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Effects of gravity and density stratification on the asymptotic representation of p-modes in stars

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Abstract. The second-order asymptotic theory for low-degree p-modes in a star developed by Smeyers et al. (1996) is reconsidered, especially for lower-frequency modes. The investigation is undertaken in analogy with an earlier investigation of Roxburgh and Vorontsov (1994), in which a generalization of the first Born approximation for the scattering, by the stellar core, of acoustic waves modified by gravity and buoyancy is applied.

A frequency-dependent velocity of propagation of acoustic waves is introduced that is affected by gravity and density gradient, mainly in the central region of the star. The time needed for an acoustic wave to propagate from the centre of the star to a given radial distance is increased, and, in the first asymptotic approximation, the oscillation frequency of a p-mode is decreased. The differences are larger for lower-frequency p-modes.

The asymptotic theory is applied to a polytropic model with index equal to 3. The relative errors on the scaled frequency separations $D_{n,\ell}$ for degrees $\ell = 0, 1, 2$ are reduced in comparison to those resulting from the usual asymptotic theory, but still amount to about 30% for modes of radial order n = 20 and to about 18% for modes of radial order n = 30.

For a normal solar model, the second asymptotic approximations of the eigenfrequencies do not lead to satisfactory results. The failure is ascribed to the behaviour of the second derivative of the mass density in the partial ionization zone of hydrogen near the solar surface. This behaviour introduces a sharp and high peak in the propagation diagram, which is not taken into account in the present asymptotic analysis.

Key words: stars: oscillations – stars: interiors – Sun: oscillations – methods: analytical

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1. Introduction

In recent years, appreciable progress has been realized in the asymptotic theory of linear, isentropic, low-degree oscillation modes of spherically symmetric stars.

Tassoul (1990) derived a second-order asymptotic representation of low-degree, high-frequency nonradial oscillation modes p with inclusion of the Eulerian perturbation of the gravitational potential in the theory. For this purpose, she adopted a fourth-order system of two differential equations in the divergence and the radial component of the Lagrangian displacement that was established earlier by Pekeris (1938). As in her previous asymptotic treatment (Tassoul 1980), Tassoul applied asymptotic techniques given by Olver (1974).

An alternative approach was introduced by Smeyers et al. (1995) for the derivation of an asymptotic representation of low-frequency g^+-modes in a star. These authors adopted procedures suited for singular perturbation problems as described by Kevorkian & Cole (1981). For distances sufficiently large from the star's centre and surface, they derived asymptotic solutions by using a two-variable expansion procedure in terms of a fast and a slow variable in the radial direction. Furthermore, they treated the regions near the star's centre and surface as boundary-layers. For the sake of simplification, the authors assumed the star to be everywhere convectively stable. Afterwards, Willems et al. (1997) extended the asymptotic representation of low-frequency g^+-modes to stars containing a convective core.

The alternative approach was used subsequently by Smeyers et al. (1996) for the derivation of a second-order asymptotic representation of high-frequency p-modes. The resulting asymptotic representation appears to be identical to Tassoul's asymptotic representation, apart from a difference in the boundary-layer solution for the radial component of the Lagrangian displacement near the star's centre. However, this difference does not affect the asymptotic representation at the degree of approximation considered.

A main conclusion from this alternative asymptotic approach is that, in the first approximation for p- and g-modes, a star can be assimilated (in the radial direction) with a high-frequency linear oscillator displaying a small damping, when the divergence of the Lagrangian displacement is used as the dependent variable.

Despite the progress realized, the asymptotic theory continued to yield limited approximations for the small frequency separations $D_{n,\ell}$ between lower-frequency, low-degree p-modes, which are of particular interest for helioseismology. From their part, Roxburgh and Vorontsov (1994) stressed the importance of local effects of gravity and buoyancy forces on acoustic waves in the stellar core. They developed a different approach by deriving a single second-order differential equation from the governing fourth-order system of differential equations and applying a generalization of the first Born approximation for the scattering, by the stellar core, of acoustic waves modified by gravity and buoyancy. From applications to a standard solar model and to simple zero-age and evolved models of a $3\,M_\odot$ main-sequence star with a convective core, the authors inferred that their theoretical description of small frequency separations leads to accurate results. They also derived higher-order Born approximations in a later investigation (Roxburgh and Vorontsov 1996).

Such being the case, we reconsidered the asymptotic representation of lower-frequency, low-degree p-modes in stars that is developed from the exact fourth-order system of differential equations in the divergence and the radial component of the Lagrangian displacement. In this investigation, we take into consideration that lower-frequency p-modes originate from propagating waves that are not purely acoustic waves, especially in the central regions of a star. We modify the asymptotic theory from the first-order approximation on by incorporating terms that involve the gravity and the density gradient in the expression for the velocity of propagation of an acoustic wave and by considering the velocity of propagation to be frequency-dependent to some extent. The time needed for an acoustic wave to propagate from the centre of the star to a given radial distance is increased, and, in the first asymptotic approximation, the oscillation frequency of a p-mode is decreased. The increase of the propagation time of an acoustic wave and the associated decrease of the oscillation frequency are larger for lower-frequency p-modes.

The plan of the paper is as follows. In Sect. 2, we present the basic equations and introduce the modified velocity of propagation of an acoustic wave. Sect. 3 is devoted to the asymptotic expansions in the acoustic cavity, Sect. 4 to the boundary-layer expansions near $r = 0$, and Sect. 5 to the boundary-layer expansions near $r = R$. In Sects. 6 and 7, we respectively present the equation for the eigenfrequencies and the uniformly valid asymptotic expansions for the divergence and the radial component of the Lagrangian displacement. In Sects. 8 and 9, we discuss results obtained for a polytropic model with index equal to 3, and a standard solar model. Finally, Sect. 10 is devoted to some concluding remarks.

2. Basic equations

Consider a spherically symmetric, static star in hydrostatic equilibrium with mass M and radius R. We assume the star to be oscillating in a linear, isentropic, low-degree p-mode that depends on time t by $\exp(i\sigma t)$ and is associated with a spherical harmonic $Y_\ell^m(\theta,\phi)$. We adopt the same notations as Smeyers et al. (1995).

