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On collimation of the outflows in force-free magnetospheres

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Abstract. It is pointed out that "an indeterminacy" claimed to exist in the *pulsar equation* (Sulkanen & Lovelace 1990) or a "free" function $B_t\varpi \equiv \beta$ (Appl & Camenzind 1993a,b) can be uniquely related to the stream function at the boundary of the force-free region to the load region at infinity (Michel 1973b: Okamoto 1974). The form of β is given by $\beta = -(\alpha/c)(B_p\varpi^2)_\infty$ from the plasma conditions at infinity, where α is the angular velocity of field lines. This relation can be interpreted as indicating Ohm's law for the return current from the equatorial region to the polar region on the 'sphere at infinity' with finite resistivity $R_L = 4\pi/c$ = 377 Ohm, thereby ensuring the current closure. The field structure tends to be asymptotically conical, toward the 'sphere at infinity' and fields fill all the space there. It is argued that force-free models for purely cylindrical jets cannot self-consistently conserve electric current. The role of the plasma conditions in the black hole equation is mentioned as well.

Key words: MHD – ISM: jets and outflows – galaxies: jets – Black hole physics – pulsars: general

1. Introduction

The basic equation under concern is the so-called *pulsar equation* in the force-free region of the stationary axisymmetric magnetosphere, which reads in cylindrical coordinates

$$\nabla\cdot\left[\frac{1}{\varpi^2}\left(1-\frac{\alpha^2\varpi^2}{c^2}\right)\nabla P\right] + \frac{1}{2\varpi^2}\frac{d\beta^2}{dP} + \frac{\alpha^2}{c^2}\frac{d\ln\alpha}{dP}|\nabla P|^2 = 0 \qquad (1)$$

(Michel 1973a,b; Scharlemann & Wagoner 1973; Julian 1973). Here P is the stream or flux function and $\varpi = r\sin\theta$ is the axial distance. Needless to say, Eq. (1) contains two functions of P. One is the angular velocity of field lines, $\alpha(P)$, and the other is related to the angular momentum flux per unit flux tube $\beta(P)$ (Mestel 1968).

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Our main task is then to resolve whether function $\beta(P)$ is "free" or fixed under an appropriate physical condition, and whether fields can take asymptotically cylindrical geometry with $P = P(\varpi)$ and $\beta = \beta(P)$ or not. Basic quantities and relations between them, necessary to resolve these questions, are summarized in the following.

The poloidal and toroidal components of the electromagnetic field are then expressed as

$$\boldsymbol{B}_p = -\frac{\boldsymbol{t}\times\nabla P}{\varpi},\quad \boldsymbol{B}_t = \frac{\beta(P)}{\varpi}\boldsymbol{t},$$

$$\boldsymbol{E}_p = -\frac{\alpha(P)}{c}\nabla P,\quad \boldsymbol{E}_t = 0 \qquad (2)$$

where $\boldsymbol{t}$ is the unit toroidal vector (Okamoto 1974, 1978). The poloidal current is given by

$$\boldsymbol{j}_p = \frac{c}{4\pi}\frac{d\beta}{dP}\boldsymbol{B}_p. \qquad (3)$$

We assume for definiteness that field lines emanating from a central object are open in the range of $0 \le P \le \bar{P}$ with antisymmetry between the upper and lower hemispheres, where $\bar{P}$ denotes the last field line open to infinity in the upper hemisphere. The total outward electric current from $P = 0$ to P is given by integrating Eq. (3)

$$I_p(P) = \int \boldsymbol{j}_p\cdot d\boldsymbol{A} = c\int_0^P \frac{d\beta}{dP}dP = c\,[\beta(P)-\beta(0)] \qquad (4)$$

where $d\boldsymbol{A}$ is the surface element that open poloidal field lines penetrate normally (Okamoto 1974). We have to impose charge neutrality of the object as a whole in the steady state, that is, $I_p(\bar{P}) = 0$, and thus we have $\beta(\bar{P}) = \beta(0) = 0$ (note that $\beta(P) \le 0$).

All of energy and angular momentum are transported by the Poynting flux all the way to the end of the force-free region without dissipation. The rates of changes of the central object in energy and angular momentum are given by

$$\dot{J} = \int_0^{\bar{P}} \beta(P)dP,\quad \dot{E} = \int_0^{\bar{P}} \alpha(P)\beta(P)dP \qquad (5)$$

(see Okamoto 1974).

The pulsar equation certainly does not seem directly to imply particular functional forms for both $\alpha(P)$ and $\beta(P)$. The angular velocity of field lines has however been usually regarded as fixable as the angular velocity of the perfectly conducting region from which the field lines emanate. In the pulsar case, for example, $\alpha(P)$ can be taken as the angular velocity of a rigidly rotating neutron star. On the other hand, Michel (1982) observed that β is completely arbitrary except that $\beta(0) = 0$, to avoid a singular current along the rotation axis $P = \varpi = 0$, and $\beta(\bar{P}) = 0$, because the total electric current must be null. Sulkanen & Lovelace (1990) stated that, beyond these restrictions, it has been assumed previously that there are no other physical constraints on the form of β. Then in order to remove an indeterminacy from the pulsar equation, they derived it from a variational principle that minimizes the electromagnetic field energy subject to the constraints of the fixed total angular momentum and total magnetic helicity. They obtained a piecewise linear partial differential equation for the pulsar equation.

