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Collapse spectra of cloud turbulence

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Abstract. The collapse spectra of cloud turbulence are obtained from the governing equations of the self-gravitating system. The spectra can be used to explain the observed scaling behaviour of the internal velocity dispersions and macrostructural density inhomogeneity in molecular clouds.

Key words: hydrodynamics – turbulence – ISM: clouds

1. Introduction

For the ensemble of molecular clouds, it has been noted that there are power law scalings for the internal velocity dispersions with cloud size l:$v_l \propto l^{0.5}$ and macro-density inhomogeneity: $\rho_l \propto l^{-1}$ (Myers 1983). The observed velocities are mostly supersonic and the Reynolds numbers are large, even at the smallest scales. This has raised the question of explaining the rapid dissipation for the turbulent energy (Henrikson & Turner 1984; Fridman 1988; Dolotin & Fridman 1990). Understanding the spatial structure of molecular cloud complexes is essential for an understanding of star formation. It has been argued that a giant molecular cloud may be treated as a cascade of gravitationally-driven compressible turbulence, between the tidal disruption scale and the stellar formation scale (Henrikson & Turner 1984; Henrikson 1991). In this opinion, each cloud is an element of a larger one and contains subunits, as the result of a self-similar turbulent cascade through collapse and hierarchical fragmentation. The concept of a cloud thus becomes an entity of a preferred velocity or gravitational correlation length. The power law forms $v_l \propto l^{0.5}$ and $\rho_l \propto l^{-1}$ are considered as the consequence of a turbulent spectrum similar to that of Kolmogorov in the inertial range of hydrodynamic turbulence.

In this paper, we will extract information from the analysis of characteristic variations of the terms in the governing equations describing the nonlinear evolution of a self-gravitating system and construct a self-similar collapse solution that can be used to derive the turbulent spectra. It is shown that nonlinear gravity and hydrodynamical effects lead to collapse and fragmentation that may explain the observed scaling dependences of velocity dispersions and macro-density inhomogeneity in molecular cloud complexes.

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In Sect. 2, the governing equations of a self-gravitating system are obtained from fluid equations, using the two time-scale method. In Sect. 3, we present the process of deriving the self-similar collapse solution and deduce the turbulence spectra. The discussion and conclusions are given in Sect. 4.

2. Basic governing equations

The governing equations of a self-gravitating system in a stable mode have been obtained in the case of two-component, say, gas and dust (Zhang & Li 1995). Actually, this two-component restriction is not necessary, we can also derive the governing equations from single-fluid equations. The main procedure is the same.

The starting equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ , \tag{2.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{g} + \mathbf{a} - \frac{1}{\rho}\nabla P \ , \tag{2.2}$$

$$\nabla \cdot \mathbf{g} = -4\pi G \rho \ , \tag{2.3}$$

$$\nabla \times \mathbf{g} = 0 \ , \tag{2.4}$$

$$P = P(\rho) \tag{2.5}$$

where ρ is the matter density, P the pressure, $\mathbf{v}$ the velocity, and $\mathbf{g}$ the gravitational acceleration, while $\mathbf{a}$ is the non-gravitational acceleration.

As done by Zhang and Li(1995), we will analyse the above starting equations in the two time-scale method. Any quantity A is divided into a slow time-scale part A^s and a fast time-scale part A^f

$$A = (\rho, P, \mathbf{v}, \mathbf{g}) = A^s + A^f , \tag{2.6}$$

and the assembled mean of the fast time-scale part vanishes over the slow time-scale.

$$\langle A^f \rangle_s = 0 \ . \tag{2.7}$$

In order that the unperturbed state satisfy the basic equation, a non-gravitational acceleration **a**, such as the centrifugal acceleration due to self-rotation of the cloud, is introduced and maintains the relation

$$\mathbf{a} + \mathbf{g}^s = 0 \ . \tag{2.8}$$

The evolution equations of the slow part and fast part of the density can be derived from the continuity Eq. (2.1) by means of averaging and subtracting,

$$\frac{\partial \rho^s}{\partial t} + \nabla \cdot [\rho^s \mathbf{v}^s + \langle \rho^f \mathbf{v}^f \rangle_s] = 0 \ , \tag{2.9}$$

$$\frac{\partial \rho^f}{\partial t} + \nabla \cdot [\rho^s \mathbf{v}^f + \rho^f \mathbf{v}^s + \rho^f \mathbf{v}^f - \langle \rho^f \mathbf{v}^f \rangle_s] = 0 \ . \tag{2.10}$$