We start from the fourth-order system of differential equations in the divergence $\alpha(r)$ and the radial component $\xi(r)$ of the Lagrangian displacement established by Pekeris (1938) and write the system of equations in the form

$$\frac{d^2\alpha}{dr^2} + K_2(r)\,\frac{d\alpha}{dr} + \left[\frac{\sigma^2}{c^2} + K_3(r) + \frac{K_1(r)}{\sigma^2}\right]\alpha = -K_4(r)\,\frac{d\xi}{dr}, \tag{1}$$

$$\frac{d^2\xi}{dr^2} + \frac{4}{r}\,\frac{d\xi}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi = \frac{d\alpha}{dr} + \left[\frac{2}{r} - \frac{1}{\sigma^2}\,\frac{c^2}{g}\,K_1(r)\right]\alpha, \tag{2}$$

where

$$K_1(r) = \ell(\ell+1)\,\frac{N^2}{r^2}, \tag{3}$$

$$K_2(r) = \frac{2}{r} + \frac{2}{\rho\,c^2}\,\frac{d\left(\rho\,c^2\right)}{dr} - \frac{1}{\rho}\,\frac{d\rho}{dr}, \tag{4}$$

$$K_3(r) = -\frac{\ell(\ell+1)}{r^2} + \frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr} + \frac{1}{r}\right) + \frac{1}{\rho\,c^2}\,\frac{d\left(\rho\,c^2\right)}{dr}\left(\frac{2}{r} - \frac{1}{\rho}\,\frac{d\rho}{dr}\right) + \frac{1}{\rho\,c^2}\,\frac{d^2\left(\rho\,c^2\right)}{dr^2}, \tag{5}$$

$$K_4(r) = -\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr} - \frac{1}{r}\right). \tag{6}$$

In Eq. (1), σ^2 is the large parameter. The use of $1/c^2$ as coefficient of the term with the large parameter goes back to Ledoux' derivation of an asymptotic representation of high-frequency radial oscillation modes (1962). We modify this coefficient in order to take into account effects of gravity and density stratification on the velocities of propagation and the associated travel times of lower-frequency acoustic waves, especially in the central regions of stars. We consider the velocity of propagation of an acoustic wave to be frequency-dependent.

For lower-frequency p-modes, the term

$$\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr} + \frac{1}{r}\right)\alpha \tag{7}$$

in the left-hand member of Eq. (1) is, in the central region of a star, of the same order of magnitude as the term $\left(\sigma^2/c^2\right)\alpha$.

Furthermore, the divergence $\alpha(r)$ of the Lagrangian displacement and the first derivative $d\xi/dr$ of the radial component of the Lagrangian displacement are equal in their dominant asymptotic approximations in $1/\sigma$ [see, e.g., Smeyers et al. (1996), Solutions (37)]. Therefore, we decompose the right-hand member of Eq. (1) as

$$-K_4(r)\left[\alpha + \left(\frac{d\xi}{dr} - \alpha\right)\right]. \tag{8}$$

The second term inside the brackets is then of an order $1/\sigma$ higher than the first term. We bring the first term to the left-hand member of Eq. (1) and add it to the term given by Expression (7). This yields

$$\left[\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr}+\frac{1}{r}\right)+K_4(r)\right]\alpha=\frac{1}{c^2}\,\frac{4\,g}{r}\,\alpha. \tag{9}$$

Furthermore, we incorporate the term

$$\frac{K_1(r)}{\sigma^2}\,\alpha, \tag{10}$$

which contains the gravity and the density gradient, and becomes more important as the degree ℓ increases.

Consequently, at this stage, the coefficient of the large parameter takes the form

$$\frac{1}{c^2}\left[1+\frac{1}{\sigma^2}\,\frac{4\,g}{r}+\frac{\ell(\ell+1)}{\sigma^4}\,\frac{c^2\,N^2}{r^2}\right]. \tag{11}$$

In the particular case of the equilibrium configuration with uniform mass density, the expression inside the brackets can be written as

$$1+\frac{4}{\sigma^2}\,\frac{G\,M}{R^3}-\frac{\ell(\ell+1)}{\sigma^4}\left(\frac{G\,M}{R^3}\right)^2$$
$$=\frac{4}{3}\,\frac{1}{\sigma^2/(\pi\,G\,\rho)}\left[\frac{3}{4}\,\frac{\sigma^2}{\pi\,G\,\rho}+4-\frac{4}{3}\,\frac{\ell(\ell+1)}{\sigma^2/(\pi\,G\,\rho)}\right]. \tag{12}$$

Apart from a factor $2/\Gamma_1$, the expression inside the brackets in the right-hand member corresponds to Tassoul's (1980) Definition (73) of the modified large parameter

$$\lambda^2=\frac{2}{\Gamma_1}\left[\frac{3}{4}\,\frac{\sigma^2}{\pi\,G\,\rho}+4-\frac{4}{3}\,\frac{\ell(\ell+1)}{\sigma^2/(\pi\,G\,\rho)}\right]. \tag{13}$$

Tassoul observed that the use of λ^2 as a large parameter leads to much better asymptotic approximations for the eigenfrequencies than the use of σ^2 does. Our Definition (11) of the coefficient of the large parameter for any stellar model is thus in agreement with Tassoul's redefinition of the large parameter in the case of the equilibrium configuration with uniform mass density.

To Expression (11) for the coefficient of the large parameter, we add the term

$$\frac{2}{r}\,\frac{1}{\rho}\,\frac{d\rho}{dr}. \tag{14}$$

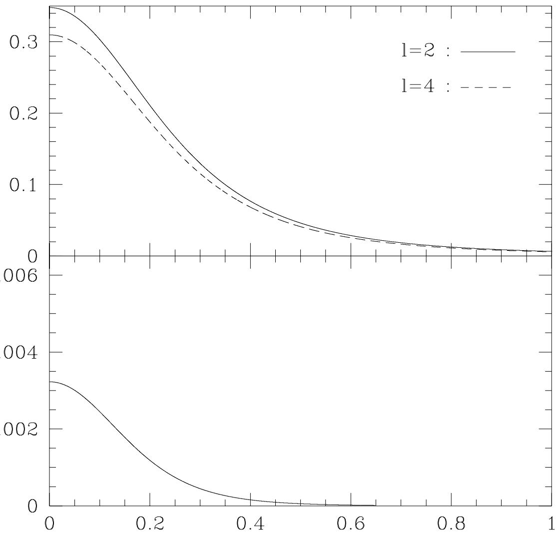
Since it contains the density gradient, this term is equal to zero in the particular case of the equilibrium configuration with uniform mass density.

