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Period changes of AR Lacertae between 1900 and 1989

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Abstract. The epochs of the primary and secondary minima of AR Lac between 1900 and 1989 are analysed with two non-parametric methods of searching for periodicity in a weighted time point series. An analytical model for transforming the O – C data of eclipsing binaries into the period domain is presented and is applied to AR Lac. The abrupt period changes of AR Lac reported in several previous studies are most probably a consequence of an oversimplified interpretation of the O – C data, since these changes are as likely to be continuous, and even the possibility of a 36^{y} cycle in the period cannot be definitely ruled out.

Key words: stars: AR Lacertae – binaries: eclipsing – methods: analytical; statistical

1. Introduction

The following physical parameters are listed for the eclipsing RS CVn star AR Lac (HR8448, HD210334) in the catalogue of chromospherically active binaries by Strassmeier et al. (1993): G2IV/K0IV, double–line spectrum with strong CaII H&K emission from both components, variable Hα emission and $P_{\rm orb}$ = 1.$^{\rm d}$983222 $\approx P_{\rm phot}$. The variability of AR Lac was first detected by Leavitt in 1907 (Pickering 1907). A continuous light curve is difficult to obtain, because the orbital period is very close to two days, and hence over two decades elapsed, before Jacchia (1929) found AR Lac to be an eclipsing binary. Since then numerous photometric studies have been made (e.g. see our Table 1) and the geometric, photometric and orbital elements of this binary system have been thoroughly examined (e.g. Chambliss 1976, Park 1984, Lee et al. 1986). Hall (1976) published the definition of the RS CVn class of binaries and classified AR Lac as one member of this class. An important detail to remember is his remark that another long period RS CVn star HK Lac (HD 209813, see Blanco & Catalano 1970) was unfortunately used as a photometric comparison of AR Lac by Wood (1946) and Kron (1947), which means that some epochs of the earlier photometric minima may not be very reliable.

Numerous studies of the orbital period variations of AR Lac have been made (e.g. Wood 1946, Cester 1967, Chambliss 1976, Lee et al. 1986). For example, Kim (1991: his Table 2) detected $P_{\rm orb}$ changes between 1.$^{\rm d}$9831674 and 1.$^{\rm d}$9832601 (i.e. about 0.$^{\rm d}$00009). The possible physical phenomena responsible for the period changes of AR Lac have been discussed by Hall & Kreiner (1980), and later by, e.g., Panchatsarem & Abhyankar (1982) and Kim (1991). The proposed models include mechanisms causing apparent (e.g. photometric starspot disturbances, light time effect) or real (e.g. rocket effect, mass transfer or loss) $P_{\rm orb}$ changes, to mention a few. Most of these models were examined with the classical step–technique by Kim (1991), who argued that the correct model to explain the apparently abrupt $P_{\rm obr}$ changes of AR Lac has not yet been found. A more general discussion of the different physical processes that may explain the short– and long–term period changes of eclipsing binaries can be found, e.g., in Hall (1990). A model where the orbital period modulations of eclipsing binaries are connected to magnetic activity, was quite recently proposed by Applegate (1992), and different types of observational data seem to support several predictions of this model (e.g. Hall 1990, Hall 1991, Rodonò et al. 1995). Other recently developed models are mass loss (e.g. Verbunt & Hut 1983, van't Veer & Maceroni 1988, 1989, Maceroni & van't Veer 1991) or mass transfer (Tout & Hall 1991) driven by stellar winds. None of the above models has been tested for AR Lac. Our paper presents a method for determining the period variations of eclipsing binaries from the epochs of the primary and secondary minima, and this is applied to the currently available data for AR Lac.

2. Observations

The epochs of the primary and secondary minima of AR Lac in Table 1 were collected from numerous sources. The earliest

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Table 1. The epochs of the primary and secondary minima of AR Lac [HJD–2400000]: the references are [1]Dugan & Wright (1939), [2]Jacchia (1930), [3]Loreta (1930), [4]Schneller & Plaut (1932), [5]Parenago (1930), [6]Parenago (1938), [7]Rügemer (1931), [8]Zverev (1936), [9]Himpel (1936), [10]Wood (1946), [11]Gainullin (1943), [12]Ahnert (1949), [13]Svechnikov (1955), [14]Wroblewski (1956), [15]Makarov et al. (1957), [16]Karetnikov (1959), [17]Aleksandrovich (1959), [18]Karetnikov (1961), [19]Karle (1962), [20]B.A.N. observers, [21]Oburka (1964), [22]Ahnert (1965), [23]Oburka (1965), [24]Hall (1968), [25]Pohl & Kizilirmak (1966), [26]Ahnert (1966), [27]Kizilirmak & Pohl (1968) [28]Cester (1967), [29]Karle et al. (1977), [30]Pohl & Kizilirmak (1970), [31]Oburka & Silhan (1970), [32]Nha & Kang (1982), [33]Battistini et al. (1973), [34]Pickup (1972), [35]Peter (1972), [36]Kizilirmak & Pohl (1974), [37]Chambliss (1974), [38]Isles (1973), [39]Isles (1975), [40]Pokorny (1974), [41]Scarfe & Barlow (1978), [42]Srivastava (1981), [43]Kurutac et al. (1981), [44]Ertan et al. (1982), [45]Pohl et al. (1982), [46]Park (1984), [47]Caton (1983), [48]Evren et al. (1983), [49]Kim (1991), [50]Pagano (1990), [51]Nezry (1988), [52]Martignoni (1995) and [53]Panov (1987). The notations for different systems (v, f, p, pv and e) are explained in the text (Sect. 2). The epochs of the secondary minima are denoted by •

15300.0590[1v]	29404.8470[10e]	38670.4815[22v,23v]	41274.4170[34v]	42319.5750[39v]	44550.6417[46e]	46349.3690[51v]
17005.6490[1v]	29535.7390[10e]	38672.4435[22v,23v]	41506.4430[35v]	42634.8850[29e]	44809.4452•[44e,45e]	46351.3620[51v]
18889.7290[1v]	29692.4110[11f]	38680.3820[22v]	41513.3920•[36e]	42715.2081•[42e]	44817.3817•[45e]	46397.9601•[49e]
20991.9730[1v]	32889.3360[12v]	38975.8796[24e]	41592.7219•[37e]	42716.1968[42e]	44818.3702[45e]	46726.1722[49e]
22995.0380[1v]	34646.4700[13v]	38996.7067•[24e]	41593.7124[37e]	42717.1861•[42e]	44823.3279•[44e]	46728.1583[49e]
25196.4440[1v]	34888.4290[14v]	39019.5255[25v]	41604.6224•[37e]	43084.0812•[32e]	44898.6836•[47e]	46738.0734[49e]
25555.4280[2f]	34890.4060[14v]	39027.4425[20v,26f]	41635.3860[38v]	43420.2290[32e]	44977.0223[46e]	46742.0394[49e]
25573.2780[3v]	34902.3230[14v]	39029.4260[20v,26f]	41637.3440[38v]	43427.1700•[32e]	45163.4375[48e]	46745.0181•[49e]
25712.0860[4f]	36080.3340[15v]	39259.4913[26f,27v]	41639.3505[38v]	43428.1606[32e]	45164.4375•[48e]	46752.9532•[49e]
25801.3280[5v]	36082.3220[15v]	39376.4926[28e]	41911.0170[39v]	43739.5194[43e]	45165.4189[48e]	47042.5320•[52v]
25803.3200[6v]	36084.3030[15v]	39383.4386•[28e]	41920.9480[39v]	43740.5104•[43e]	45166.4098•[48e]	47045.5290[52v]
26146.4050[7p]	36766.5250[16v]	39386.4040[26v]	41922.9270[39v]	43745.4701[43e]	45296.3065[48e]	47053.3907[50e,53e]
26489.5260[7p]	36774.4740[16v]	39691.8224[29e]	41936.8022[37e]	43747.4535[43e]	45333.9946[49e]	47054.3796•[50e]
26592.6510[7f]	36778.4460[16v]	39695.7941[29e]	41938.7874[37e]	43750.4274•[43e]	45337.9580[49e]	47055.3720[53e]
26604.5480[8v]	36784.4175[16v,17v]	39699.7590[29e]	41962.6030[40v]	43751.4182[43e]	45612.6224•[50e]	47056.3610•[53e]
26622.3950[7v]	37137.4410[18v]	39701.7410[29e]	41968.5640[40v]	43752.4108•[43e]	45674.1062•[49e]	47494.6518•[50e]
26624.3780[7f]	37569.7977[19e]	39876.2680[30e]	41972.5360[40v]	43755.3858[43e]	45680.0533•[49e]	47495.6385[50e]
26626.3610[7v]	37623.3320[20v]	40046.8186[29e]	41974.5170[40v]	44113.3584•[43e]	45683.0263[49e]	47733.6202[50e]
26842.5420[4v]	37958.4740[20v]	40443.4710[31v]	42285.8406[29e]	44114.3380[43e]	45691.9558•[49e]	
26991.2860[9pv]	38315.4520[21v]	40546.5834[32e]	42287.8274[29e,41e]	44449.4985[44e,45e]	46032.0635[49e]	
29178.7610[10e]	38321.3930[21v]	40932.3168•[32e,33e]	42288.8215•[29e,41e]	44451.4809[44e,45e]	46041.9815[49e]	
29186.6930[10e]	38323.3910[20v,21v]	41268.4610[34v]	42289.8120[29e]	44458.4232•[44e,45e]	46344.4039•[50e]	
29188.6750[10e]	38333.3450[20v]	41270.4350[34v]	42292.7872•[29e]	44549.6484•[46e]	46345.4280[51v]	