Prior to this, Lovelace et al. (1987) looked for solutions of the Grad-Shafranov equation in the force-free limit and, based on its linearized version, claimed that self-collimated jets are only possible within the light cylinder (LC). On the other hand, Appl & Camenzind (1993a,b) claimed that physically meaningful self-collimated equilibria that extend beyond the LC are possible, under the assumption that the spatial distribution of the poloidal current $c\beta(P)/2$ is *given*. They derived the restrictions for $\beta(P)$ from Lovelace et al.'s equation for the plasma motion and imposed the 'regularity condition' at the LC. In subsequent papers (Fendt et al. 1995; Fendt & Camenzind 1996; Fendt 1996), $\beta(P)$ is treated as a *free* function, and a cylindrical shape is *a priori* assumed in the 'asymptotic'[1] force-free domains.

As we see in Eqs. (4) and (5), $\beta(P)$ is the crucial quantity which governs the electric current distribution and thereby the rates of energy and angular momentum losses from the central object. We argue here that $\beta(P)$ cannot be "free" but should be determined by loading the plasma inertia on the poloidal field line with P. The relation thus obtained in terms of α and the gradient of the flux function P itself naturally requires some sort of consistency to the pulsar equation for P with the term including $\beta(P)$ as seen in Eq. (1). This fact is reflected as a particular functional form for P, that is, as a function of θ only, indicating a *conical* field structure there.

The physical meaning and the limitation of the *force-free* treatment will be clarified in this paper. In Sect. 2 we derive the plasma conditions for $\beta(P)$ in the force-free treatment and also the same conditions from the force-free limit of the criticality condition at the fast magnetosonic point in the cold stellar wind theory. It is then shown that the plasma conditions are consistent to conical geometry in the *asymptotic* limit of the pulsar equation. In Sect. 3 we clarify physical implications of the plasma conditions in the force-free theory. One can then postulate existence of a dissipative membrane on the 'sphere at infinity', through which the Poynting flux would be converted to the kinetic energy of plasma motion. In Sect. 4 we make critical review of force-free models for cylindrical jets, and point out basic difficulties in the asymptotic domain. In the last section we summarize the results and mention the similar situations in the *black hole equation* (as opposed to the pulsar equation).

[1] In the following 'asymptotic' stands for the limit of $z \gg \varpi$, while *asymptotic* denotes the limit of $\varpi \gg \varpi_{\rm L}$.

2. The plasma conditions

We have so far tacitly presumed that all of field lines are to cross the LC (Okamoto 1974). Now let us argue that this must be the case at least in the force-free treatment in the theory of stellar winds from rapidly rotating objects. Needless to say that the toroidal component of magnetic field is a component bent and swept back by loading the inertia of outgoing particles from the poloidal component. In the force-free approximation one usually neglects inertial terms in the energy and momentum equations. This means that the plasma inertia is thrown away beyond the LC onto the 'sphere at infinity' (Phinney 1982). Then the presence of the toroidal component automatically indicates that the field lines related must reach the 'sphere at infinity' at $\varpi \gg \varpi_{\rm L}$. If the field shape were cylindrical in the domain of $\varpi \lesssim \varpi_{\rm L}$, it can be shown in Eq. (2) that for $\beta(P) \neq 0$ the toroidal field would remain finite for $z \gg \varpi$, or even for $z \to \infty$. This situation would be unphysical unless $\beta(P) \approx 0$ for cylindrical field lines in $\varpi \lesssim \varpi_{\rm L}$.

Before considering a means of making massless particles carrying charges in the force-free region appreciate their inertia far beyond the LC, we remark that the condition of $\beta(0) = \beta(\bar{P}) = 0$ requires that $\beta(P)$ has a minimum at $P = P_c$ (say), and that $\boldsymbol{j}_{\rm p} < 0$ in $0 < P < P_c$ and $\boldsymbol{j}_{\rm p} > 0$ in $P_c < P < \bar{P}$ in Eq. (3). Because we are assuming perfect conductivity in the force-free region, we cannot expect electrons to migrate equatorwards and positrons to migrate polewards across field lines at finite distances. Then when determining $\beta(P)$, we have also to contrive the *surface* return current flowing from the equatorial region to the polar region on the sphere at infinity.

In order for massless particles to restore their inertia, we require the plasma velocity to *asymptotically* tend to the light velocity with the Lorentz factor tending to infinity. The plasma velocity is given in terms of $B_{\rm p}$ and $B_{\rm t}$ by

$$\boldsymbol{v} = \boldsymbol{v}_{\rm p} + \left(v_{\rm p} \frac{B_{\rm t}}{B_{\rm p}} + \alpha\varpi \right) \boldsymbol{t} \tag{6}$$

under the 'frozen-in' condition, and the solutions for $v_{\rm p}$ become in terms of v, $B_{\rm p}$ and $B_{\rm t}$

$$\frac{v_{\rm p}}{c} = \frac{\alpha\varpi}{c} \frac{B_{\rm p}}{B} \left(-\frac{B_{\rm t}}{B} \pm \sqrt{\frac{v^2}{\alpha^2\varpi^2} - \frac{B_{\rm p}^2}{B^2}} \right). \tag{7}$$

(see Michel 1973b; Okamoto 1974, 1992). Eqs. (6) and (7) are completely equivalent to Lovelace et al.'s expressions (49a,b) and (50) and to Appl & Camenzind's equation (33). We require as the *asymptotic* condition

$$v \to c \quad \text{for } \varpi \gg \varpi_{\rm L} \tag{8}$$

and impose the condition that the radical tend to zero for $\varpi \gg \varpi_{\rm L}$, demanding that the two solutions, *physical* and *unphysical* solutions of Eq. (7) coincide each other, that is, infinity to be an *X*-type critical point. We have then $B_{\rm p}/B \to c/\alpha\varpi$, and see that the poloidal field $\boldsymbol{B}_{\rm p}$ decreases faster than the toroidal field $B_{\rm t} = \beta(P)/\varpi$. From this one has

$$\beta(P) = -\frac{\alpha(P)}{c}(B_{\rm P}\varpi^2)_\infty. \tag{9}$$

We refer to conditions (8) and (9) as the *plasma conditions* in the *force-free* magnetosphere. These are equivalent to the demand that $v_{\rm t} \to 0$ and $v_{\rm p} \to c$ for $\varpi \to \infty$ in Eq. (6).