Therefore, we bring Eq. (1) to the form

$$\frac{d^2\alpha}{dr^2}+K_2(r)\,\frac{d\alpha}{dr}+\left[\sigma^2\,K_7\left(r;\sigma^2\right)+K_3^*(r)\right]\alpha$$
$$=-K_4(r)\left(\frac{d\xi}{dr}-\alpha\right), \tag{15}$$

where

$$K_3^*(r)\equiv K_3(r)-\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr}+\frac{1}{r}\right)-\frac{2}{r}\,\frac{1}{\rho}\,\frac{d\rho}{dr}$$



3. Asymptotic expansions in the acoustic cavity

We define the fast variable $\tau(r)$ in the radial direction as

$$\tau(r) = \frac{1}{\varepsilon}\int_0^r K_7^{1/2}\left(r';\sigma^2\right)\,dr'. \tag{19}$$

The functions $h(r)$ and $F(r)$ are now given by

$$h(r) = \left(\rho\, r^2\, c^4\, K_7^{1/2}\right)^{-1/2}, \tag{20}$$

$$F(r) = \frac{1}{2}\int_{r_0}^r K_7^{-1/2}\left(r';\sigma^2\right)\, W\left(r'\right)\,dr', \tag{21}$$

where r_0 is an arbitrary value of the radial coordinate r in $(0, 1)$. The function $W(r)$ is defined as

$$W(r) = \frac{1}{h}\frac{d^2h}{dr^2} + K_2\,\frac{1}{h}\frac{dh}{dr} + K_3^* \tag{22}$$

and can be written explicitly as

$$W(r) = -\frac{\ell(\ell+1)}{r^2} - \frac{1}{r}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{3}{4}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2 + \frac{1}{2}\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{5}{16}\left(\frac{1}{K_7}\frac{dK_7}{dr}\right)^2 - \frac{1}{4}\frac{1}{K_7}\frac{d^2K_7}{dr^2}. \tag{23}$$

With the foregoing modifications, the asymptotic expansions for the divergence and the radial component of the Lagrangian displacement that are valid to order ε in the star's acoustic cavity can be written as

$$\left.\begin{aligned} \alpha^{(o)}(r;\varepsilon) &= h(r)\left[\left(A_0^*\,\cos\tau + B_0^*\,\sin\tau\right)\right. \\ &\quad \left.+\varepsilon\, F(r)\,\left(B_0^*\,\cos\tau - A_0^*\,\sin\tau\right)\right] + O\left(\varepsilon^2\right), \\ \xi^{(o)}(r;\varepsilon) &= -\varepsilon\, h(r)\, K_7^{-1/2}(r)\left(B_0^*\,\cos\tau - A_0^*\,\sin\tau\right) \\ &\quad +C_0^*\, r^{\ell-1} + D_0^*\, r^{-(\ell+2)} + O\left(\varepsilon^2\right). \end{aligned}\right\} \tag{24}$$

Here, A_0^*, B_0^*, C_0^*, D_0^* are yet undetermined constants.

4. Boundary-layer expansions near $r = 0$

The transformation from the functions $\alpha(r)$ and $\xi(r)$ to the functions $v(r)$ and $w(r)$ now takes the form

$$\left.\begin{aligned} \alpha(r) &= h(r)\, v(r), \\ \xi(r) &= h(r)\, K_7^{-1/2}\left(r;\sigma^2\right)\, w(r). \end{aligned}\right\} \tag{25}$$

The Taylor series of the function $K_7\left(r;\sigma^2\right)$ for small values of r can be written as

$$K_7\left(r;\sigma^2\right) = K_{7,c}\,\left[1 + O\left(r^2\right)\right], \tag{26}$$

The dominant boundary-layer equation for the function $v(r)$ is given by

$$\frac{d^2v_0^{(c)}}{d\tau^2} + \left[1 - \frac{\ell(\ell+1)}{\tau^2}\right] v_0^{(c)} = 0, \tag{27}$$

which corresponds formally to Eq. (50) of Smeyers et al. (1996).

The boundary-layer expansions for $v^{(c)}(\tau;\varepsilon)$ and $w^{(c)}(\tau;\varepsilon)$ to order ε correspond formally to boundary-layer Expansions (61) of Smeyers et al. (1996) and contain the yet undetermined functions $\mu_0^{(c,1)}(\varepsilon)$ and $\mu_0^{(c,2)}(\varepsilon)$ and the yet undetermined constants $A_{0,c}$ and $C_{0,c}$.

In view of the matching of the boundary-layer expansions, it is appropriate to decompose the function $F(r)$ determined by Definition (21) as

$$F(r) = F^{(c)}(r) - F^{(c)}\left(r_0\right), \tag{28}$$

where

$$F^{(c)}(r) = K_{7,c}^{-1/2}\,\frac{\ell(\ell+1)}{2\,r} + \frac{1}{2}\int_0^r \left[K_7^{-1/2}\left(r'\right)\, W\left(r'\right) + K_{7,c}^{-1/2}\,\frac{\ell(\ell+1)}{r'^2}\right] dr'. \tag{29}$$

The integrand in the second term remains finite as $r \to 0$.

The matching, to order ε, of the boundary-layer expansion for $v^{(c)}(\tau;\varepsilon)$ and the asymptotic expansion for $\alpha^{(o)}(r;\varepsilon)/h(r)$ leads to conditions corresponding formally to Conditions (71)–(73) in Smeyers et al. (1996). By these conditions, the oscillatory parts in the boundary-layer expansion for $w^{(c)}(\tau;\varepsilon)$ and in the asymptotic expansion for $\xi^{(o)}(r;\varepsilon)/\left[h(r)\,K_7^{-1/2}\left(r;\sigma^2\right)\right]$ are also matched to order ε.

The matching of the non-oscillatory parts leads to conditions similar to Conditions (77), (80), and (81) of Smeyers et al. (1996).

5. Boundary-layer expansions near $r = 1$

For the construction of boundary-layer solutions near $r = 1$, we introduce the coordinate $z = a - r$. Here a is the radial distance from the centre to the point where the pressure becomes equal to zero. We let a be larger than or equal to unity, in order to take into account that, for realistic stellar models, $P(R) \neq 0$. The basic assumption adopted is that the mass density ρ behaves as z^{n_e} for small values of z. The Taylor series of the pressure for small values of z is derived by integration of the condition of hydrostatic equilibrium under the assumption that the layers near the star's surface do not contribute to the total mass so that $m(r) = 1$. For the Taylor series of the various quantities, we refer to Smeyers et al. (1995). The Taylor series of $K_7\left(r;\sigma^2\right)$ can be written as

$$K_7\left(r;\sigma^2\right) = K_{7,s}\,\frac{1}{z}\,[1 + O(z)]. \tag{30}$$

We define the boundary-layer coordinate as

$$\tau_s(z) = \frac{1}{\varepsilon}\int_0^z K_7^{1/2}\left(r';\sigma^2\right)\,dz'. \tag{31}$$