observations were compiled from Hall & Kreiner (1980), and their values with zero weights have been omitted. Some modifications have been made, the most important being that when several values were available for the same minimum, the values in Table 1 are then mean epochs. The original references for the data denoted by Hall & Kreiner (1980) as "B.B.S.A.G observers" are given, but the first of these values has been omitted (i.e. HJD2440933.330), because it is not listed in the BBSAG Bulletin. Although we did not find the original references for the data referred to as "B.A.N. observers" in Hall & Kreiner (1980), these data have been included in Table 1. The more recent observations in Table 1 were compiled from Kim (1991), who revised the epochs of some previously published minima, as well as discarding some values by Srivastava (1981) and Nha & Kang (1982). However, the data by Ishchenko (1963) were omitted, because Hall & Kreiner (1980) gave zero weights to these data. The new epochs in Table 1 are from Panov (1987), Nezry (1988), Pagano (1990) and Martignoni (1995).

The data in Table 1 have been obtained with different techniques, and we use the notations by Hall & Kreiner (1980): visual estimates ("v"), a series of photographic exposures ("f"), mid-time of exposures on which the object appeared faint ("p"), a series of measurements with a visual polarizing photometer or similar device ("pv") and, finally, photoelectric measurements ("e"). Because the accuracy of the data is not the same in all systems and most of the references do not contain an error estimate, Hall & Kreiner (1980) gave the weight $w=3$ for the "e" data, while observations in all "other" systems were given the weight $w=1$. Therefore, the measurements performed in "v", "f", "p" and "pv" are also referred to as "other" systems in this paper. For reasons to be discussed later (see Table 3), our error estimates are $\sigma_{\rm e}=0.^{\rm d}004$ and $\sigma_{\rm other}=0.^{\rm d}011$.

3. The ephemeris of AR Lac

The standard technique to determine the ephemerides of eclipsing binaries is to make linear or quadratic least squares fits to the O−C data. This procedure has been applied for AR Lac, e.g., by Chambliss (1976) and Hall & Kreiner (1980). Even higher order polynomials have been used in deriving the O−C data of some eclipsing binaries. For example, Wood & Forbes (1963) derived ephemerides based on third order polynomials. In this paper the ephemeris of AR Lac is determined with two nonparametric methods of searching for periodicity in a weighted time point series, presented by Jetsu & Pelt (1996: hereafter Pa-

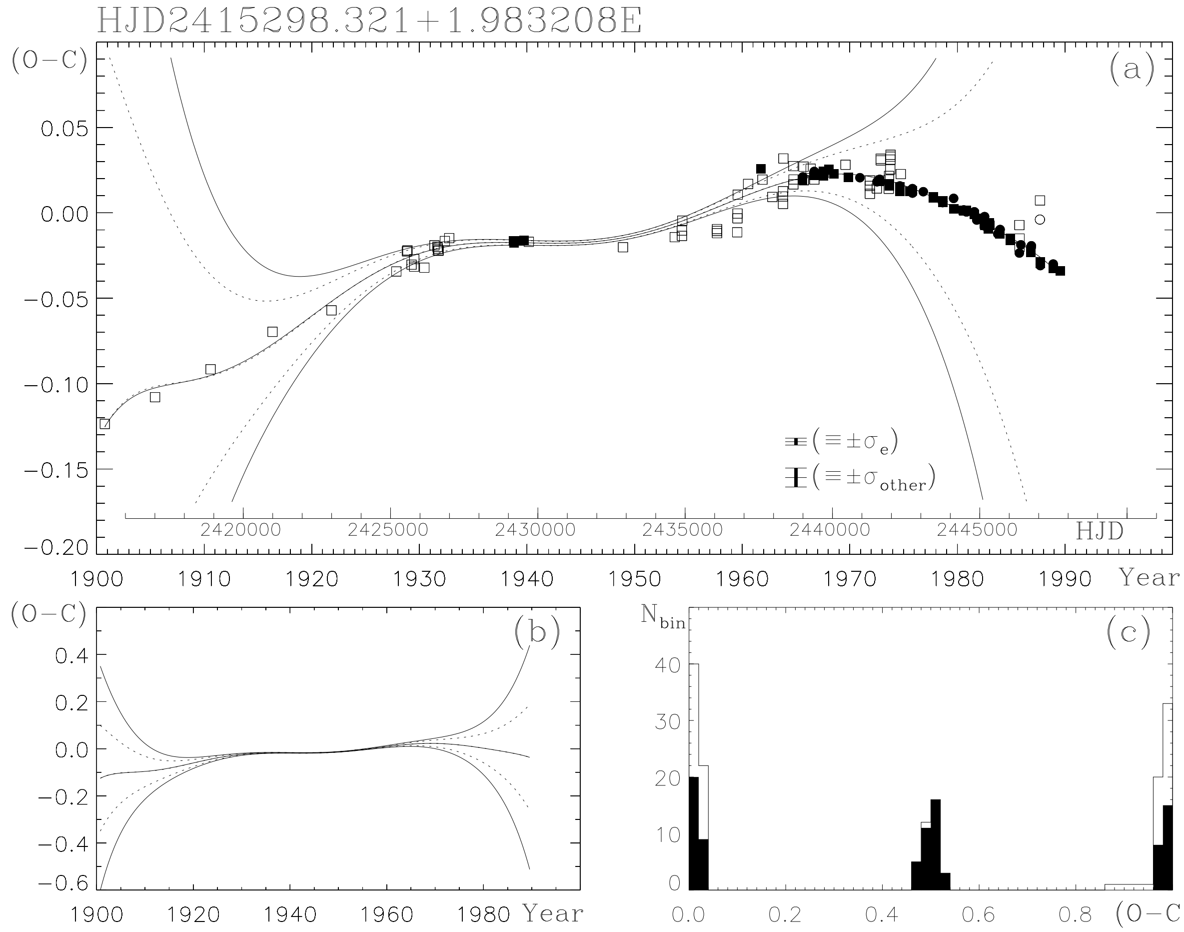

Fig. 1. a The O−C variations of AR Lac with the ephemeris (1): primary minima ("e"≡ closed squares, "other" ≡ open squares), secondary minima ("e"≡ closed circles, "other" ≡ open circles). The (O−C)(T, P_0) curves (Eq. 3: $C_i(P_0)$ from Table 2) for $K=8$ (dotted line) and $K=9$ (continuous line) and their 1σ error limits (Eq. 5). The assigned errors in "e" ($\pm 0.^{\rm d}004 \approx \pm 0.0020$ in phase) and "other" ($\pm 0.^{\rm d}011 \approx \pm 0.0055$ in phase) are shown separately in the lower right corner. **b** The total range of 1σ error limits of (O−C)(T, P_0). **c** The O−C distribution in bins of 0.02 for the "e" (dark) and "other" data (white)

per I), which were already applied in Jetsu (1996) and Jetsu et al. (1997). Our Appendix summarizes the necessary information of the WK– and WSD–methods, as well as the determination of the following two ephemerides

$$\mathrm{HJD}2415298.321(\pm.003)+1.983208(\pm.000006)\mathrm{E}_{\mathrm{WK}} \quad (1)$$

$$\mathrm{HJD}2415298.646(\pm.005)+1.983184(\pm.000007)\mathrm{E}_{\mathrm{WSD}}. \quad (2)$$

The O−C variations for (1) and (2) are shown in Figs. 1a and 2a, where 0.5 has been subtracted from the O−C values of the secondary minima to shift them to the level of the primary minima. Note that the unit of the O−C data is phases (i.e. not days), because these phases represent *circular data* (see Appendix) and thus the modelling presented below is formulated for phases.