Now let us show that the plasma conditions can be derived by taking the massless limit of the criticality condition at the fast magnetosonic point at infinity for cold relativistic winds. Okamoto (1978) extended Michel's (1969) cold wind theory with split-monopolar geometry to the one with general field topology. The criticality condition yields [see equations (6.1), (6.2) and (5.24a,b) in Okamoto (1978)]

$$\beta(P) = -\frac{\alpha(P)}{c}(B_{\rm P}\varpi^2)_\infty\lambda, \quad \gamma_\infty = (1+\sigma^{2/3})^{1/2},$$

$$\lambda = \frac{1}{\sigma}\left[(1+\sigma^{2/3})^{3/2} - \mu(P)\right], \tag{10}$$

where γ_∞ is the Lorentz factor at the magnetosonic point at infinity and $\mu(P)$ is the energy integral of the momentum equation. Also σ is Michel's magnetization parameter defined at the magnetosonic point at infinity

$$\sigma \equiv \frac{\alpha^2}{4\pi c^3\eta(P)}(B_{\rm p}\varpi^2)_\infty, \quad \eta(P) \equiv \frac{\rho v_{\rm p}}{B_{\rm p}}. \tag{11}$$

Now one can easily see that the massless limit of Eqs. (10) produces the plasma conditions (8) and (9), that is, taking $\eta(P) \to 0$, and hence $\sigma \to \infty$ yield $\lambda \to 1$ and $\gamma_\infty \to \infty$ in Eqs. (10) (see also equations (7.4a,b) and (10.1a,b) in Okamoto 1978).

Next we examine the *asymptotic* behavior of the pulsar equation (see Macdonald & Thorne 1982; Okamoto 1992 for the similar behavior of the black hole equation). For $\varpi \gg \varpi_{\rm L}$, Eq. (1) reduces to

$$\boldsymbol{\nabla}^2 P - \boldsymbol{\nabla}\ln B_{\rm p}\varpi^2\cdot\boldsymbol{\nabla}P = \frac{c^2}{2\alpha^2\varpi^2}\frac{\partial}{\partial P}\left[\beta^2 - \left(\frac{\alpha}{c}B_{\rm P}\varpi^2\right)^2\right]. \tag{12}$$

where $\partial/\partial P \equiv (1/B_{\rm P}^2\varpi^2)(\boldsymbol{\nabla}P\cdot\boldsymbol{\nabla})$. The left hand side is equal in the spherical coordinates to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) - \frac{1}{2\varpi^2}\frac{\partial}{\partial P}\left(\varpi\frac{\partial P}{\partial r}\right)^2.$$

If $\partial P/\partial r = 0$, i.e. $P = P(\theta)$ in the *asymptotic* domain, one has $B_{\rm P}\varpi^2 = \sin\theta(dP/d\theta)$ and then the right hand side of Eq. (12) reduces to

$$\frac{d}{dP}\left[\beta^2 - \left(\frac{\alpha}{c}B_{\rm P}\varpi^2\right)^2\right] = 0,$$

that is, to condition (9).

We expand the stream function P in the spherical coordinates far out from the LC

$$P(r,\theta) = P^{(0)}(\theta) + \frac{c}{\alpha r}P^{(1)}(\theta) + \frac{c^2}{\alpha^2 r^2}P^{(2)}(\theta) + \cdots \tag{13}$$

where $P^{(n)}(\theta)$'s ($n \geq 0$) have the same dimension as $P(r,\theta)$. Then one has

$$\beta(P) = -\frac{\alpha(\theta)}{c}\frac{dP^{(0)}(\theta)}{d\theta}\sin\theta. \tag{14}$$

Expression (14) coincides with the solution which Michel (1991) has given in the 'far-zone limit' where the field becomes asymptotically radial.[2]

It will be worthwhile remembering that an exact solution of Eq. (1) is given by

$$P(r,\theta) \equiv P_c(1-\cos\theta) \tag{15}$$

and

$$\beta(P) = -\frac{\alpha}{c}B_{\rm p}\varpi^2 = -\frac{\alpha}{c}P_c\sin^2\theta = -\frac{\alpha}{c}P\left(2-\frac{P}{P_c}\right). \tag{16}$$

Eq. (15) is valid not only at infinity but everywhere in the force-free magnetosphere with $d\alpha/dP = 0$. Note that $\beta(0) = \beta(\bar{P}) = 0$ where $\bar{P} = 2P_c$. The field structure is everywhere radial, and therefore the plasma motion is also radial, with velocity equal to the light velocity everywhere (Michel 1973b; Okamoto 1974).

We need make remark here that condition (9) allows not only conical geometry but also cylindrical geometry in the *asymptotic* domain. It can be shown that the left hand side of Eq. (12) is equal to $B_{\rm P}(\varpi/R)$, where R is the curvature radius of a given point on a field line. If the right hand side of Eq. (12) vanishes, it requires $R = \infty$, that is, the field line be straight at least in the vicinity of the point. Heyvaerts & Norman (1989) called the condition corresponding to Eq. (9) in nonrelativistic winds the *solvability* condition for *conical* geometry (see their equation (25) or (26)), but their condition itself allows not only conical but cylindrical geometry in the domain far out from the Alfvén point. In our case too, instead of Eq. (14), it is possible to choose

$$\beta(\varpi) = -\frac{\alpha\varpi^2}{c}B_z = -\frac{\alpha\varpi}{c}\frac{dP}{d\varpi} \tag{17}$$

for $\varpi \gg \varpi_{\rm L}$, which integrates to yield

$$\ln\frac{\varpi}{\varpi_{\rm a}} = -\frac{1}{c}\int_{P_{\rm a}}^{P}\frac{\alpha(P)}{\beta(P)}dP \tag{18}$$

