_2^2)}{(\epsilon_1\epsilon_2)^4}\right\} \\ & \times \left\{2\ln\frac{\epsilon_1\epsilon_2 + p_1p_2 - 1}{k} - \frac{p_1p_2}{\epsilon_1\epsilon_2}\right. \\ & \left.\times\left[1 + \frac{1}{\epsilon_1\epsilon_2} + \frac{7}{20}\frac{p_1^2+p_2^2}{(\epsilon_1\epsilon_2)^3} + \left(\tfrac{9}{28}k^2 + \tfrac{263}{210}p_1^2p_2^2\right)\frac{1}{(\epsilon_1\epsilon_2)^3}\right]\right\}.\end{aligned} \tag{4}$$

Expanding the cross section (4) into powers of p_i^2 up to order p_i^6, it takes the form (2). Thus it is of similar accuracy as Eq. (2) up to semirelativistic energies (relative error $< 1\%$ for $E_1 < 300$ keV). But it is also valid at relativistic energies with maximum relative errors of less than 5.5% (around E_1 = 1800 keV).

All the formulae quoted above were derived from the bremsstrahlung cross section in Born approximation which does not take into account the distortion of the electron wave functions by the nuclear Coulomb field. These cross sections can be improved by the multiplicative factor (Elwert, 1939)

$$f_E = \frac{a_2}{a_1}\,\frac{1-\exp(-2\pi a_1)}{1-\exp(-2\pi a_2)}, \tag{5}$$

where $a_1 = \alpha Z\epsilon_1/p_1$ and $a_2 = \alpha Z\epsilon_2/p_2$ are Coulomb parameters. In particular, this factor gives rise to a finite cross section at the short-wavelength limit of the spectrum ($p_2 = 0$). Contrary to the statement of Koch and Motz (1959) it yields accurate results in the whole energy range (Pratt and Tseng 1975) for low-Z target nuclei which are dominant in astrophysical applications.

Considering the bremsstrahlung cross section differential in photon angle and energy, the nonrelativistic approximation is still poorer than in case of the total cross section. Whereas the relativistic corrections to the photon spectrum (1) are of order p_i^2 they are of order p_i for the photon angular distributions. Thus their shapes are strongly distorted even at low energies (see Elwert and Haug 1971). Consequently it is inappropriate to use the nonrelativistic limit of the differential cross section for electron energies above 10 keV. Instead the fully relativistic Sauter cross section, reproduced in Elwert and Haug (1971) has to be applied.[1]

References

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[1] Note that there is a misprint in the cross section (6) of this paper: the second term in curly braces should read $-(\epsilon^2 + 1 + \frac{1}{2}\epsilon' p\cos\theta)/p^2u^2$.