4. The model

The determination of an ephemeris for an eclipsing binary does not represent an analysis of $P_{\rm orb}$ changes, but only yields the period P_0 that minimizes the O−C residuals with respect to a linear or higher order least squares fit for the whole data (e.g. Hall & Kreiner 1980 for AR Lac). The classical technique to study $P_{\rm orb}$ changes (i.e. the "step–technique"), is to determine linear least squares fits for parts of the O−C data, and the previous studies of AR Lac contain examples of this approach (e.g. Chambliss 1976, Kim 1991). Another technique is to obtain higher order polynomial least squares fits for the whole or parts of the O−C data (Kalimeris et al. 1994ab, 1995), and this type of a technique is hereafter referred to as "continuous–technique".

One alternative continuous–technique for modelling of the period changes from the epochs of the primary and secondary minima of eclipsing binaries is presented in this section, and the results for AR Lac are discussed. The possible physical interpretations of these results are presented separately in Sect. 5.2. Although a thorough comparison between our method and that by Kalimeris et al. (1994ab, 1995) is not presented, we wish to emphasize the following two main differences. Firstly, the O−C data are analysed in phase domain, while Kalimeris et al. formulated their method in the time domain. Secondly, the

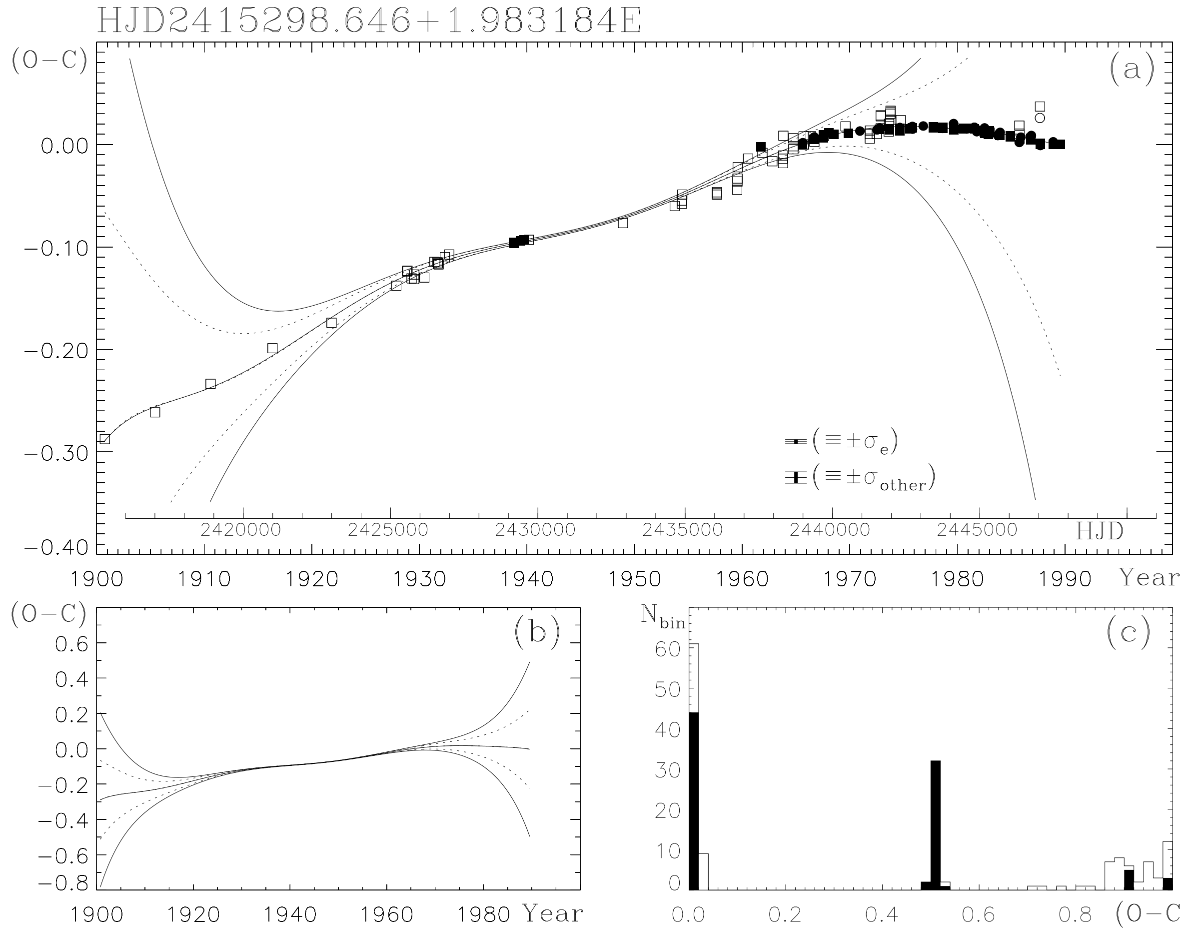

Fig. 2. The same as Fig. 1, using the ephemeris (2)

period changes are solved directly from the free parameters of the model (i.e. C_i of Eq. 3). As for the main similarities, the solution for P_{orb} changes is independent of the value of P_0 applied in deriving the O−C data for both methods (see Kalimeris et al. 1994b: Appendix I, and Eq. 7 in our paper). Furthermore, when the whole O−C data could not be adequately modelled with one polynomial fit, Kalimeris et al. (1994ab) correctly emphasized the uncertainties connected with the modelling of the data in parts. In such cases, they ascertained the continuity of the O−C modelling by spline interpolation. Except for SZ Piscium (Kalimeris et al. 1995), the orbital period changes of no other RS CVn system have been studied earlier with a continuous–technique.

4.1. *The* O−C *variations*

Before modelling of the O−C data of AR Lac with higher order polynomials (i.e. Eq. 3 below), we briefly discuss how the step–technique could be applied to these data. For example, visual inspection might suggest that the variations in Fig. 1a could be adequately presented by five or six linear fits, which would more or less resemble the step–technique applied by Kim (1991: his Fig. 1). On the other hand, perhaps four or five linear fits might suffice for rough modelling of Fig. 2a. The interpretations would then be that the period has been decreasing from the beginning of 1960 (Fig. 1a), or increasing from the beginning of the century to 1980 (Fig. 2a). One might even claim that Fig. 1a may indicate periodicity in the O−C variations. But the main problem with these interpretations is that abrupt and discontinuous period changes seem to occur at the epochs where the lines fitted to the O−C data intersect. Furthermore, there is plenty of freedom in choosing the intervals for fitting. Unfortunately, the results of the step–technique also depend on the chosen P_0 to derive the O−C values.