[2] There are some misprints in his equations (38) and (39) of Chapter 4, p. 260. These equations should read

$$(1-\mu^2)^2 f'' - 2\mu(1-\mu^2)f' - \tfrac{1}{2}\frac{dA^2}{df} = 0, \tag{fn1}$$

and

$$A = -(1-\mu^2)f'. \tag{fn2}$$

Then, Eq. (fn1) can be transformed to $(d/d\mu)[A^2 - (1-\mu^2)^2 f'^2] = 0$, which reproduces Eq. (fn2), i.e. Eq. (14).

where $P_{\rm a}$ is the field line at $\varpi = \varpi_{\rm a}$ which is in the innermost of the *asymptotic* domain. One would construct the field structure with cylindrical geometry in some regime in the *asymptotic* domain, but may not cover the whole domain with it, because of difficulty in satisfying the charge neutrality condition of $\beta(0) = \beta(\bar{P}) = 0$. Also the cylindrical regime could not probably match with the conical regime (cf. Heyvaerts & Norman 1989). It also seems to be difficult to set up a dissipative layer to convert the Poynting flux to the kinetic flux in the cylindrical structure.

3. Implications of the plasma conditions

One can now make use of Eq. (9) or (14), to derive the *asymptotic* expressions of significant quantities. From Eq. (2) one has

$$\boldsymbol{E}_\theta = \left(\frac{\beta}{\varpi}\right)_\infty \boldsymbol{e}_\theta, \quad \boldsymbol{B}_{\rm t} = \left(\frac{\beta}{\varpi}\right)_\infty \boldsymbol{t}. \tag{19}$$

The surface charge density on the sphere at infinity is given by terminating all the electric fluxes that intersect the sphere, i.e.

$$\sigma_\infty = -\left(\frac{E_{\rm r}}{4\pi}\right)_\infty = -\left(\frac{P^{(1)}(\theta)}{4\pi r^2}\right)_\infty . \tag{20}$$

One can also define the surface electric current flowing across the unit length of the circle of $2\pi\varpi$ with the same P from the equator side to the pole side by normalizing $c\beta(P)/2$ with $2\pi\varpi$, i.e.

$$\mathscr{I}_\infty = \left(\frac{c\beta(P)}{4\pi\varpi}\right)_\infty \boldsymbol{e}_\theta = \frac{c}{4\pi}\boldsymbol{E}_\theta. \tag{21}$$

This surface current terminates the tangential component of magnetic field.

It then turns out from Eqs. (19) that $\boldsymbol{E}_\theta$ and $\boldsymbol{B}_{\rm t}$ satisfy the relation $\boldsymbol{B}_{\rm t} = \boldsymbol{n}\times\boldsymbol{E}_\theta$, where $\boldsymbol{n}$ is the unit outward normal vector on the sphere at infinity. This is nothing but the 'outgoing-wave' or 'radiative' boundary condition. Eq. (21) indicates that the surface electric current flows polewards, driven by the electric field $\boldsymbol{E}_\theta$ against the surface resistivity $R_\infty = 4\pi/c = 377$ Ohm.

In derivation of $\beta(P)$, we have used the condition of $v \to c$, to make the force-free approximation break down at $\varpi \gg \varpi_{\rm L}$, and to restore the inertia to massless charged particles. This leads automatically to requirement of break-down of the frozen-in condition, and finite electric conductivity enables charged particles to cross the radial field lines on the sphere at infinity, following Ohm's law. The resulting Joule dissipation of the surface electric current equals the total Poynting flux

$$\oint R_\infty |\mathscr{I}_\infty|^2 dA_\infty = \frac{c}{4\pi}\oint (\boldsymbol{E}\times\boldsymbol{B})_\infty \cdot \boldsymbol{n} dA_\infty$$
$$= \frac{1}{c}\int_0^{\pi/2} \alpha(\theta)^2 \left(\frac{dP^{(0)}}{d\theta}\right)^2 \sin\theta d\theta \tag{22}$$

where dA_∞ is the surface element on the sphere at infinity. One can interpret this as conversion of the electromagnetic energy extracted from the central object to the kinetic energy of accelerated *non*-massless particles.

The location of the dissipation region is however at infinity, and one can replace this load region with such a membrane with electric resistivity as Thorne et al. (1986) introduced at the stretched horizon of a Kerr black hole (Okamoto 1992). One can think of this L-membrane (say) as a kind of substitute of the load to be imposed on the force-free region, thereby determining the current distribution flowing in the whole electric circuit (see Fig. 1).

One can also think of the surface Lorentz force acting on the L-membrane

$$\sigma_\infty \boldsymbol{E}_\theta + \frac{\mathscr{I}_\infty}{c} \times B_{\rm r}\boldsymbol{n}$$
$$= \frac{\alpha}{c}\left[\frac{P^{(1)}(\theta)}{4\pi r^2}(B_{\rm r}\varpi)\right]_\infty \boldsymbol{e}_\theta - \left(\frac{\beta(P)}{4\pi\varpi}B_{\rm r}\right)_\infty \boldsymbol{t} \tag{23}$$

(see equation (8.26) in Okamoto 1992).[3] The surface torque exerted on the L-membrane per unit time is given by multiplying the lever arm length ϖ and integrating on the whole of the surface, i.e.

$$-\oint \left(\frac{\beta(P)}{4\pi}B_{\rm r}\right)_\infty dA_\infty = -\frac{1}{2}\oint \beta(P)dP$$
$$= \frac{1}{c}\int_0^{\pi/2} \alpha(\theta)\left(\frac{dP^{(0)}}{d\theta}\right)^2 \sin\theta d\theta. \tag{24}$$