The dominant boundary-layer equation for the function $v(r)$ is given by

$$\frac{d^2v_0^{(s)}}{d\tau_s^2} + \left[1 - \frac{(n_e+1)^2 - 1/4}{\tau_s^2}\right] v_0^{(s)} = 0. \tag{32}$$

The boundary-layer expansions for $v^{(s)}(\tau_s;\varepsilon)$ and $w^{(s)}(\tau_s;\varepsilon)$ to order ε correspond formally to boundary-layer Expansions (106) of Smeyers et al. (1996):

$$\left.\begin{aligned} v^{(s)}(\tau_s;\varepsilon) &= \mu_0^{(s,1)}(\varepsilon)\, A_{0,s}\, \tau_s^{1/2}\, J_{n_e+1}(\tau_s) \\ &+\varepsilon\, \mu_0^{(s,2)}(\varepsilon)\, 2\, C_{0,s}\, \frac{K_{4,s}}{K_{7,s}}\, \tau_s^{n_e+3/2}, \\ w^{(s)}(\tau_s;\varepsilon) &= \mu_0^{(s,2)}(\varepsilon)\, C_{0,s}\, \tau_s^{n_e+5/2} \\ &+\mu_0^{(s,3)}(\varepsilon)\, D_{0,s}\, \tau_s^{n_e+1/2} \\ &+\varepsilon\, \mu_0^{(s,1)}(\varepsilon)\, A_{0,s}\, \tau_s^{1/2}\, J_{n_e}(\tau_s)\,. \end{aligned}\right\} \quad (33)$$

Here $\mu_0^{(s,1)}(\varepsilon)$, $\mu_0^{(s,2)}(\varepsilon)$, $\mu_0^{(s,3)}(\varepsilon)$ are yet undetermined functions of ε, and $A_{0,s}$, $C_{0,s}$, $D_{0,s}$ yet undetermined constants.

In view of the matching of the oscillatory parts of the boundary-layer expansion for $v^{(s)}(\tau_s;\varepsilon)$ and the asymptotic expansion for $\alpha^{(o)}(r;\varepsilon)/h(r)$, it is appropriate to decompose the function $F(r)$ given by Definition (21) as

$$F(r) = F^{(s)}(z) - F^{(s)}(z_0) \quad (34)$$

with

$$F^{(s)}(z) = -K_{7,s}^{-1/2}\, \frac{(2\,n_e+1)(2\,n_e+3)}{16\, z^{1/2}} - \frac{1}{2}\int_0^z \left[K_7^{-1/2}(r')\, W(r') + K_{7,s}^{-1/2}\, \frac{(2\,n_e+1)(2\,n_e+3)}{16\, z'^{3/2}} \right] dz'. \quad (35)$$

The integrand in the second term of the function $F^{(s)}(z)$ behaves as $z'^{-1/2}$ for small values of z'. The coordinate z_0 is related to the coordinate r_0 by the relation $z_0 = a - r_0$.

The matching, to order ε, of the oscillatory parts in the boundary-layer expansion for $v^{(s)}(\tau_s;\varepsilon)$ and in the asymptotic expansion for $\alpha^{(o)}(r;\varepsilon)/h(r)$ leads to conditions corresponding formally to Conditions (117)–(119) of Smeyers et al. (1996). By these conditions, the matching of the oscillatory parts in the boundary-layer expansion for $w^{(s)}(\tau_s;\varepsilon)$ and in the asymptotic expansion for $\xi^{(o)}(r;\varepsilon)/\left[h(r)\, K_7^{-1/2}(r;\sigma^2)\right]$ is also performed to order ε.

The matching of the non-oscillatory parts leads to conditions similar to Conditions (125)–(127) of Smeyers et al. (1996). It then follows that the boundary-layer expansion for $v^{(s)}(\tau_s;\varepsilon)$ to order ε does not contain any non-oscillatory part.

6. The equation for the eigenfrequencies

Still proceeding as in Smeyers et al. (1996), we derive the equation for the eigenfrequencies

$$\sigma \int_0^a K_7^{1/2}(r;\sigma^2)\, dr = \left(2\,n + \ell + n_e + \frac{1}{2}\right)\frac{\pi}{2} + \frac{T_\ell(\sigma)}{\sigma} + O\left(\sigma^{-2}\right), \quad (36)$$

where n is the radial order of the p-mode. The quantity $T_\ell(\sigma)$ is defined as

$$T_\ell(\sigma) = F^{(s)}(z_0) - F^{(c)}(r_0) \quad (37)$$

and can be written as

$$T_\ell(\sigma) = -\frac{1}{2}\left[\int_0^a I(r)\, dr + K_{7,c}^{-1/2}\, \frac{\ell(\ell+1)}{a} + K_{7,s}^{-1/2}\, \frac{(2\,n_e+1)(2\,n_e+3)}{8\, a^{1/2}}\right], \quad (38)$$

where the integrand $I(r)$ is given by

$$I(r) = K_7^{-1/2}(r;\sigma^2)\; W(r) + K_{7,c}^{-1/2}\, \frac{\ell(\ell+1)}{r^2} + K_{7,s}^{-1/2}\, \frac{(2\,n_e+1)(2\,n_e+3)}{16\,(a-r)^{3/2}}. \quad (39)$$

7. Uniformly valid asymptotic expansions

As in Smeyers et al. (1996), the condition on the gravitational potential at $z = 0$ leads to the consequence that the radial component of the Lagrangian displacement is purely oscillatory.

The asymptotic expansions for the divergence and the radial component of the Lagrangian displacement that are uniformly valid to order ε, from $r = 0$ to the outer boundary of the acoustic cavity, then take the form

$$\left.\begin{aligned} \alpha^{(c,u)}(r;\varepsilon) &= A_{0,s}\,(-1)^{n+1}\, h(r)\Bigg\{\tau^{1/2}\, J_{\ell+1/2}(\tau) \\ &-\varepsilon\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{1}{\varepsilon}\,\frac{\ell(\ell+1)}{2\,\tau} - F^{(c)}(r)\right] \\ &\cos\left(\tau - \frac{\ell\,\pi}{2}\right)\Bigg\} + O\left(\varepsilon^2\right), \\ \xi^{(c,u)}(r;\varepsilon) &= -\varepsilon\, A_{0,s}\, \frac{(-1)^{n+1}}{2\,\ell+1}\, h(r)\, K_7^{-1/2}(r) \\ &\tau^{1/2}\left[\ell\, J_{\ell-1/2}(\tau) - (\ell+1)\, J_{\ell+3/2}(\tau)\right] + O\left(\varepsilon^2\right), \end{aligned}\right\} \quad (40)$$

where we have used the relation $A_{0,c} = (-1)^{n+1}\, A_{0,s}$ [Smeyers et al. (1996), Eq. (134)].