In reality, the function for modelling the O−C data is unknown. Furthermore, the shape of this function depends on P_0. The models for the O−C data shown in Figs. 1a and b and 2a and b are

$$(\mathrm{O-C})(T, P_0) = \sum_{\mathrm{i}=0}^{\mathrm{K}} C_{\mathrm{i}}(P_0) T^{\,\mathrm{i}}, \tag{3}$$

where $K = 8$ or 9. The standard least squares fit method was applied to solve the best values for the free parameters $C_0(P_0)$, ..., $C_{\mathrm{K}}(P_0)$. The time scale is $T = (t - t'')/T_{\mathrm{scale}}$, where $t'' =$ HJD2431516.8396 is the mid point of the time interval of the data and time is transformed into centuries by $T_{\rm scale}$ = 36525 (i.e. $-1 < T < 1$), which simplifies the computation of the free parameters for higher order models. The values $C_0(P_0)$, ..., $C_{\rm K}(P_0)$ for the curves in Fig. 1a and b are given in Table 2. A simple connection exists between the models (Eq. 3) for the data in Figs. 1a and b, and 2a and b, because the transformation between them is

$$(\mathrm{O-C})(T,P_1)-(\mathrm{O-C})(T,P_2) = \frac{P_2-P_1}{P_1P_2}T+\frac{P_1T_2-P_2T_1}{P_1P_2}, \quad (4)$$

where T_1 and P_1, and T_2 and P_2 are the zero points and periods of the ephemerides (1) and (2), respectively. The model gives different values only for $C_0(P_0)$ and $C_1(P_0)$, while values of $C_2(P_0), ..., C_{\rm K}(P_0)$ are the same for $P_0 = P_1$ and P_2. The coefficients on the right side of Eq. 4 determine the differences in $C_0(P_0)$ and $C_1(P_0)$. Although the O$-$C variations in Figs. 1a and 2a do not appear similar, Eq. 4 shows that the model does not strongly depend on the choice of P_0.

Why should the model of Eq. 3 be better than some more arbitrary model? What is the reason for choosing K = 8 or 9 in Figs. 1a and b, and 2a and b? The answer to the first question is that whatever the form of the unknown function suitable for modelling the O$-$C data, it can be expanded in a Taylor series, unless it or its derivatives are discontinuous. The coefficients of Eq. 3 can be interpreted as coefficients of the Taylor expansion of this unknown function. In conclusion, this is all the information that can be extracted from the O$-$C data with an unknown model. The O$-$C variations of AR Lac may indeed be discontinuous, but the observations in "e" system certainly do not suggest this (Figs. 1a and 2a: closed symbols). It rather seems that the accuracy of the data in "other" systems is relatively low. While the continuity of the O$-$C curve of AR Lac can not be proved beyond doubt, we will continue by assuming that this is the case. Some physical considerations are also mentioned in connection with Eqs. 6 and 7. The 1σ error limits for (O$-$C)(T, P_0) in this model (Eq. 3) are

$$\sigma_{\rm O\text{-}C}(T,P_0) = \sqrt{\sum_{\rm i=0}^{\rm K}(T^{\rm i}\sigma_{\rm C_i})^2}, \quad (5)$$

where $\sigma_{\rm C_i}$ are the 1σ errors of the free parameters $C_{\rm i}(P_0)$. Because only a Taylor expansion of the unknown model is available, these error estimates diverge strongly at both ends of the time interval of the data, as shown in Figs. 1b and 2b.

The question of choosing the order (K) in Eq. 3 can be settled by setting all weights to unity and modelling Eq. 3 to different orders. The mean residuals given in Table 3 stop decreasing at $K = 9$ for both the "e" and the "other" data. These mean residuals with weights of unity should satisfy an approximation $\langle\sigma_{\rm e}\rangle \approx \langle\epsilon_{\rm e}\rangle = 0.^{\rm d}004$ and $\langle\sigma_{\rm other}\rangle \approx \langle\epsilon_{\rm other}\rangle = 0.^{\rm d}011$ (see, e.g., Press et al. 1986) and thus they were chosen as error estimates for the data in Table 1. Inspection of Table 3 also reveals that it is unnecessary to model the data in higher orders (i.e. $K > 9$), because the mean residuals will not decrease. The weighted modelling for $K = 9$ with the above error estimates

Table 2. The coefficients $C_{\rm i}(P_0)$ of Eq. 3 in Figs. 1a and b (i.e. $P_0 = P_1$)

K = 9	$C_0 = -0.0174 \pm 0.0013$	$C_4 = -23.4 \pm 3.0$	$C_8 = -362 \pm 74$
	$C_1 = 0.036 \pm 0.012$	$C_5 = -41 \pm 12$	$C_9 = -77 \pm 220$
	$C_2 = 0.91 \pm 0.11$	$C_6 = 158 \pm 27$	
	$C_3 = 4.15 \pm 0.68$	$C_7 = 126 \pm 88$	
K = 8	$C_0 = -0.0175 \pm 0.0012$	$C_4 = -24.3 \pm 2.3$	$C_8 = -383 \pm 55$
	$C_1 = 0.038 \pm 0.010$	$C_5 = -37.7 \pm 5.2$	
	$C_2 = 0.94 \pm 0.10$	$C_6 = 166 \pm 20$	
	$C_3 = 4.01 \pm 0.45$	$C_7 = 96 \pm 17$	

Table 3. The mean residuals $\langle\epsilon_{\rm e}\rangle$ and $\langle\epsilon_{\rm other}\rangle$ [d] for the nonweighted fits to the data of Fig. 1a in orders K of Eq. 3

K	$\langle\epsilon_{\rm e}\rangle$	$\langle\epsilon_{\rm other}\rangle$	K	$\langle\epsilon_{\rm e}\rangle$	$\langle\epsilon_{\rm other}\rangle$	K	$\langle\epsilon_{\rm e}\rangle$	$\langle\epsilon_{\rm other}\rangle$
4	0.006	0.018	8	0.005	0.012	12	0.004	0.011
5	0.007	0.017	9	0.004	0.011	13	0.004	0.011
6	0.005	0.014	10	0.004	0.011	14	0.004	0.011
7	0.006	0.012	11	0.004	0.011	15	0.004	0.011

yields $\chi^2 = 287$ for the data of Fig. 1a. The probability for this or a smaller value of χ^2 is unity, i.e. the fit is not significant. However, this result is due to outliers. For example, the three residuals for the data by Karle (1962) and Martignoni (1995) contribute 31% to this χ^2 of 156 residuals, while the mean "e" residual is already $\langle\epsilon_{\rm e}\rangle = 0.^{\rm d}0031$. But the elimination of outliers is a questionable procedure, especially for an unknown model. Therefore we conclude that modelling the data in Fig. 1a with $K = 9$ is sufficient, although the fit is not significant due to outliers in the "other" systems. However, we do note that another approach to decide the limiting order K in models similar to our Eq. 3 has been proposed by Kalimeris et al. (1995: their Fig. 3), which is also based on the residuals of the modelling. Alternative approaches might be to formulate criteria based on the "expected noise" level (e.g. correlations between residuals or χ^2) but this is quite difficult, if the residuals are unevenly spaced in time and/or the errors of the data are unknown.

Although trivial, we note that the χ^2 values for the data in Figs. 1 and 2 are equal for $K = 8$ and 9 (see Eq. 4), and the residuals are also the same (i.e. Table 3). The results for the model with $K = 8$ are also outlined in all our figures, because they show how strongly the error estimates depend on the order of the model (Eq. 3), although the solutions for (O$-$C)(T, P_0), $P(T, P_0)$ and $dP(T, P_0)/dT$ remain practically unchanged, since the curves for $K = 8$ and 9 in Figs. 1a, 2a, 3a and 4a nearly overlap each other.