This is the rate of the angular momentum transfer from the force-free region to the load region. Finally we note in Eq. (23) that not only the $\boldsymbol{t}$-component but also $\boldsymbol{e}_\theta$-component of the surface Lorentz force are unbalanced, i.e. the first term of Eq. (23) $\neq 0$. If we require the total charges over the L-membrane to vanish, i.e. from Eq. (20)

$$\oint \sigma_\infty dA_\infty = -\int_0^{\pi/2} P^{(1)}(\theta)\sin\theta d\theta = 0, \tag{25}$$

then $P^{(1)}(\theta)$ and hence $\sigma_\infty(\theta)$ itself must change sign somewhere (say $\theta = \theta_c$). It turns out thus that if $P^{(1)} < 0$ and $\sigma_\infty > 0$ in $0 < \theta < \theta_c$, and $P^{(1)} > 0$ and $\sigma_\infty < 0$ in $\theta_c < \theta < \pi/2$, the surface Lorentz force acts polewards in the former domain and equatorwards in the latter. Further details must be clarified, explicitly including inertial terms.

For the exact monopole solution given in Eqs. (15) and (16), one has from Eq. (5)

$$\dot{J} = -\frac{2\alpha}{3c}P_c^2, \quad \dot{E} = -\frac{2\alpha^2}{3c}P_c^2. \tag{26}$$

Because $P^{(1)}(\theta) \equiv 0$ in Eqs. (15), there is no unbalanced component of the surface Lorentz force in the $\boldsymbol{e}_\theta$-direction.

The plasma conditions by which one can avoid singular behaviors of the plasma motion at infinity is consistent to the

[3] There is a trivial error in the second term of (8.26) in Okamoto (1992) for the surface Lorentz force; the last factor $(B_r\varpi^2)$ should be replaced by $(B_r\varpi)$.

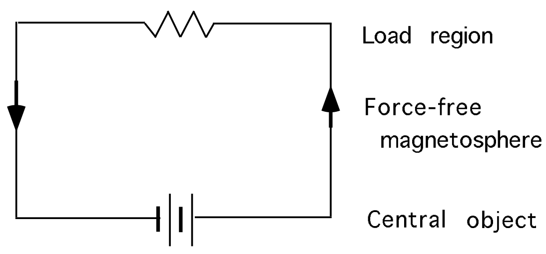

Fig. 1. The current flowing along wires of this DC circuit remains indeterminate, unless the resistance of the load to be imposed is specified.

asymptotic behavior of the field in the pulsar equation that the stream function be a function of θ only. The field shape must be *conical* at infinity, and the force-free structure naturally fills all space. Thus one cannot expect self-collimation of field lines due to the pinching effect in the force-free region into a cylindrical shape, unless one resorts to some confining external forces.

4. Force-free region with a cylindrical shape

Heyvaerts & Norman (1989) considered the asymptotic structure of the transfield equation for non-relativistic winds, and Chiueh et al. (1991) extended their results to special-relativistic winds. These authors showed that the flux surfaces either collimate to current-carrying cylinders or to current-free paraboloids. The point is that it is in the *asymptotic* domains of winds having passed through the three critical points that Heyvaerts & Norman's and Chiueh et al.'s results exactly hold.

Appl & Camenzind (1993a) studied the 'asymptotic' pulsar equation for relativistic jets in the assumption that the spatial distribution of the poloidal current $c\beta(P)/2$ is *given*. They claimed that physically meaningful self-collimated equilibria that extend beyond the LC are possible contrary to Sulkanen & Lovelace (1990). They *adopted* a cylindrical field shape for the 'asymptotic' force-free region of $z \gg \varpi$ for the MHD winds. This seems to be an assumption led by Chiueh et al.'s result that current-carrying jets *asymptotically* collimate to cylinders. But, pointing out that Chiueh et al.'s field structures *fill all space only*, they emphasized that their jets carry a finite flux and have a finite distance.

Their 'asymptotic' transfield equation is obtained by just dropping the z-dependence of P in Eq. (1), that is,

$$\left(1 - \frac{\alpha^2\varpi^2}{c^2}\right)\frac{d^2P}{d\varpi^2} - \left(1 + \frac{\alpha^2\varpi^2}{c^2}\right)\frac{1}{\varpi}\frac{dP}{d\varpi} + \frac{1}{2}\frac{d\beta^2}{dP} + \frac{\alpha^2\varpi^2}{c^2}\frac{d\ln\alpha}{d\varpi}\frac{dP}{d\varpi} = 0 \tag{27}$$

which can be transformed to a similar form to Eq. (12),

$$\varpi\frac{dB_z}{d\varpi} + \frac{1}{2}\frac{d}{dP}\left[\beta^2 - \left(\frac{\alpha}{c}B_z\varpi^2\right)^2\right] = 0. \tag{28}$$

If we put α = constant, Eq. (27) and (28) are equivalent to Appl & Camenzind's 'asymptotic' equation (21) and (25). We stress here that Eq. (28) was obtained without taking the 'asymptotic' limit of $z \gg \varpi$ in Eq (1), whereas Eq. (12) was derived in the *asymptotic* limit and its left hand side vanishes for any form of $P(\varpi)$.

In the following subsections we consider the two cases for $\beta(P)$ in cylindrical geometry. One is the case with $\beta = -(\alpha/c)B_z\varpi^2$ and the other is the solution given by Appl & Camenzind (1993b). In the last subsection we discuss difficulties related to the cylindrical structure in the 'asymptotic' domain in these two cases.