The asymptotic expansions for the divergence and the radial component of the Lagrangian displacement that are uniformly valid to order ε, from $z = 0$ to the inner boundary of the acoustic cavity, take the form

$$\left.\begin{aligned} \alpha^{(s,u)}(r;\varepsilon) &= A_{0,s}\, h(r)\Bigg\{\tau_s^{1/2}\, J_{n_e+1}(\tau_s) \\ &-\varepsilon\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{1}{\varepsilon}\,\frac{(2\,n_e+1)(2\,n_e+3)}{8\,\tau_s} + F^{(s)}(z)\right] \\ &\cos\left[\tau_s - (2\,n_e+1)\,\frac{\pi}{4}\right]\Bigg\} + O\left(\varepsilon^2\right), \\ \xi^{(s,u)}(r;\varepsilon) &= \varepsilon\, A_{0,s}\, h(r)\, K_7^{-1/2}(r)\, \tau_s^{1/2}\, J_{n_e}(\tau_s) \\ &+O\left(\varepsilon^2\right). \end{aligned}\right\} \quad (41)$$

Under the appropriate limit processes, these uniformly valid expansions reduce to the respective asymptotic expansions valid in the different regions considered (Kevorkian & Cole 1981, Sect. 1.3). For example, for the divergence of the Lagrangian displacement, one has

$$\left.\begin{aligned} &\lim_{\substack{\varepsilon \to 0 \\ r \neq 0 \text{ fixed}}} \frac{\alpha^{(c,u)}(r;\varepsilon) - \alpha^{(o)}(r;\varepsilon)}{\varepsilon} = 0, \\ &\lim_{\substack{\varepsilon \to 0 \\ \tau \text{ fixed}}} \frac{\alpha^{(c,u)}(r;\varepsilon) - \alpha^{(c)}(r;\varepsilon)}{\varepsilon} = 0. \end{aligned}\right\} \quad (42)$$

One may observe that, in the limit for $\varepsilon \to 0$ with $r \neq 0$ fixed, $\tau \to \infty$, so that Hankel's asymptotic approximation for Bessel functions of the first kind for large values of their argument can be used [Abramowitz & Stegun 1965, (9.2.5)]. Furthermore, in the limit for $\varepsilon \to 0$ with τ fixed, $r \to 0$.

8. Application to a polytropic model

We applied the asymptotic theory to lower-frequency, low-degree p-modes of a polytropic model with index $n_e = 3$ and generalized isentropic coefficient $\Gamma_1 = 5/3$. For this model, $a = 1$. We considered radial orders from $n = 10$ to $n = 40$. In the presentation of the results in the figures, we have used dimensionless quantities.

In Fig. 2, the variation of the velocity of propagation of an acoustic wave modified by gravity and density stratification, $K_7^{-1/2}\left(r;\sigma^2\right)$, is displayed as a function of the radial distance from the centre, for a frequency corresponding to that of the radial mode ($\ell = 0$) of order $n = 10$. For comparison, the variation of the isentropic sound velocity $c(r)$ is also displayed. The velocity of propagation $K_7^{-1/2}\left(r;\sigma^2\right)$ differs most from the isentropic sound velocity $c(r)$ in the central region of the model and is smaller there. In the region near the surface, both velocities are almost equal to each other.

In Figs. 3 and 4, relative errors of the *first asymptotic approximations* of eigenfrequencies are represented for the degrees $\ell = 0, 1$ and $\ell = 2, 3, 4$, respectively. The approximations of the eigenfrequencies determined by means of the usual asymptotic theory [Tassoul (1990), Smeyers et al. (1996)] are also represented. From the comparison, it follows that the relative errors on the asymptotic approximations of the eigenfrequencies are sensibly reduced by the use of $K_7^{-1/2}\left(r;\sigma^2\right)$ for the velocity of propagation of the acoustic waves. The reduction is larger for lower-order modes. By the lower velocity of propagation in the central region, the propagation time of an acoustic wave, which is proportional to the variable $\tau(r)$ determined by Expression (19), is somewhat increased, and the frequency decreased. Another consequence is that, for a given frequency, the inner boundary of the acoustic cavity is displaced slightly towards the star's centre [see Eq. (27)].

Even for radial modes of orders as low as $n = 10$, the relative errors on the eigenfrequencies are now smaller than 0.2%. For non-radial p-modes of a given order n, the relative error of the asymptotic approximation increases with the degree ℓ. For example, for the p_{15}-mode belonging to $\ell = 4$, the relative error already amounts to about 1.8%. We ascribe this increase to the growth of the domain extending from the centre to the inner boundary of the acoustic cavity. The treatment of this domain as a boundary layer becomes less adequate as the inner boundary of the acoustic cavity is situated at larger distances from the centre.

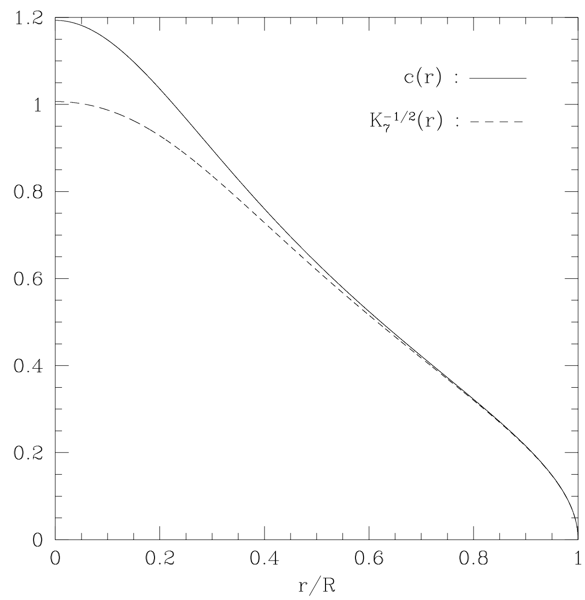


Fig. 2. Variation of the velocity of propagation of an acoustic wave modified by gravity and density stratification, in a polytropic model with index equal to 3. The variation of the isentropic sound velocity $c(r)$ is also displayed for comparison.