The nonparametric bootstrap for regression coefficients formulated by Efron & Tibshirani (1986) was chosen to estimate the errors $\sigma_{\rm C_i}$ of the free parameters $C_{\rm i}(P_0)$, because outliers are present and the error estimates for the data are based on the residuals of the model. An example of this method can be found in Jetsu (1993) and the same notation is adopted here. If the model g of Eq. 3 is denoted as $g = (T, P_0, \bar{a})$, the vector of the free parameters is $\bar{a} = [C_0(P_0), C_1(P_0), ..., C_K(P_0)]$. The

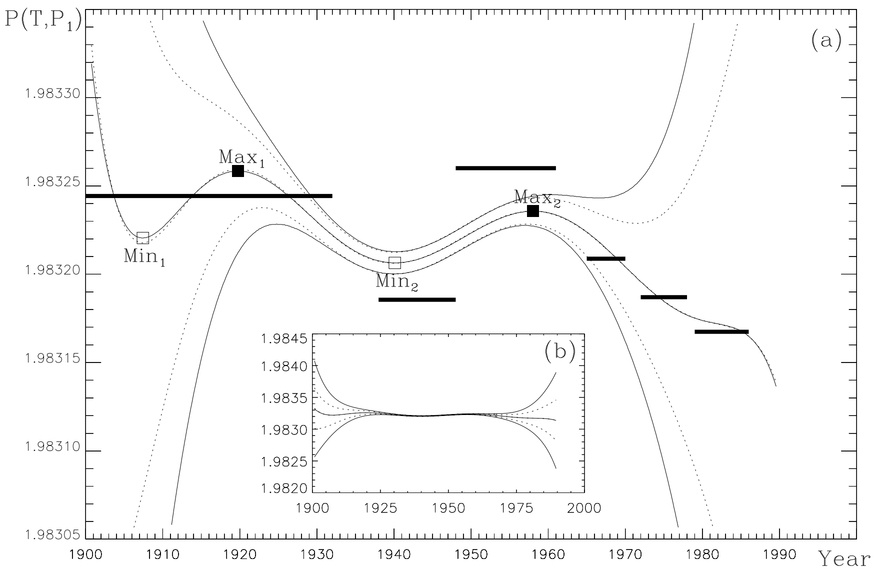

Fig. 3. a The period variations $P(T, P_1)$ and their 1σ error limits (Eqs. 7 and 8) with $K = 8$ (dotted line) and $K = 9$ (continuous line). Note that the two highest and the two lowest curves represent $P(T, P_1) + 1\sigma_{\mathrm{P(T,P_1)}}$ and $P(T, P_1) - 1\sigma_{\mathrm{P(T,P_1)}}$, respectively. The epochs of the two maxima and minima of $P(T, P_1)$ for $K = 9$ are marked with closed and open squares (Max_1, Max_3, Min_1 and Min_2). Note also that the solutions for $P(T, P_1)$ with $K = 8$ and 9 nearly overlap each other. The results by Kim (1991: his Table 2) are outlined with thick horizontal lines. **b** The total range of 1σ error limits of $P(T, P_1)$

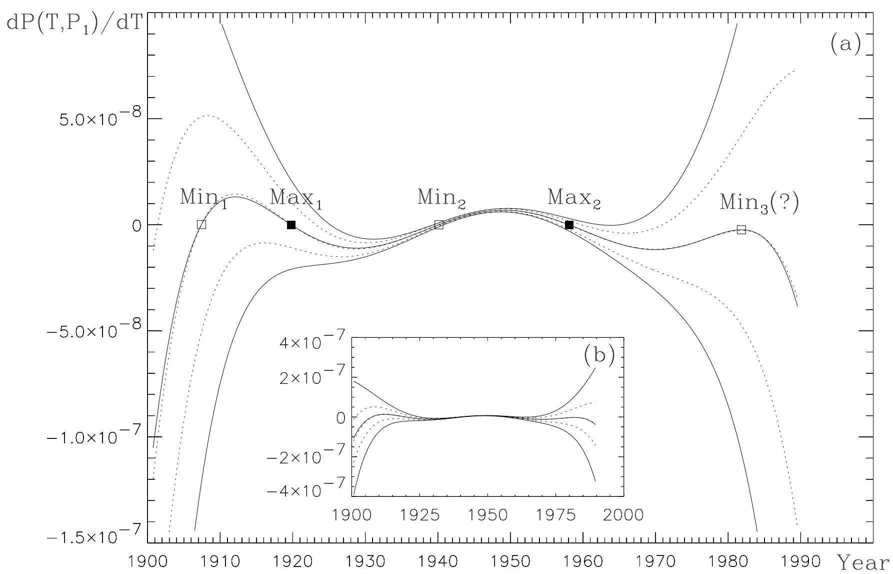

Fig. 4. a The $\dot{P} = dP(T, P_1)/dT$ changes and their 1σ error limits (Eqs. 10 and 11), where the notations for $K = 8$ and 9 are as in Fig. 3. The maxima and minima of $P(T, P_1)$ coincide with $\dot{P} = 0$, while $\dot{P}$ nearly reaches zero at Min_3(?). **b** The total range of $\dot{P} \pm \sigma_{\dot{\mathrm{P}}\mathrm{(T,P_1)}}$

random samples $y^*(T_i)$ for a fixed P_0 are derived as explained in Jetsu (1993: Eq. 5). The only difference is that the correspondence that w_i is the weight of the residual ϵ_i is conserved when selecting the random samples $\bar{\epsilon}^*$ and the modelling is performed for the weighted samples $\bar{y}^*$. The estimates of $\sigma_{\mathrm{C_i}}$ obtained with this bootstrap procedure for the models in Fig. 1a and b are given in Table 2.

4.2. The $P(T, P_0)$ variations

Because the variations of $(\mathrm{O-C})(T, P_0)$ for $P_0 = P_1$ and P_2 are not very large over 90 years, the ratio $P(T, P_0)/P_0$ remains close to unity, and the relation

$$\Delta(\mathrm{O-C})(T, P_0) = \frac{\Delta T}{P_0} - \frac{\Delta T}{P(T, P_0)}, \tag{6}$$

is satisfied during any short time interval ΔT. This relation is valid even in the more general case, i.e. when $P(T, P_0)/P_0$ is not close to unity. But in this case the choice of P_0 should be unique to ensure the continuity of the O−C data (see Appendix). In conclusion, a sufficient general condition for the validity of Eq. 6 is that the O−C variations and their higher order derivatives are continuous. Apart from mathematical considerations, most of the earlier models (e.g. coronal mass ejections, rocket effect, sudden mass transfer) predicting abrupt P_{orb} changes have been found inconsistent (e.g. Hall 1990). If Eq. 6 is valid, then the period variation for the model (Eq. 3) is

$$P(T, P_0) = \left[\frac{1}{P_0} - \sum_{\mathrm{i=1}}^{\mathrm{K}} i C_{\mathrm{i}}(P_0) T^{\,\mathrm{i}-1} \right]^{-1}, \tag{7}$$

while the error estimate of $P(T, P_0)$ is

$$\sigma_{\mathrm{P(T,P_0)}} = P^2(T, P_0)\sqrt{s_1 + s_2} \tag{8}$$

where $s_1 = \sum_{i=1}^{\mathrm{K}} \left(i T^{\,\mathrm{i}-1} \sigma_{\mathrm{C_i}} \right)^2$ and $s_2 = \left(\sigma_{\mathrm{P_0}} P_0^{-2} \right)^2$.

The transformation of Eq. 4 shows that the difference between the coefficients $C_1(P_1)$ and $C_1(P_2)$ for the model of Eq. 3 is $P_1^{-1} - P_2^{-1}$. Because all other coefficients $C_2(P_0)$, ..., $C_K(P_0)$ are equal for both periods, it is quite easy to show that

$$P(T, P_1) = P(T, P_2). \tag{9}$$

In conclusion, the results for the period variations are the same for the O − C data of Figs. 1 and 2.

The $P(T, P_1)$ changes of AR Lac display two maxima at 1919.8 and 1958.1 (Fig. 3: Max_1 and Max_2), the respective values of $P(T, P_1)$ being $1.^{d}983258 \pm 0.^{d}000046$ and $1.^{d}9832357 \pm 0.^{d}0000085$. There are also two minima at 1907.4 and 1940.1, where $P(T, P_1) = 1.^{d}98322 \pm 0.^{d}00030$ and $1.^{d}9832065 \pm 0.^{d}0000063$ (Fig. 3: Min_1 and Min_2). These epochs yield two estimates for the possible cycle in period changes: $38.^{y}3$ ($\mathrm{Max}_2 - \mathrm{Max}_1$) and $32.^{y}7$ ($\mathrm{Min}_2 - \mathrm{Min}_1$), i.e. a $P_{\mathrm{cycle}} = 35.^{y}5$ mean. This value agrees well with 35^{y} proposed by Hall & Kreiner (1980). The above epochs also yield three estimates for the amplitude of the period changes [d]: $4 \pm 30 \cdot 10^{-5}$, $5.2 \pm 4.6 \cdot 10^{-5}$, and $2.9 \pm 1.0 \cdot 10^{-5}$, the last one being the most accurate. Finally, the full scale of $P(T, P_1)$ changes is $1.8 \cdot 10^{-4}$ [d] during 90 years.