4.1. The first case

Now let us see what we have if we require

$$\beta(P) = -\frac{\alpha}{c}B_z\varpi^2 \tag{29}$$

in Eq. (28) in spite of having not taken the *asymptotic* limit of $\varpi \gg \varpi_{\rm L}$. Obviously we have

$$\frac{dB_z}{d\varpi} = 0, \quad \text{i.e. } B_z = \frac{1}{\varpi}\frac{dP}{d\varpi} = B_{z,0}$$

and then

$$P = \frac{1}{2}B_{z,0}\varpi^2 + P_0,$$

$$\beta = -\frac{\alpha}{c}B_{z,0}\varpi^2 = -\frac{2\alpha}{c}(P - P_0), \quad B_{\rm t} = -\frac{\alpha\varpi}{c}B_{z,0} \tag{30}$$

where $B_{z,0}$ and P_0 are integration constants which must be fixed at the boundary to this 'asymptotic' region. We thus see that $\beta(P)$ must be a linear function of P, which means no return currents to close the current system in the cylindrical domain. Eq. (6) reduces to

$$\boldsymbol{v} = \boldsymbol{v}_{\rm p} + \alpha\varpi\left(1 - \frac{v_{\rm p}}{c}\right)\boldsymbol{t} \tag{31}$$

and solving this for $v_{\rm p}$, one has

$$\frac{v_{\rm p}}{c} = \frac{x^2}{1+x^2}\left[1 \pm \frac{1}{x}\sqrt{\left(1 + \frac{1}{x^2}\right)\frac{v^2}{c^2} - 1}\right] \tag{32}$$

where $x = \varpi/\varpi_{\rm L}$. If one confines the outflow to a jet with a finite distance with $x = \bar{x}$, then the two solutions for $v_{\rm p}$ cannot constitute an *X*-type crossing point even if one demands $v = c$ at $x = \bar{x}$. Thus the choice of Eq. (29) for $\beta(P)$ and the resultant field structure lead to no criticality condition to reproduce Eq. (29) for $\beta(P)$. Thus one cannot set up at a suitable location the transition zone or membrane where Ohm's law holds and the Poynting energy is transformed to the kinetic energy of plasma motion. In the context of DC circuit theory, one can not say that the leads of cylindrical field lines in the force-free magnetosphere can be connected smoothly to the load region, to ensure the current closure.

4.2. The second case

Following Appl & Camenzind (1993a), let us next discuss what comes from the cylindrical pulsar equation with β as a *free* function and with $d\alpha/dP = 0$. Then Eq. (27) reduces to

$$(1-x^2)\frac{dB_z^2}{dx} - 4xB_z^2 + \frac{1}{x^2}\frac{d}{dx}\left(\frac{\alpha\beta}{c}\right)^2 = 0 \tag{33}$$

which is the same as their equation (21). Using the variation of constants and imposing what they called the 'regularity' condition at the 'critical' point at $x = 1$, we arrive at their solution

$$B_z^2 = \frac{1}{(1-x^2)^2}\int_x^1 \frac{1-x^2}{x^2}\frac{d}{dx}\left(\frac{\alpha\beta}{c}\right)^2 dx,$$
$$B_z^2(1) = \frac{1}{4}\left[\frac{d}{dx}\left(\frac{\alpha\beta}{c}\right)^2\right]_{x=1} \tag{34}$$

where the value of B_z^2 at $x = 1$ is determined applying l'Hôpital's rule. They derived the constraint for $\beta(P)$ from the discriminant of the plasma equation being positive, i.e. from Eq. (7)

$$\beta^2 \geq \frac{\alpha^2}{c^2}\left(\frac{dP}{dx}\right)^2 \frac{\gamma^2(x^2-1)-1}{\gamma^2-1} \tag{35}$$

which corresponds to their equation (35). If one abandons a cylindrical shape, putting $\gamma \to \infty$ for $x \gg 1$ along a given field line, one has $\beta = -(\alpha/c)(B_{\rm p}\varpi^2)_\infty$, taking equality sign in Eq. (35). Instead, Appl & Camenzind (1993a) retained β still *free* and proceeded further, selecting the 'screw pitches' and 'reversed field pitches' for β.

In the next paper, Appl & Camenzind (1993b) obtained an exact solution of Eq. (27) with $d\alpha/dP = 0$

$$\frac{P(x)}{P_{\rm max}} = \frac{1}{b}\ln\left(1+\frac{x^2}{a^2}\right) \tag{36}$$

and

$$\frac{\beta(P)}{\beta_{\rm max}} = -\frac{1-e^{-bP/P_{\rm max}}}{1-e^{-b}} = -\frac{1}{1-e^{-b}}\frac{\frac{x^2}{a^2}}{1+\frac{x^2}{a^2}} \tag{37}$$

which satisfy $P = \beta = 0$ at $x = 0$ and $P = P_{\rm max}$, $\beta = \beta_{\rm max}$ at $x = x_{\rm jet} \equiv a\sqrt{e^b - 1}$, where a is related to b, $P_{\rm max}$ and $\beta = \beta_{\rm max}$ through g as follows:

$$a^2 = \frac{g_\infty}{g - g_\infty} \tag{38}$$

and

$$g \equiv \frac{1}{2}\left(\frac{c\beta_{\rm max}}{\alpha P_{\rm max}}\right)^2, \quad g_\infty \equiv 2\left(\frac{1-e^{-b}}{b}\right)^2. \tag{39}$$

They called g the coupling constant.