In this case of the fluid description, fluid velocity is much smaller than the phase velocity of the mode, $v \ll v_\phi \equiv \omega/k$ which can be supersonic for the driven perturbation mode of interest to us. If we consider the perturbations of the form e^{iz}, along the x-direction, $z = x - Mt$ with $M = \frac{v_\phi}{c}$, from the driven sound Eq. (2.31) below, we then find $M^2 - 1 = |\mathbf{g}|^2/n > 0$, with proviso that $n > 0$ (Zhang & Li 1995), although then the fluid velocity is less then sound velocity(Zakharov 1984). Then we have the following relations

$$\left| \frac{\nabla \cdot (\rho^f \mathbf{v}^f)}{\partial \rho^f / \partial t} \right| \sim \frac{k}{\omega} v^f \sim \frac{v^f}{v_\phi} \ll 1 \ , \tag{2.11}$$

$$\left| \frac{\nabla \cdot (\rho^f \mathbf{v}^s)}{\partial \rho^f / \partial t} \right| \sim \frac{k}{\omega} v^s \sim \frac{v^s}{v_\phi} \ll 1 \ , \tag{2.12}$$

Thus, Eq. (2.10) can be simplified to

$$\frac{\partial \rho^f}{\partial t} + \nabla \cdot (\rho^s \mathbf{v}^f) = 0 \ . \tag{2.13}$$

With the aid of Eq. (2.13), we can show that

$$\frac{\rho^f}{\rho^s} \sim \frac{\mathbf{k}}{\omega} \cdot \mathbf{v}^f \leq \frac{v^f}{v_\phi} \ll 1 \tag{2.14}$$

and

$$\left| \frac{\rho^f v^f}{\rho^s v^s} \right| \ll 1 \ . \tag{2.15}$$

Then, the slow time-scale continuity Eq. (2.9) becomes

$$\frac{\partial \rho^s}{\partial t} + \nabla \cdot (\rho^s \mathbf{v}^s) = 0 \ , \tag{2.16}$$

Similarly, by examining the momentum Eq. (2.2) together with the principles of order-of-magnitude approximation, we find that the fast time-scale and the slow time-scale momentum equations are, respectively

$$\frac{\partial \mathbf{v}^f}{\partial t} = \mathbf{g}^f - \frac{c^2}{\rho^s} \nabla \rho^f \ , \tag{2.17}$$

$$\frac{\partial \mathbf{v}^s}{\partial t} + (\mathbf{v}^s \cdot \nabla) \mathbf{v}^s = -\frac{c^2}{\rho^s} \nabla \rho^s - \frac{1}{2} \nabla \langle (v^f)^2 \rangle_s \ , \tag{2.18}$$

where c is the sound velocity which is simply suggested to be a constant. From Eq. (2.3), we get the fast time-scale field equation

$$\nabla \cdot \mathbf{g}^f = -4\pi G \rho^f \tag{2.19}$$

Differentiating Eq. (2.19) with respect to time, and substituting $\mathbf{g}^f$ and $\partial \rho^f / \partial t$ by Eq. (2.17) and Eq. (2.13), we obtain the fast time-scale equation of motion

$$\frac{\partial^2 \mathbf{v}^f}{\partial t^2} - c^2 \nabla (\nabla \cdot \mathbf{v}^f) - \omega_0^2 \mathbf{v}^f - \omega_0^2 \mathbf{v}^f (\frac{\delta \rho}{\rho_0}) = 0 \ ; \tag{2.20}$$