4.3. The $dP(T, P_0)/dT$ variations

The first derivative of the period changes is useful in testing some physical models (see Sect. 5.2.), and has a form

$$\dot{P}(T, P_0) = \frac{dP(T, P_0)}{dT} = \frac{\sum_{i=2}^{K} i(i-1)C_i(P_0)T^{i-2}}{\left[P_0^{-1} - \sum_{i=1}^{K} iC_i(P_0)T^{i-1}\right]^2} = P^2(T, P_0)\sum_{i=2}^{K} i(i-1)C_i(P_0)T^{i-2}. \tag{10}$$

The error estimate for $\dot{P}(T, P_0)$ is

$$\sigma_{\dot{P}(T,P_0)} = \sqrt{s_3 + s_4}, \tag{11}$$

where

$$s_3 = P^4(T, P_0)\sum_{i=2}^{K}(i(i-1)T^{i-2}\sigma_{C_i})^2 \text{and } s_4 = \left[\frac{2\dot{P}(T, P_0)\sigma_{P(T,P_0)}}{P(T, P_0)}\right]^2.$$

The $\dot{P}(T, P_1)$ [d^0] changes for AR Lac are shown in Fig. 4. It is unnecessary to display the function $\dot{P}/P$ separately, since the maximum difference between $\dot{P}/P$ and $\dot{P}/P_1$ is only $3.0 \cdot 10^{-12}$ [d^{-1}]. In other words, a very accurate estimate for $\dot{P}/P$ is obtained by dividing $\dot{P}$ with the constant $P_1 = 1.^{d}983208$, i.e. the functions $\dot{P}$ and $\dot{P}/P$ have practically the same shape. The epochs of the maxima and minima of $P(T, P_1)$ are also displayed in Fig. 4, since these correspond to the epochs where $\dot{P} = 0$ and changes its sign. Note that $\dot{P}$ nearly reaches zero at 1982, but does not change sign (Fig. 4: Min_3(?)). Considering the errors $\sigma_{\dot{P}(T,P_1)}$ at Min_3(?), this epoch might represent a third minimum of $P(T, P_1)$, which could be verified with new data.

The smallest and largest values of $\dot{P}$ [d^0] are $-1.05 \cdot 10^{-7}$ and $1.32 \cdot 10^{-8}$, which correspond to $-5.30 \cdot 10^{-8}$ and $6.63 \cdot 10^{-9}$ for $\dot{P}/P$ [d^{-1}]. The orbital period of AR Lac may have been gradually decreasing during this century, because $\dot{P} < 0$ during most of the observed time interval. The difficulties in establishing such a trend are discussed in Sect. 4.5.

4.4. Some special cases of the O − C variations

Our O − C data are phases. The O − C data were days in Kalimeris et al. (1994a), where two special cases were discussed: linear and quadratic O − C changes in the time. Analogous cases for our model (Eq. 3) are

(I) $(O - C)(T, P_0) = C_0 + C_1 T$
(II) $(O - C)(T, P_0) = C_0 + C_1 T + C_2 T^2$.

It is convenient to study these cases in the frequency domain

$$f(T, P_0) = P^{-1}(T, P_0) = P_0^{-1} - \sum_{i=1}^{K} iC_i T^{i-1}.$$

Case (I) reduces to

$$f(T, P_0) = P_0^{-1} - C_1,$$

i.e. $f(T, P_0) = P^{-1}(T, P_0)$ is constant, as in Kalimeris et al. (1994a) for linear O − C changes in time.
Case (II) yields

$$\dot{f}(T, P_0) = \frac{df(T, P_0)}{dT} = -2C_2,$$

i.e. the frequency change $\dot{f}(T, P_0) = -\dot{P}(T, P_0)P^{-2}(T, P_0)$ is constant. It is important to realize that the rate of the period change is not constant when the O − C changes are quadratic in *phase*.

4.5. Linear trend in $P(T, P_0)$

Finally, we will discuss the question of detecting linear trends in $P(T, P_0)$, since Fig. 3 implies a gradual decrease of $P(T, P_1)$ during this century. This is also indicated by Fig. 4, since $\dot{P}$ is mainly negative. Should a trend exist, several physical mechanisms could be responsible for it (see Sect. 5.2.). One estimate for the linear trend can be obtained by expanding $P(T, P_0)$ into a Taylor series at $T = 0$, which yields the linear part

$$P_L(T, P_0) = P(T = 0, P_0) + \dot{P}(T = 0, P_0)T = a(P_0) + b(P_0)T. \tag{12}$$

The linear coefficients given by Eqs. 7 and 10 are

$$a(P_0) = [P_0^{-1} - C_1(P_0)]^{-1} \tag{13}$$

and

$$b(P_0) = 2C_2(P_0)a^2(P_0). \tag{14}$$

This linear trend is independent of P_0, which can be either inferred directly from Eq. 9 or verified from the relation $C_1(P_1)=C_1(P_2)+P_1^{-1}-P_2^{-1}$. The errors

$$\sigma_{\mathrm{a(P_0)}} = a^2(P_0)\sqrt{(P_0^{-2}\sigma_{\mathrm{P_0}})^2 + \sigma^2_{\mathrm{C_1}}} \quad (15)$$

and

$$\sigma_{\mathrm{b(P_0)}} = 2a(P_0)\sqrt{(a(P_0)\sigma_{\mathrm{C_2}})^2 + (2C_2(P_0)\sigma_{\mathrm{a(P_0)}})^2}. \quad (16)$$

for $a(P_0)$ and $b(P_0)$ are useful, since especially $\sigma_{\mathrm{b(P_0)}}$ reveals directly, if this linear trend is statistically significant.

The C_1 and C_2 from Table 2 (K=9) yield $a(P_1) = 1.9832119 \pm 0.0000061$ [d] and $b(P_1)=15\pm 2\cdot 10^{-14}$ [d^0]. Surprisingly, this statistically significant slope $b(P_1)$ of $P_{\mathrm{L}}(T,P_1)$ is positive! Furthermore, a linear least squares fit to $P(T,P_1)$ and $\sigma_{\mathrm{P(T,P_1)}}$ ($K=9$) yields a slope of 3.0x10^{-14} [d^0], which is also positive, but not equal to $b(P_1)$. How should one interpret these results when $\dot{P}$ is mostly negative, i.e. if any linear trend was present, it ought be negative! The solution for this apparent paradox is simple. The linear trend $P_{\mathrm{L}}(T,P_1)$ represents the tangent of the $P(T,P_1)$ curve at $T=0$. For example, $P_{\mathrm{L}}(T,P_1)$ would be zero, if the Taylor expansion of $P(T,P_1)$ would be determined at $T=\mathrm{Max}_1, \mathrm{Max}_2, \mathrm{Min}_1$ or Min_2. The errors $\sigma_{\mathrm{P(T,P_0)}}$ are smallest at the chosen t'' ($\equiv T=0$). Thus any linear least squares fit to the $P(T,P_1)$ and $\sigma_{\mathrm{P(T,P_0)}}$ will always more or less imitate $P_{\mathrm{L}}(T=0,P_1)$. Furthermore, when performing a linear least squares fit to $P(T,P_1)$ and $\sigma_{\mathrm{P(T,P_1)}}$, one inevitably disregards the fact that such a model is not appropriate in describing this curve. The $\sigma_{\mathrm{P(T,P_1)}}$ distribution in Fig. 3 is not gaussian and the correct statistical term to characterize these errors is *heteroscedasticity*, since their scale is nonuniform. Thus a linear least squares fit with these errors is a questionable procedure. One alternative approach would be to determine the weighted average of $\dot{P}(T,P_1)$ and claim that it can reveal the linear trend, but again the result would depend on the chosen t''. The conclusion is that the determination of a unique linear trend in the $P(T,P_1)$ is impossible, since the result depends on t'', which determines where the errors $\sigma_{\mathrm{P(T,P_1)}}$ are smallest.