For the p_{14}-mode belonging to $\ell = 2$, the first asymptotic approximations of the divergence and the radial component of the Lagrangian displacement are displayed in Fig. 5 as a function of the radial distance from the centre. The eigenfunctions obtained by integration of the full fourth-order system of governing differential equations and by means of the usual asymptotic theory are shown for comparison. As normalization, we set the radial component of the Lagrangian displacement equal to unity at the star's surface. The eigenfunctions derived by means of the asymptotic theory resemble the exact eigenfunctions more closely than the eigenfunctions obtained by means of the usual asymptotic theory do. In particular, the positions of the zeros correspond better to the actual positions of the zeros. Furthermore, the amplitudes of the eigenfunctions are somewhat smaller than those obtained by means of the usual asymptotic theory, as can be seen from comparing the expressions for the amplitudes [see Eqs. (24)] with those of the usual asymptotic theory [see Smeyers et al. 1996, Eqs. (37)].

Relative errors of the *second asymptotic approximations* of eigenfrequencies belonging to $\ell = 2, 3, 4$ are represented in Fig. 6. The relative errors of the second approximations that are obtained by the usual asymptotic theory are also shown. The relative errors on the asymptotic approximations of the eigenfrequencies are negative, as they are in the case of the usual

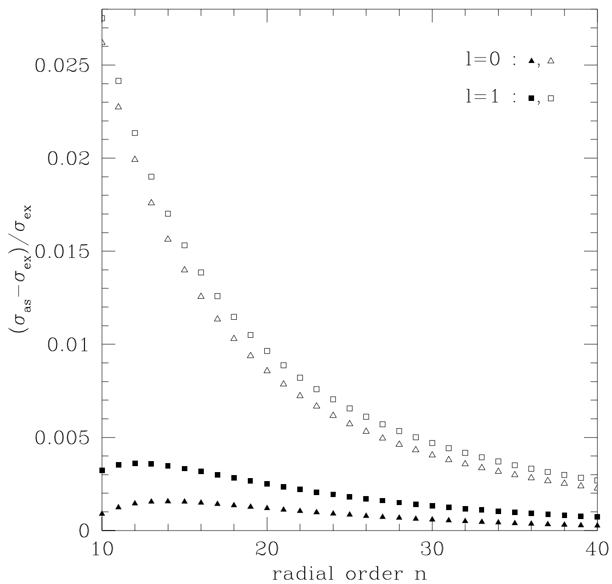


Fig. 3. Relative errors of the first asymptotic approximations of eigenfrequencies for the degrees $\ell = 0, 1$ (solid symbols), for a polytropic model with index equal to 3. Relative errors of the first asymptotic approximations of eigenfrequencies determined by means of the usual asymptotic theory are shown for comparison (open symbols).

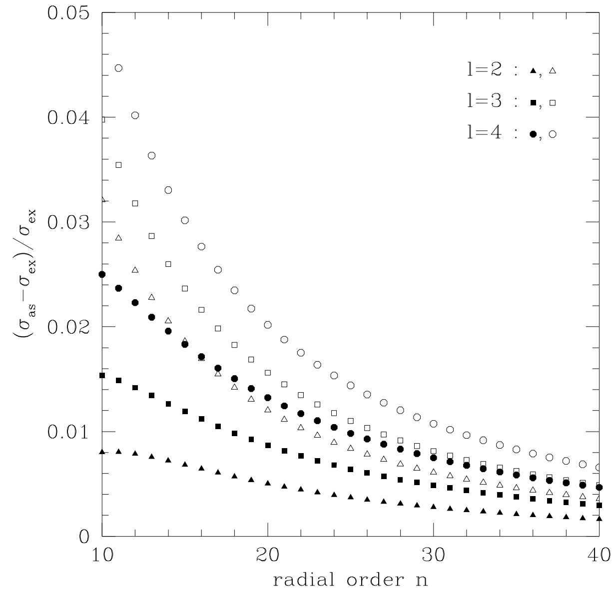


Fig. 4. Relative errors of the first asymptotic approximations of eigenfrequencies for the degrees $\ell = 2, 3, 4$ (solid symbols), for a polytropic model with index equal to 3. Relative errors of the first asymptotic approximations of eigenfrequencies determined by means of the usual asymptotic theory are shown for comparison (open symbols).

asymptotic theory, but they are smaller in absolute value. The gain in accuracy is small for $\ell = 0$ and $\ell = 1$, and increases as the degree ℓ increases. For example, for the eigenfrequency of the p-mode of order $n = 15$ belonging to $\ell = 4$, a relative error of about 1.3 % is found in the case of the usual asymptotic theory, while the relative error is reduced to about 0.8% in the present theory.

We also considered the scaled frequency separations $D_{n,\ell}$ defined as

$$D_{n,\ell} = \frac{1}{2\,\ell + 3}\,\frac{\sigma_{n,\ell} - \sigma_{n-1,\ell+2}}{2\,\pi}. \tag{43}$$

Because of the term containing the factor $\ell(\ell + 1)$ in the definition of the modified velocity of propagation of an acoustic wave, we now have a contribution to the scaled frequency separations already from the first asymptotic approximation on. However, satisfactory results are only obtained by means of the second asymptotic approximation. In Fig. 7, relative errors of scaled frequency separations obtained from the second asymptotic approximation of the eigenfrequencies are displayed for the degrees $\ell = 0, 1, 2$. The relative errors are reduced in comparison to those found by the application of the usual asymptotic theory. However, the relative errors still amount to about 30 % for modes of order $n = 20$ and to about 18 % for modes of order $n = 30$. For modes of orders $n < 20$, the relative errors of the scaled frequency separations decrease somewhat as the degree ℓ increases.

9. Application to a standard solar model

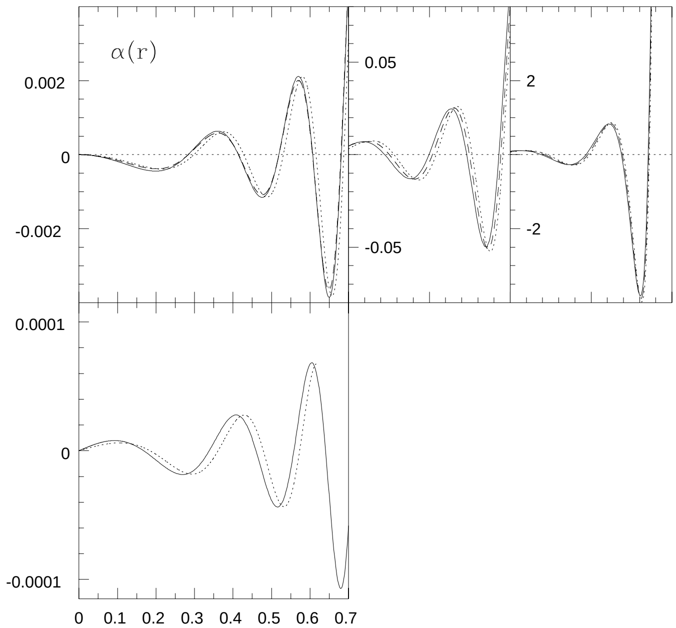
We also tried to apply our modified asymptotic theory to a normal solar model without helium diffusion [Christensen-Dalsgaard, Proffitt, Thompson (1993)]. We considered modes with radial orders from $n = 10$ to $n = 35$, which is the range of modes of interest for helioseismology. As in the previous section, we use dimensionless quantities in the presentation of the results.