In the above, we have written $\rho^s = \rho_0 + \delta\rho$, where $\delta\rho$ is the perturbation of the slow time-scale density, and $\omega_0 = (4\pi G \rho_0)^{1/2}$. The slow time-scale equation of motion can be derived from Eqs. (2.16) and (2.18) linearized with respect to **v** and $\delta\rho$; it is

$$\frac{\partial^2}{\partial t^2} (\frac{\delta \rho}{\rho_0}) + c^2 \nabla^2 (\frac{\delta \rho}{\rho_0}) + \frac{1}{2} \nabla^2 \langle (\mathbf{v}^f)^2 \rangle_s = 0 \ . \tag{2.21}$$

This equation is treated as the driven equation of the slow time-scale disturbed density by the ponderomotive force $\frac{1}{2} \nabla \langle (\mathbf{v}^f)^2 \rangle_s$. To transform Eqs. (2.20) and (2.21) into envelope equations, we suppose

$$\mathbf{v}^f = \frac{1}{2} [\mathbf{v}(\mathbf{r}, t) exp(i\omega t) + \mathbf{v}^*(\mathbf{r}, t) exp(-i\omega t)] \ , \tag{2.22}$$

$$\mathbf{g}^f = \frac{1}{2} [\mathbf{g}(\mathbf{r}, t) exp(i\omega t) + \mathbf{g}^*(\mathbf{r}, t) exp(-i\omega t)] \ , \tag{2.23}$$

where real ω is the main frequency of the mode and $\mathbf{v}(\mathbf{r}, t)$ and $\mathbf{g}(\mathbf{r}, t)$ are slowly varying amplitudes. On the other hand, from Eqs. (2.13) and (2.19), we obtain an approximate relation between $\mathbf{v}^f$ and $\mathbf{g}^f$

$$\frac{\partial \mathbf{g}^f}{\partial t} \simeq \omega_0^2 \mathbf{v}^f \ . \tag{2.24}$$

Then we have

$$\mathbf{v}(\mathbf{r}, t) \simeq \frac{i\omega}{\omega_0^2} \mathbf{g}(\mathbf{r}, t) \ , \tag{2.25}$$

and

$$\langle (\mathbf{v}^f)^2 \rangle_s = \frac{1}{2} |\mathbf{v}(\mathbf{r}, t)|^2 \simeq \frac{\omega^2}{2\omega_0^4} |\mathbf{g}(\mathbf{r}, t)|^2 \ . \tag{2.26}$$

Therefore, Eqs. (2.20) and (2.21) become (Zhang & Li 1995)

$$\frac{\partial^2 \mathbf{g}}{\partial t^2} + 2i\omega \frac{\partial \mathbf{g}}{\partial t} - c^2 \nabla^2 \mathbf{g} - \omega^2 (1 + \alpha^2) \mathbf{g} - \omega_0^2 \mathbf{g} (\frac{\delta \rho}{\rho_0}) = 0 \ ; \tag{2.27}$$

$$\frac{\partial^2}{\partial t^2}(\frac{\delta\rho}{\rho_0}) - c^2\nabla^2(\frac{\delta\rho}{\rho_0}) = \frac{\omega^2}{4\omega_0^4}\nabla^2|g|^2 , \quad (2.28)$$

where $\alpha^2 = \frac{\omega_0^2}{\omega^2}$. Making the following transformations

$$\mathbf{g}' = \frac{\mathbf{g}}{\mathbf{4}\omega_0\mathbf{c}} , \mathbf{r}' = \frac{2\omega}{c}\mathbf{r} , \ t' = 2\omega t ,$$

$$n = \frac{\alpha^2}{4}\frac{\delta\rho}{\rho_0} , \quad (2.29)$$

and dropping the small term $\partial^2\mathbf{g}/\partial t^2$ as $\mathbf{g}(\mathbf{r}, t)$ varies slowly: $|\frac{\partial \ln g}{\partial t}| \ll \omega$, we obtain the governing equations of the self-gravitating system in a stable mode

$$i\frac{\partial\mathbf{g}}{\partial t} - \nabla^2\mathbf{g} - \frac{1+\alpha^2}{4}\mathbf{g} - \mathbf{g}n = 0 , \quad (2.30)$$

$$(\frac{\partial^2}{\partial t^2} - \nabla^2)n = \nabla^2|\mathbf{g}|^2 . \quad (2.31)$$

where the marks $'$ have been omitted for simplicity. As regards Eqs. (2.30) and (2.31), they could be used to describe the nonlinear structures and evolution of the cloud complexes.