5. Discussion and conclusions

5.1. The direct results of the modelling

Our paper has concentrated mainly on developing a method for extracting information about the period changes of AR Lac, as well as of other eclipsing binaries, based on analysing the epochs of the primary and secondary minima. However, we do note that it is possible to explain the period variations of AR Lac without abrupt changes. In fact, our Fig. 3a does not even rule out the possibility of a cyclic variation in $P(T,P_0)$. The period changes of AR Lac may well be continuous and Fig. 3a might indicate the presence of *quasiperiodic* trends with a period of about 36 years. Unfortunately, the error estimates for the period variations are as yet too large to verify this possibility, but a few decades of new photoelectric determinations of the epochs of the primary and secondary minima will certainly determine whether a cycle in $P(T,P_0)$ exists. In general, it might be more fruitful to transform the O$-$C data to the period domain and then consider the alternative physical processes, because direct interpretations of the O$-$C data with any P_0 have their disadvantages, as explained below.

The time series analysis of the epochs of the primary and secondary minima of AR Lac with the nonparametric WK– and WSD–methods gave two different periods for studying the O$-$C variations. This contradiction is apparent for three reasons. Firstly, although both methods are nonparametric (i.e. model independent), they are not equally sensitive to different types of distributions, as discussed in greater detail in Jetsu et al. (1997). Secondly, the period of AR Lac is variable, and the available data cannot be used to decide whether these variations are stationary. If they are nonstationary, i.e. both the mean and the standard deviation of the period are not constant over a longer time interval, then a unique period for determining the O$-$C values does not exist or, equivalently, any time series analysis method will fail. Thirdly, although the different periods (P_0) detected with the WK– and WSD–methods yield different O$-$C values, the temporal variations of the period of AR Lac with the model of Eq. 3 are the same for both values of P_0.

All O$-$C data of AR Lac was modelled (Eq. 3) with K+1 =9 or 10 free parameters in Figs. 1 and 2. However, the error estimates of (O$-$C)(T,P_0), $P(T,P_0)$ and $dP(T,P_0)/dT$ diverge strongly at both ends of the time interval of data. But this is unavoidable, if only a Taylor series of an unknown model is available, and the aim is to analyse the whole time series simultaneously. Naturally, the centre (t'') of the Taylor expansion can be shifted, for example, closer to the recent data, but the cost will be that the error estimates diverge even more strongly for the earlier data. In any case, the results of the modelling of Eq. 3 are independent of the chosen t''. Another approach would be to determine these expansions for parts of the data, but this may yield discontinuous period curves, which have been amply discussed in the earlier literature of AR Lac, as well as that of other eclipsing binaries. Although, the selection of these parts of the data for modelling may sometimes be subjective, the spline interpolation procedure discussed by Kalimeris et al. (1994ab) could be applied to ascertain the continuity between these models for parts of the data.

For example, Kim (1991) applied the step–technique to study the P_{orb} changes of AR Lac with six linear fits for separate parts of the data and a similar approach can also be found in Chambliss (1976: Table 2). The six linear fits in Kim (1991) required 12 free parameters, while the epochs selected to subdivide the data represented 5 additional free parameters, giving total of 17 free parameters. Moreover, the subdivision was made by eye and would most probably have been different, for example, if the value of P_0 in our Fig. 2 had been used to derive the O$-$C values. That model results in discontinuous period variations, interpreted as abrupt period changes (see Fig. 3: thick horizontal lines). The results by Kim (1991) are in quantitative agreement with our $P(T,P_0)$ curve, except between 1938 and 1961. This is mainly due to omitting the data by Ishchenko

(1963) from our Table 1, while these data were analysed by Kim (1991). As already mentioned earlier, Hall & Kreiner (1980) gave zero weights to these data. Had we included the data by Ishchenko (1963), then the other data with zero weights in Hall & Kreiner (1980) should have also been included, but this would have only introduced more outliers to the modelling. Note that this problem of outliers was discussed in Sect. 4.1., in connection with the data of Karle (1962) and Martignoni (1995).

There are at least two approaches that could reduce the uncertainties in the determination of the period variation. If the model remains unknown, *new data* will eventually reveal more details of the period variation of AR Lac, and there are certainly eclipsing binaries where the model of Eq. 3 will not require so many orders (K) and/or the quality of the data is better. Hall & Kreiner (1980) note that AR Lac has one of the most "baffling" O$-$C curves among eclipsing binaries, which in terms of our model means a high value of K in Eq. 3. For example, modelling the O$-$C variations of RT And or SV Cam would most probably succeed with $K<9$ in Eq. 3 and give more accurate estimates of the period changes (see Hall & Kreiner 1980: Figs. 1 and 5). Another possibility is that a *known* model with less free parameters than in ours is developed, that is, some parameters can be fixed on physical or other grounds. The partial derivatives of this hypothetical model with respect to the free parameters should not have a similar time dependence as in our model (Eq. 3), e.g. trigonometric functions might be utilized. In any case, the method outlined in this paper ought to motivate observers to obtain new epochs of the minima of eclipsing binaries.

The most important conclusions of this study are:

1. The period variations $P(T,P_0)$ of AR Lac can be modelled (Eq. 3) with a continuous curve, and the results are independent of the period P_0 used to derive the O$-$C values.
2. A decrease or increase of O$-$C values should not be interpreted as a decrease or increase of the period, respectively. These changes of O$-$C depend on the chosen P_0, while the real period changes depend on the derivative of the O$-$C curve, and are independent of P_0. For example, the O$-$C values will always decrease when $P(T,P_0)<P_0$, regardless of whether $P(T,P_0)$ itself is decreasing or increasing.
3. Different types of regularities, as for example periodic variations of the O$-$C values, are meaningless, because the period is not constant. These apparent regularities depend solely on the chosen P_0, which can not be uniquely determined. Moreover, the period changes may be nonstationary, i.e. the long–term mean and variance may not be constant. Even periodic changes of the orbital period do not necessarily show as periodic changes in the O$-$C data. For example, let us assume that T_1 is the time interval for the period to increase from the minimum ($P_{\rm min}$) to the maximum ($P_{\rm max}$), and the corresponding time interval for the decrease back to $P_{\rm min}$ is T_2. If the period changes are not symmetric, e.g. $T_1 \neq T_2$, then the O$-$C data do not necessarily show this cycle (i.e. T_1+T_2), although the O$-$C values were derived with the mean period $(P_{\rm max}+P_{\rm min})/2$.
4. Linear fits to parts of the data (i.e. the step–technique) are oversimplifications which give misleading results for the period variation. Moreover, because the whole data set is not analysed simultaneously, discontinuous period changes will result.

5.2. The physical interpretation

Hall & Kreiner (1980) reviewed several physical phenomena that might explain the period variations of AR Lac: third body, magnetically driven anisotropic mass ejection, effects of starspots, apsidal motion, etc. More recently, Kim (1991) suggested a new alternative, a beat phenomenon, where several periodicities due to different physical mechanisms interact to produce the observed aperiodic and seemingly irregular O$-$C variations. He also summarized the reasons for rejecting most of the previously proposed models for explaining the period changes. However, since the analysis by Kim (1991) was based on the step–technique, it concentrated on explaining the apparently abrupt $P_{\rm orb}$ changes.

The study of SZ Psc (HD 219113) by Kalimeris et al. (1995) is the only earlier case, where a RS CVn system has been studied with a continuous–technique. They concluded that the Applegate (1992) mechanism might be mainly responsible for the short–term $P_{\rm orb}$ changes of SZ Psc, while the stellar wind driven mass transfer (Tout & Hall 1991) could induce a weaker long–term decrease of $P_{\rm orb}$. Our interpretation for the results obtained in Sect. 4 is limited those physical mechanisms that might explain the apparently periodic and continuous $P_{\rm orb}$ changes of AR Lac. These include apsidal motion, third body and the Applegate mechanism connected to magnetic activity. The mechanisms related to stellar wind driven mass transfer or loss are also mentioned, since these might cause a trend in $P(T,P_0)$.