The rapid variation of the logarithmic derivative of the mass density near the solar surface makes it difficult to unambiguously determine the value of the effective polytropic index n_e. To keep the calculations based upon the asymptotic theory internally consistent, we determined the values of the effective polytropic index, n_e, and the coordinate z at the solar surface by means of the logarithmic derivatives of the mass density ρ and the pressure P. Near the star's surface, these derivatives lead to the system of two algebraic equations

$$\left.\begin{aligned} z &= \left(\frac{1}{P}\frac{dP}{dz} - \frac{1}{\rho}\frac{d\rho}{dz}\right)^{-1}, \\ \frac{n_e}{z} &= \frac{1}{\rho}\frac{d\rho}{dz}. \end{aligned}\right\} \tag{44}$$

The right-hand members of these equations are evaluated at $r = 1$. This leads to the values $n_e = 3.49$ for the effective polytropic index and $z = 8.67 \times 10^{-4}$ for the coordinate z at the solar surface. Thus, the fact that the pressure does not vanish at the star's surface implies a value of z different from zero at $r = R$. As a consequence, the surface boundary conditions in the asymptotic theory are not exactly imposed at the star's surface.

In Fig. 8, we present relative errors of the *first asymptotic approximations* of eigenfrequencies for the degrees $\ell = 0, 1, 2$. The relative errors of the eigenfrequencies determined by means of the usual asymptotic theory are also shown. For the lowest-order


α(r)
0.002
0
-0.002
0.05
-0.05
2
-2
0.0001
0
-0.0001
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

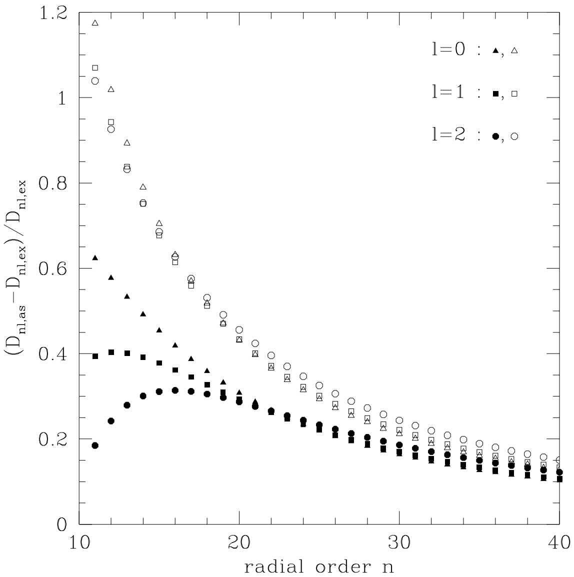


Fig. 7. Relative errors of the scaled frequency separations $D_{n,\ell}$ obtained from the second asymptotic approximation of the eigenfrequencies for the degrees $\ell = 0, 1, 2$ (solid symbols), for a polytropic model with index equal to 3. Relative errors of the scaled frequency separations determined by means of the usual asymptotic theory are shown for comparison (open symbols).

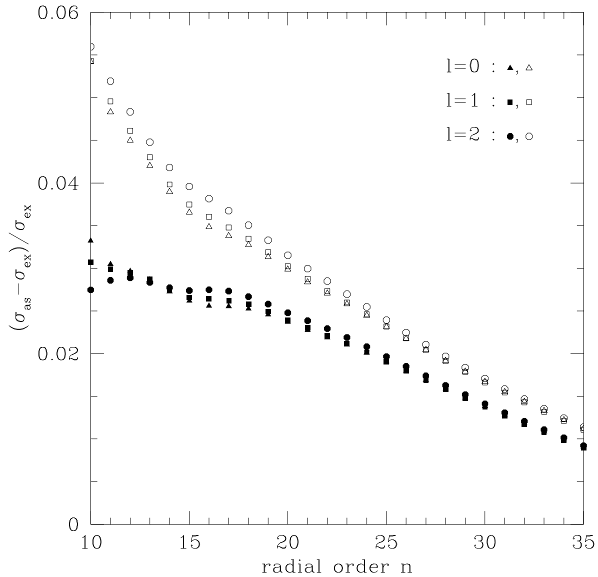


Fig. 8. Relative errors of the first asymptotic approximations of eigenfrequencies for the degrees $\ell = 0, 1, 2$ (solid symbols), for a standard solar model. Relative errors of the first asymptotic approximations of eigenfrequencies determined by means of the usual asymptotic theory are shown for comparison (open symbols).

frequencies $\sigma^2 > 70$. From $r/R = 0.98$ to $r/R = 1$, the horizontal scale is expanded by a factor 5. For comparison, we also show a propagation diagram obtained by means of a dispersion relation that is given by Deubner & Gough (1984),

$$\frac{\sigma^2}{c^2} - \frac{\ell(\ell+1)}{r^2} - \frac{3}{4}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2 + \frac{1}{2}\frac{1}{\rho}\frac{d^2\rho}{dr^2} + \frac{\ell(\ell+1)}{\sigma^2}\frac{N^2}{r^2} = 0, \quad (49)$$

and a propagation diagram obtained by means of a dispersion relation that is given by Smeyers (1984),

$$\frac{\sigma^2}{c^2} - \frac{\ell(\ell+1)}{r^2} - \frac{1}{4}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2 + \frac{\ell(\ell+1)}{\sigma^2}\frac{N^2}{r^2} = 0. \quad (50)$$

Eq. (47) admits of no real solutions for σ^2 in the region $0.19 \le r/R \le 0.73$. As can be seen from Fig. 9, the propagation diagram is then determined for p-modes of radial orders higher than $n = 5$. The same phenomenon occurs in the propagation diagram obtained from the dispersion relation that is given by Deubner & Gough, but only in a small region just below the base of the convective envelope.