3. Self-similar collapse solution

It is shown (Zhang & Li 1995) that the system may collapse, and the collapses are anisotropic. Hence pancake-like structures can be formed. Now let us find a self-similar solution of those equations, assuming that the nonlinear entity has a pancake form with a scale ξ and radius $R \gg \xi$. Putting $\mathbf{g} = \mathbf{a}e^{i\varphi}$, they then become

$$\mathbf{a}\frac{\partial\varphi}{\partial t} + \nabla^2\mathbf{a} - (\nabla\varphi)^2\mathbf{a} + \frac{1+\alpha^2}{4}\mathbf{a} + \mathbf{a}n = 0, \quad (3.1)$$

$$\frac{\partial\mathbf{a}}{\partial t} - \mathbf{a}\nabla^2\varphi - 2(\nabla\varphi\cdot\nabla)\mathbf{a} = 0 , \quad (3.2)$$

$$(\frac{\partial^2}{\partial t^2} - \nabla^2)n = \nabla^2 a^2 . \quad (3.3)$$

We assume that the φ and a have characteristic time-space scales (τ, ξ). For supersonic motion ($\tau \ll 1$), it is possible to neglect the term $\nabla^2 n$ in Eq. (3.3). Then one has

$$n \sim \frac{\tau^2}{\xi^2}a^2 , \quad (3.4)$$

where $\tau = t_0 - t$, and t_0 corresponds to the singularity (at which, as mentioned earlier, the perturbed fields become too strong for Eqs. (2.30) and (2.31) to be valid).

We write the order of magnitudes of terms in Eqs. (3.2) and (3.1):

$$\frac{1}{\tau} \approx c_1\frac{\varphi(y)}{\xi^2}, \ (y = y(\xi,\tau)), \quad (3.5)$$

$$\frac{1}{\xi^2} \approx c_1'\frac{a^2\tau^2}{\xi^2} + c_2'\frac{\varphi(y)}{\tau} + c_3'\frac{\varphi^2(y)}{\xi^2} + c_4'\frac{1+\alpha^2}{4}, \quad (3.6)$$

where $c_1, c_1', \cdots$ are constants of order unity. It follows from Eq. (3.5)

$$\varphi(y) \approx c_2\frac{\xi^2}{\tau}. \quad (3.7)$$

Multiplying Eq. (3.6) by the factor $(\frac{\xi^2}{\tau^2})$ and using Eq. (3.7) yield

$$\frac{1}{\tau^2} \approx \frac{1}{\tau^2}[c_1'a^2\tau^2 + c_2''\frac{\xi^4}{\tau^4}\tau^2 + c_3''\frac{1+\alpha^2}{4}\frac{\xi^2}{\tau^2}\tau^2], \quad (3.8)$$

where c_2'' and c_3'' are also constants of order unity. Thus we obtain from Eq. (3.8) for supersonic collapse ($\tau \ll 1$):

$$a \sim \frac{1}{\tau} \quad (3.9)$$

provided that the last two terms in Eq. (3.8) are balanced, which implies

$$\xi \sim \tau(1+\alpha^2)^{\frac{1}{2}}. \quad (3.10)$$

In addition, the conservation of action (Zhang & Li 1995)

$$\int |\mathbf{g}|^2 d\mathbf{r} = const.$$

gives $a^2\xi\pi R^2 = const.$, i.e.,

$$a\sqrt{\xi}R = const.. \quad (3.11)$$

We can try to find a solution in the form

$$a = \tau^\mu F(\frac{\xi^\beta}{\tau}); \quad (3.12)$$

Eqs. (3.9) and (3.11) become

$$\tau^{\mu+1}F(\frac{\xi^\beta}{\tau}) = const.;$$

$$\frac{\sqrt{\xi}R}{\tau}F(\frac{\xi^\beta}{\tau}) = const..$$

In consideration of Eq. (3.10), we find

$$\mu = -1, \ \beta = 1, \ R \sim \sqrt{\tau}.$$

Therefore, a self-similar solution can be asymptotically shown as

$$a \approx \frac{1}{\tau}F(\frac{\xi}{\tau} , \frac{R}{\sqrt{\tau}}), \quad (3.13)$$

where F is a function determined by the initial conditions.