Apsidal motion is proportional to the eccentricity of the orbit (e.g. Hall 1990), and hence it can not cause the 35 year cycle in $P_{\rm orb}$ of AR Lac with $e\approx 0.0$ (Strassmeier et al. 1993). On the other hand, the *Applegate* (1992) *model* can not be tested for AR Lac, because both binary components are about equally magnetically active (e.g. Walter et al. 1983, Neff et al. 1989). This model relies on the assumption that *one* of the binary components is more active, while the other one can be treated as a point mass.

Could a *third body* induce this $35^{\rm y}$ cycle? Strassmeier et al. (1993) gave $M_1\approx M_2=1.3M_\odot$ and $i=87°$, where the masses for the G2 IV and K0 IV components are lower limits. The most accurate value obtained for the half of the total amplitude of the $P(T,P_1)$ variations is ΔP = 1.46 $\cdot$ 10^{-5} [d] (Sect. 4.2.), the cycle length being $T=35.5$ [y]. The relations by Kalimeris et al. (1995: their Eqs. 7 and 8) give a mass function $f(m)=3\cdot10^{-6}M_\odot$ and $M_3=0.03M_\odot$ for the hypothetical third body. The semimajor axis for the M_3 orbit is $a_3=0.16$ [a.u.], i.e. an angular separation 0.″002 at the distance of 47 Pc. Thus a_3 is of the same order as $a_1\approx a_2$ = 0.021 [a.u] (Strassmeier 1993). In conclusion, the existence of such M_3, being about 30 times more massive than Jupiter ($0.001 M_\odot$), can not be verified nor rejected with modern observational techniques. Our estimate is 25 times smaller than $M_3 \approx 0.75 M_\odot$ by Hall & Kreiner (1980).

The models of mass loss or transfer driven by stellar winds do not predict periodic $P_{\rm orb}$ changes, but a decreasing trend in $P_{\rm orb}$. But as shown in Sect. 4.5., a detection of such a trend would not be unique. Finally, Tout & Hall (1992) noted that U Cephei is perhaps the only binary system, where such a trend has been established.

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Appendix A

The reasons for selecting nonparametric methods to determine P_0 in the ephemerides (1) and (2) are discussed first. Then the technical details of the determination of the parameters P_0 and t_0 of the ephemerides. Nonparametric methods have not been applied earlier to study the $P_{\rm orb}$ changes of eclipsing binaries. We showed in Sect. 4. that the $P(T, P_0)$ changes are independent of the chosen P_0. Why then apply nonparametric methods, instead of simply adopting an existing ephemeris?

The standard technique to determine the ephemerides for eclipsing binaries is to obtain linear or quadratic least squares fits to the O$-$C data (e.g. Chambliss (1976) and Hall & Kreiner (1980) for AR Lac). First, a preliminary P_0 can be chosen to derive the O$-$C data (e.g. P_0 for part of the data), and then the final ephemeris is searched with a linear or quadratic least squares fit. Yet, the O$-$C changes of eclipsing binaries can not always be convincingly modelled with linear or quadratic least squares fits, and this problem certainly arises with AR Lac. One example of how P_0 uncertainties may mislead $P(T, P_0)$ studies is discussed below.

If the time distribution of the data contains long gaps, it may be impossible to ascertain whether E or $E \pm 1$ orbital periods have elapsed during these gaps, i.e. the O$-$C continuity is not unique. If $P(T, P_0)/P_0$ is not sufficiently close to unity (Eq. 6), an arbitrary epoch after a long gap may represent, e.g., $E-0.9$, E+0.1 or E+1.1. This problem is more serious, if the light curve shapes at the primary and secondary eclipse are similar, i.e. this epoch may represent a primary or secondary minimum. Linear or quadratic least squares fits to determine P_0 are insufficient in such cases. In conclusion, the standard ephemeris determination may sometimes yield unreal O$-$C discontinuities that mislead the studies of $P_{\rm orb}$ changes.

Nonparametric (i.e. model independent) methods are an ideal choice to determine P_0 for the following reasons: (1) A model (e.g. a polynomial) is unnecessary. (2) The detected P_0 minimizes the dispersion of O$-$C, or equivalently maximizes continuity of O$-$C assumed in Eq. 6. (3) The primary and secondary minima are analysed simultaneously, if the chosen method is sensitive to multimodal distributions (see Figs. 1c and 2c), as is the case for both methods discussed below (4) No a priori information of whether an arbitrary epoch represents a primary or secondary minimum is needed. (5) The errors of the data are utilized in the period analysis. (6) If abrupt $P_{\rm orb}$ changes do occur in eclipsing binaries, then these methods can detect them. Periodicity detection does succeed even if both phase shifts and period changes occur in the data (Jetsu 1996). (7) The significance of the detected periodicity, and t_0 and P_0 error estimates are obtained.

The applied weighted nonparametric methods are: the WK–method (weighted Kuiper (1960) test: Sect. 3.3. in Paper I) and the WSD–method (weighted Swanepoel & De Beer (1990) test: Sects. 2.2.–2.3. in Paper I). These methods analyse circular data, i.e. a random sample of single measurements representing directions in a plane or phases at different time values folded with a fixed period. The data in Table 1 are circular when folded with any arbitrary period, and are typical of a case where the model is unknown.

The number of analysed time points ($t_{\rm i}$) with errors $\sigma_{\rm i}$ ($0.^{\rm d}004$ or $0.^{\rm d}011$) is $n = 156$. The ratio of the weights ($w_{\rm i} = \sigma_{\rm i}^{-2}$) between "e" and "other" systems is ~ 7.6, which is more than twice as large as that of Hall & Kreiner (1980). The means of all available values for any particular minimum were derived, because all values for the same minimum would be close in phase to any tested period, which might mislead the period analysis. The results for the WK– and WSD–methods using trial periods between $1.^{\rm d}96$ and $2.^{\rm d}00$ were given in the ephemerides (1) and (2). Both periods are extremely significant, because the solutions for the critical levels of Eqs. 19 and 30 in Paper I exceed the available computing capacity, i.e. they are below 10^{-200}! No other significant periodicities are present within the above period interval, since all other periods revealed by the periodograms are spurious values resulting from the data spacing of $365^{\rm d}$ (see Jetsu 1996: Eq. 6).

The error estimates for the periods (P_0) of the ephemerides (1) and (2) were determined with the bootstrap approach explained in Sect. 4 of Paper I. However, the method for determining the zero point (t_0) for these ephemerides is formulated here for the first time. The notation is the same as in Paper I. The original data are denoted by the vectors $\bar{t}$ and $\bar{w}$, and a large number (q_1) of random samples are drawn from these original data vectors, and are denoted as $\bar{t}^*_1, ..., \bar{t}^*_{q_1}$ and $\bar{w}^*_1, ..., \bar{w}^*_{q_1}$. This random selection procedure is the same as explained in Paper I, i.e. the connection that $w_{\rm i}$ is the weight of $t_{\rm i}$ is preserved in every random sample. Each random sample (k=1,...,q_1) yields the following estimate for the mean of the O$-$C values for a constant P_0 and a fixed zero point (t')

$$\phi_{\rm k} = \left[\sum_{\rm i=1}^{\rm n} w^*_{\rm i} \left({\rm FRAC} \left[(t^*_{\rm i} - t')P_0^{-1} \right] - a \right) \right] \left[\sum_{\rm i=1}^{\rm n} w^*_{\rm i} \right]^{-1}, \quad (17)$$

where the notation FRAC means that the integer part of $(t^*_{\rm i} - t')P_0^{-1}$ is removed, while $a = 0.0$ and 0.5 for the primary and secondary minima, respectively. The final result for the zero point of the ephemeris is

$$t_0 = t' + P_0(\langle \phi_{\rm k} \rangle \pm \sigma_{\phi_k}), \quad (18)$$

where $\langle \phi_{\rm k} \rangle$ and σ_{ϕ_k} are the average and the standard deviation of the q_1 estimates of $\phi_{\rm k}$.

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