A main difference between the propagation diagrams determined by means of dispersion Relation (47) and those determined by means of dispersion Relations (49) and (50) is that the inner boundary of the acoustic cavity is shifted towards the star's centre. The terms responsible for this shift are $(1/r)(d\ln\rho/dr)$ and $4g/(rc^2)$ [see Eq. (48)]. When these terms are removed from dispersion Relation (47), the propagation diagram shows a close resemblance to the propagation diagram resulting from the dispersion relation of Deubner & Gough. The shift of the inner boundary of the acoustic cavity towards the star's centre is consistent with the shift resulting from Eq. (27).

In Fig. 10, we focus on the propagation diagram near the solar surface. Near $r/R = 0.9999$, the propagation diagram displays a sharp and high peak. A similar peak was found by Deubner & Gough (1984), and, more recently, by Buchler et al. (1997) in an investigation on the nature of strange modes in classical variable stars. The complex structure near $r/R = 0.9999$ is related to the hydrogen partial ionization zone and is caused mainly by the second derivative of the mass density. It may be observed that the peak is not found in the propagation diagram determined by means of the dispersion relation of Smeyers, which contains no second derivative of the mass density.

As far as the asymptotic theory is concerned, one may note that the second derivative of the mass density does not appear in the asymptotic theory before the second-order asymptotic solutions. In particular, the sharp peak in the propagation diagram near the solar surface is not taken into account when the turning point is determined by means of Eq. (32). This is probably the reason why no satisfactory results are obtained for the second-order asymptotic approximations of eigenfrequencies for the standard solar model. In their procedure, Roxburgh and Vorontsov (1996) avoided the difficulties related to the existence of the high peak in the propagation diagram near the star's surface by integrating the appropriate differential equations from the solar surface to a sufficiently large distance below the partial ionization zones and joining these solutions to the asymptotic solutions valid in the acoustic cavity.

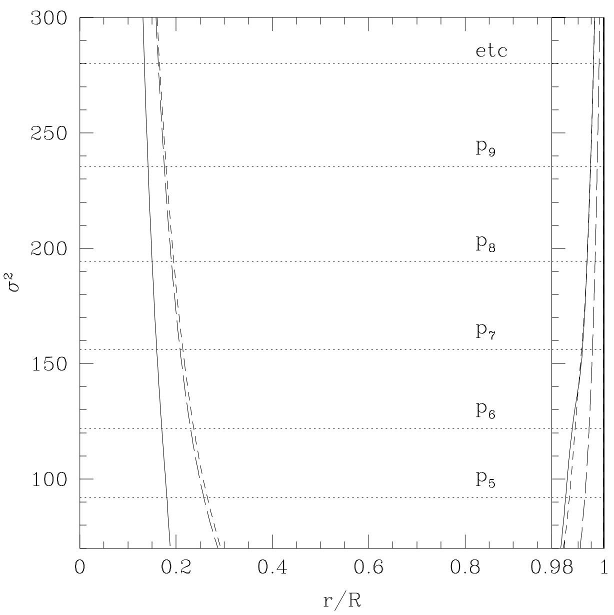


Fig. 9. Propagation diagram (—) for the Sun for p-modes belonging to ℓ = 2 for squared frequencies $\sigma^2 > 70$. Propagation diagrams determined by means of dispersion relations that are given by Deubner & Gough (- -) and Smeyers (— —) are shown for comparison. From r/R = 0.98 to r/R = 1, the horizontal scale is expanded by a factor 5.

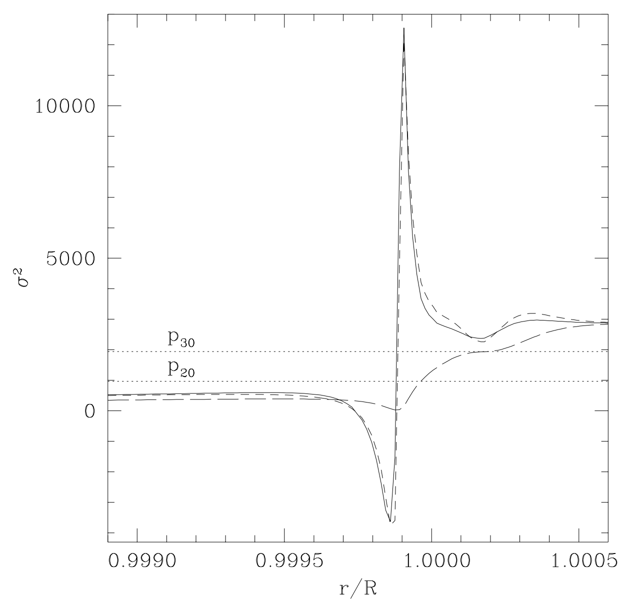


Fig. 10. Propagation diagram (—) near the solar surface for p-modes belonging to ℓ = 2. The propagation diagrams determined by means of the dispersion relations of Deubner & Gough (- -) and Smeyers (— —) are shown for comparison.

10. Concluding remarks

We reconsidered the second-order asymptotic theory for low-degree p-modes in a star developed by Smeyers et al. (1996), especially for lower-frequency modes. We introduced a frequency-dependent velocity of propagation of acoustic waves that is affected by gravity and density gradient mainly in the central region of the star.

We applied the second-order asymptotic theory to low-degree p-modes of a polytropic model with index equal to 3. The first asymptotic approximations of the eigenfrequencies, especially of the lowest-degree p-modes, are greatly improved. In addition, the phase shifts between the first asymptotic approximations of the divergence and the radial component of the Lagrangian displacement and the exact eigenfunctions, which are determined by integration of the full fourth-order system of governing differential equations, are appreciably reduced.

For the second asymptotic approximations of the eigenfrequencies, the improvement in the relative errors is small in the cases of the lowest-degree p-modes and amounts to about a factor 2 for the lowest-order p-modes belonging to ℓ = 3 and ℓ = 4. The relative errors on the scaled frequency separations $D_{n,\ell}$ for $\ell = 0, 1, 2$ are reduced in comparison to those resulting from the usual asymptotic theory, but still amount to about 30% for modes of radial order n = 20 and to about 18% for modes of radial order n = 30.

We also applied the modified asymptotic theory to a normal solar model without helium diffusion. For this model too, the relative errors of the first asymptotic approximations of the eigenfrequencies are reduced in comparison to those found by means of the usual asymptotic theory. However, the reduction is sensibly smaller than for the corresponding modes of the polytropic model.

The second asymptotic approximations of the eigenfrequencies do not lead to satisfactory results. We ascribe this failure to the behaviour of the second derivative of the mass density in the partial ionization zone of hydrogen near the solar surface. This behaviour introduces a sharp and high peak in the propagation diagram, which is not taken into consideration in the present asymptotic analysis.

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