Appl & Camenzind (1993b) defined the jet by the set of nested flux surfaces, $0 \leq P \leq P_{\rm max}$, or $0 \leq x \leq x_{\rm jet}$. They gave to the parameter b a role of controlling the concentration of current to the axis. For $b \to 0$, one has $g_\infty \to 2$, $a^2 \to 2/(g-2)$ and $x_{\rm jet} \approx \sqrt{2b/(g-2)} \to 0$, while for $b \to \infty$, one has $g_\infty \to 0$, $a^2 \to 0$ and $x_{\rm jet} \approx \sqrt{2/g}\exp(b/2)/b \to \infty$, although they called the case of $b \ll 1$ 'diffuse pinch' and the case of $b \gg 1$ 'sharp pinch'. Then using Eqs. (4) and (37), one can easily evaluate the loss rates of angular momentum and energy in terms of $\beta_{\rm max}$, $P_{\rm max}$ and b

$$\dot{J} = -\beta_{\rm max}P_{\rm max}f(b), \quad \dot{E} = -\alpha\beta_{\rm max}P_{\rm max}f(b) \tag{40}$$

where

$$f(b) \equiv \frac{1}{1-e^{-b}} - \frac{1}{b} = \begin{cases}\frac{1}{2}, & \text{for } b \ll 1\\ 1, & \text{for } b \gg 1\end{cases} \tag{41}$$

(see Fendt 1996). This solution is extensively utilized in the numerical calculations of jet magnetospheres in the subsequent papers (Appl & Camenzind 1993b; Fendt et al. 1995: Fendt & Camenzind 1996; Fendt 1996).

4.3. Difficulties with 'asymptotic' *cylindrical shapes*

Now one can point out some difficulties in jet magnetospheres with a 'asymptotic' cylindrical shape in Appl & Camenzind's force-free treatment.

(i) The solutions for the plasma motion with Eqs. (36) and (37) for $P(x)$ and $\beta(P)$ become from Eq. (7)

$$\frac{v_{\rm p}}{c} = \sqrt{\frac{g_\infty}{g}}\frac{x}{\sqrt{\frac{g_\infty}{g}+x^2}} \cdot \left[\frac{x}{\sqrt{\frac{g_\infty}{g}+x^2}} \pm \frac{1}{x}\sqrt{\frac{v^2}{c^2} - \frac{\frac{g_\infty}{g}x^2}{\frac{g_\infty}{g}+x^2}}\right]. \tag{42}$$

If v is given, one can easily evaluate $v_{\rm p}/c$ at the jet boundary by substituting $x = \sqrt{(e^b-1)/\{(g/g_\infty)-1\}}$ from Eq. (38) along the physical and unphysical solutions. On the other hand, neglecting the angular velocity sufficiently far from the LC, Appl & Camenzind (1993b) calculated the poloidal jet velocity

$$\frac{v_{\rm p}}{c} \approx -\frac{1}{\beta}\frac{\alpha}{c}B_{\rm p}\varpi^2 = \sqrt{\frac{g_\infty}{g}}. \tag{43}$$

Obviously their procedure is not justified, for in order to obtain the result given in Eq. (43), one must take the *asymptotic* limit of $x \gg 1$ beyond the jet boundary in Eq. (42). The toroidal velocity becomes from Eq. (6)

$$\frac{v_{\rm t}}{c} = x\left[1 - \sqrt{\frac{g}{g_\infty}}\frac{v_{\rm p}}{c}\right] \tag{44}$$

and therefore for $x \gg 1$ one has certainly $v_{\rm t}/c \approx (g_\infty/g)(1/x) \approx 0$ and $v_{\rm p}/c \approx (v/c) \approx \sqrt{(g_\infty/g)}$, and the radical in Eq. (42) *asymptotically* vanishes. Then, if one considers cylindrical field lines well beyond the jet radius, the plasma flow may thus have an *X*-type critical point, but with $v_{\rm p}/c \approx \sqrt{(g_\infty/g)} < 1$ unless $g \approx g_\infty$ and hence $a \to \infty$ and $x_{\rm jet} \to \infty$.

If, following Appl & Camenzind (1993b), one confines outflow inside their jet boundary, one cannot use the criticality condition of $v_{\rm p}/c \to 1$ for $\varpi \gg \varpi_{\rm L}$, which is necessary for transforming the electromagnetic energy to the kinetic energy of the plasma jet.

(ii) In their solution (36) for $P(x)$ with Eq. (37) for $\beta(P)$, one cannot connect their current distribution to a plausible load region where the electromagnetic energy has to be dissipated to become the kinetic energy of plasma motion. This disability to make use of Ohm's law will lead to indeterminacy of $\beta_{\rm max}$, and therefore b, $x_{\rm jet}$, $\dot{J}$ and $\dot{E}$. As shown in Eq. (41), parameter b gives rise to an ambiguity of factor 2 in the extraction rates of angular momentum and energy from the central object, but one perhaps cannot fix b without consulting with astrophysical loads outside the force-free region.

It thus turns out that it is difficult to match both distributions given in Eq. (29) and (37) in the 'asymptotic' cylindrical geometry to a plausible model of load region, to ensure the current closure.

(iii) We point again out that to produce the toroidal component, loading of the plasma inertia is necessary on the poloidal field lines, and to fix the poloidal current distribution, one need to connect the field lines to the load region at $\varpi \gg \varpi_{\rm L}$, where one can presume the L-membrane with finite electrical conductivity. But one cannot expect to fulfill these basic requisites for the force-free field structure of jets with a cylindrical shape.

(iv) Appl & Camenzind's expression $\beta(P)$ in Eq. (37) obviously does not satisfy $\beta(P_{\rm max}) = \beta(0) = 0$, that is, one has $\beta(P_{\rm max}) = -\beta_{\rm max}$. The extraction rates given in Eq. (40) are obtained by integrating $\beta(P)$ and $\alpha\beta(P)$ from $P = 0$ to $P = P_{\rm max}$. If $\beta(P_{\rm max}) = 0$, the areas covered would surely differ from those given in Eq. (40) and (41). Also, in order to keep charge neutrality of the entire system, the central object will need some equipment to realize a compensating return current, through perhaps a *non*-force-free, and hence dissipative, region. This return current may considerably reduce the extraction rates from the central object, thereby giving rise to uncertainties in the theory.