As usual, the constancy of energy flow along the spectrum yields

$$W_k\frac{dk}{dt(k)} = const.$$

where W_k is the energy density of collapsing cavities within the turbulence scale interval $(k, \ k + dk)$, and the dependence of energy transport time on wavenumber $dt(k)$ is determined by the time of collapse within the given interval of scales. In the case that collapse occurs preferentially in the longitudinal direction, leading to elements of fragmentation with scale ξ, it follows from the self-similar solution (3.13) that $\xi \sim k^{-1} \sim \tau$, or

$$dt(k) \sim k^{-2}dk \ . \tag{3.14}$$

This allows the turbulence spectrum to be found within the scales of the inertial subrange:

$$W_k \sim k^{-2} \ . \tag{3.15}$$

The relation of the velocity dispersion v_k with turbulence spectrum W_k is written as (Panchev 1971)

$$v_k^2 \sim \int_k^\infty W_k dk \ ,$$

yielding $v_k^2 \sim k^{-1}$, or

$$v_l \sim l^{0.5} \ . \tag{3.16}$$

The turbulent cascade through collapse and fragmentation will lead to fragmentation of macrostructural density inhomogeneity into small-scale disturbances of the density field ρ. The fragmentation of the density inhomogeneity will thus result in that the total measure of the inhomogeneity will increasingly concentrate in small-scale disturbances. By nature, due to the turbulence mixing of portions of the fluid, the macrostructural density inhomogeneity may be considered as the result of scale averaging of disturbed density inhomogeneity of different scales. In other words, for the supersonic motion of interest to us in which $n \sim \delta\rho \geq 0$ (Zhang & Li 1995), we have

$$\rho_l \sim \overline{\delta\rho_\xi} = \frac{1}{\xi_0 - l}\int_l^{\xi_0} \delta\rho_\xi d\xi \ , \tag{3.17}$$

where ξ_0 is the characteristic length scale of the cloud (tidal disruption scale) and ρ_l the macrostructural density inhomogeneity of scale l -a typical change in the mean density over the scale l in the cloud ($\xi_0 \gg l$). Using Eq. (3.9), it follows from Eq. (3.4) that

$$n \sim \delta\rho_\xi \sim \xi^{-2}. \tag{3.18}$$

Thus, we finally obtain from Eq. (3.17)

$$\rho_l \sim l^{-1} \tag{3.19}$$

In addition, if we use the Navier-Stokes equation assuming that all terms are of the same order: $v\nabla v \sim g$, or $v_l^2/l \sim l4\pi G\rho_l$. From Eq. (3.16), one also gets the above correlation (Fridman 1988). In view of Eqs. (3.16) and (3.19), it follows that

$$P = \rho_l \cdot v_l^2 \sim const..$$

This is just the important assumption of Henrikson and Turner (1984).

We can see that the observed dependences for the internal velocity dispersions and the macrostructural density inhomogeneity of scale l are obtained. By use of the nonlinear governing equations for the unstable mode case (Li 1990; Li & Li 1992), similar results can also be found.

4. Discussion and conclusions

In a self-gravitating system, such as a large cloud consisting of gas and/or dust grains, modulational instabilities due to nonlinear interactions of gravity will occur; in such case, the system could collapse and fragment. As a result, a giant pancake-like shaped cloud complex, composed of larger and smaller subunits, can form.

We show that the cascade fragmentation by collapse may lead to structures of disturbed density concentration $\delta\rho$ in clouds for a dynamical admixture, as shown from Eq. (3.18). On the basis of correlation, we obtain the basic observed scaling law(3.19) for macrostructural density inhomogeneity of scale l.

For the region of scales where turbulent energy is conserved under cascade transport, we find from the cloud collapse law (3.14) the turbulence energy spectrum, which yields another observed dependence (3.16) for the internal velocity dispersions of scale l.

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