Recently Fendt (1996) solved the black hole equation with Eq. (37) for the poloidal current distribution, that is, for such a boundary condition that the flux function tend to the 'asymptotically' collimated cylindrical shape given in Eq. (36). His numerical calculations for 'collimated jet magnetospheres around rotating black holes' will not be free from such difficulties as described above.

5. Discussion

One will be able to attach a force-free magnetosphere to a rotating central object, to extract its angular momentum and energy in the form of electric current. The amount of current to flow through the non-resistive force-free region must be fixed by connecting to astrophysical loads, that is, by the plasma conditions given in Eqs. (8) and (9), which require the flow to be non-singular at the *asymptotic* zone of $\varpi \gg \varpi_{\rm L}$. One can regard this *asymptotic* zone as located at the 'sphere at infinity' or the L-membrane having finite electrical resistivity (see Fig. 1). The Joule dissipation of the current on the L-membrane can be interpreted as indicating conversion of energy from electromagnetic form to kinetic form of plasma motion. It thus seems to be possible to construct a self-consistent theory of extracting rotational energy from the central object within the framework of force-freeness.

It will be worthwhile emphasizing here that there is certainly at least one *indeterminate* or *free* function in the force-free magnetosphere, but it is not the poloidal current distribution $\beta(P)$, but the stream function at infinity as a function of θ, i.e. $P^{(0)}(\theta)$ related to β as given in Eq. (14). This indeterminacy is not a fault of force-free theory, but just a limitation resultant from throwing the plasma inertia away beyond the LC to infinity.

The function $P^{(0)}(\theta)$ is in reality not *indeterminate*, but can and must be determined from physical considerations of the domain of field-plasma interactions, such as acceleration of *non*-massless particles and collimation of field and/or stream lines outside the force-free region, and also from possible observations of the extraction rates expressed in Eq. (22) and (24) as integrated quantities of $P^{(0)}(\theta)$ over open field lines.

The same is true for function $P^{(1)}(\theta)$ with a factor $(1/r)$ in the expansion of $P(r, \theta)$ in Eq. (13). The zeroth order function $P^{(0)}(\theta)$ indicates the *conical* field topology on the L-membrane, but the first order function $P^{(1)}(\theta)$ shows a sign of collimation or divergence of field and/or stream lines even in the force-free treatment.

One can also install a force-free magnetosphere to a rotating black hole (plus the surrounding accretion disc). The related equation is the pulsar equation with general-relativistic effects taken into account through inertial-frame-dragging factors α and ω (Blandford & Znajek 1976: Macdonald & Thorne 1982: Thorne et al. 1986: Okamoto 1992). We refer to this as the *black hole equation*, as opposed to the pulsar one. This equation of course contains two unknown functions, $\Omega_{\rm F}(\equiv \alpha)$ and $I(\equiv -c\beta/2)$ as functions of $\Psi(\equiv 2\pi P)$. In the former one must fix not only the current distribution but also the angular velocity of field lines by the two loads on the two membranes, the H- and L-membranes, with finite resistivity $R_{\rm H} = R_{\rm L} = 4\pi/c$. The L-membrane is situated on the 'sphere at infinity' and the H-membrane is at the stretched horizon (Thorne et al. 1986; Okamoto 1992). The plasma conditions at both membranes yield

$$I(\Psi) = \frac{1}{2}\Omega_{\rm F}(B_{\rm P}\varpi^2)_{\rm L} = \frac{1}{2}(\Omega_{\rm H} - \Omega_{\rm F})(B_{\rm P}\varpi^2)_{\rm H} \qquad (45)$$

from which one has

$$\Omega_{\rm F}(\Psi) = \frac{\Omega_{\rm H}}{1+\zeta}, \quad I(\Psi) = \frac{1}{2}\frac{\zeta}{1+\zeta}\,\Omega_{\rm H}(B_{\rm P}\varpi^2)_{\rm H} \qquad (46)$$

where $\zeta(\Psi) \equiv (B_P\varpi^2)_L/(B_P\varpi^2)_H$. One can transform Eqs. (45) into two expressions in form of Ohm's law on the resistive membranes

$$\left(\frac{I(\Psi)}{2\pi\varpi}\right)_L = \frac{E_L}{R_L}, \quad \left(\frac{I(\Psi)}{2\pi\varpi}\right)_H = \frac{E_H}{R_H} \tag{47}$$

where $E_L = (\Omega_F/c)(B_P\varpi)_L$ and $E_H = [(\Omega_H - \Omega_F)/c]\,(B_P\varpi)_H$ are the surface electric field, polewardly directed on the L-membranes and equatorwardly on the H-membrane, respectively. One can then close the circuit of the electric current system. ζ is referred to as a parameter expressing the degree of collimation of field lines in the force-free black hole magnetosphere (Okamoto 1992). The output power is usually maximized with impedance matching, i.e. $\zeta = 1$ (Blandford & Znajek1976: Macdonald & Thorne 1982: Thorne et al. 1986). But this is far from obvious in the actual situation, and one may need carefully determine ζ, i.e. $(B_P\varpi^2)_L$ as functions of Ψ, taking into account inertial effects.

In conclusion, the poloidal current distribution not only in the pulsar equation but also in the black hole equation is not a *free* function of the flux function, but can be determined just as the electric current in a DC circuit consisting of a battery, leads and a load. This poloidal current carried by the wind is the decisive quantity which controls the asymptotic shape of field lines in the nonrelativistic case (Heyvaerts & Norman 1989). There is no reason that this should not hold in the relativistic force-free case, for which the shape must be *conical* on the L-membrane substituting astrophysical loads, just as on the H-membrane having internal resistivity in a black hole battery.

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