Astron. Astrophys. 327, 404–415 (1997)

ASTRONOMY AND ASTROPHYSICS

Finding “absolute” parallaxes directly

Heinrich Eichhorn

Osservatorio Astronomico di Torino, Pino Torinese, Italy*

Received 3 January 1996 / Accepted 17 June 1996

Abstract. We develop the formalism for estimating parallaxes, proper motions and positions from long-focus astrometric observations as well as additional extraneous reference material whenever this is available and relax the constraints on the star parameters (basically their sums and correlations with the coordinates on the frames), thus generating (albeit correlated) quasi-observations whose inclusion would make the solutions determinate even if there were no extraneous reference material. The result are data on the same system to which the reference material is referred, so that there is no need to perform the “reduction from relative to absolute” as an afterthought.

Key words: astrometry – methods: analytical; data analysis

1. Introduction

The traditional procedure for measuring trigonometric parallaxes with respect to a set of field stars at relatively small angular distances, introduced by Bessel was universally applied until the parallaxes measured by HIPPARCOS emerged in a quasi-absolute manner (Wielen et al., 1994). Schlesinger (1911) developed the computation of parallaxes from measurements on narrow-field photographic plates to a simple routine by use of his “dependences” which drastically reduce the arithmetic effort, albeit at the expense of some accuracy.

The thus obtained raw parallax estimates of the target stars were of course “relative,” since the field stars themselves are at finite distances and thus also have positive – albeit, so one expects, small – parallaxes. Therefore, some type of mean of the (somehow obtained) expected estimated parallaxes of the field stars had to be added to each raw, “relative” parallax to get the “absolute”[1] parallax, cf. van Altena (1974). In most cases, this added mean (the “correction from relative to absolute”) was the straight arithmetic mean of the parallax estimates of the field stars [although the correction should be correctly computed by summing the products of the estimated expected parallaxes times the appropriate dependences (Eichhorn, 1996)].

The necessary process of calculating the absolute parallax from the raw reduction of the measurements has occupied astrometrists for a long time, and one occasionally meets the desire to develop a reduction routine that circumvents the calculation of a relative parallax, cf. e.g. Gatewood (1987). Murray and Corben (1979) have, in their pioneering investigation on the wholesale determination of parallaxes in a field, constrained the parallaxes (in addition to the proper motion components) of a selected number of field stars – which thereby become the equivalent to what I call “reference stars” in this paper (cf. Sect. 3.2 below) – thus fixing the system of proper motions and parallaxes, achieving a “reduction to absolute.”

Much has changed since Schlesinger’s time: Ever more sophisticated and as accurate reduction routines are needed to take full advantage of the improvements by orders of magnitude in data acquisition and mensuration [MAP, cf. Gatewood (1987) and CCD, cf. Monet et al. (1992)]. There are fewer and fewer effects which influence the result of a parallax estimation small enough to be negligible, and it is therefore the purpose of this paper to establish a reduction routine that is as rigorous and accurate as possible.

The reduction of each frame[2] by itself independently of the others with the ultimate purpose of estimating the parallax of a star ignores the kinematic constraint that the true[3] coordinates of the same star’s images on the various different frames depend ideally only on a small number of “star parameters” (e.g. zero epoch position, proper motion components, parallax); it is therefore a waste of information to allow the coordinates of a star’s images to be unrestricted on each frame. This was first recognized by Eichhorn and Jefferys (1971) who critiqued the method of dependences from the standpoint of computer-supported calculation and suggested a reduction routine that enforces globally the appropriate constraints on the stars’ positions and motions. Even though their procedure unfortunately generates a singu-

* On leave from the Department of Astronomy, University of Florida

[1] This is somewhat of a misnomer, canonical though it may have become by decades of widespread usage. Parallax, *by its very definition*, is *eo ipso* absolute. I would have difficulties to suggest a rigorous *and simple* definition of the quantity known as “relative parallax,” i.e. the raw result of measurement in a typical narrow-angle frame which I would rather call *quasi-parallax*.

[2] A set of measurements of the coordinates of the stars’ images in the field, obtained (photographically, electronically or by whatever other means) during the same exposure

[3] in the technical sense of statistics and not of spherical astronomy

lar system of normal equations, it still converges toward good relative parallaxes, cf. Eichhorn (1975). This technique is essentially the simultaneous estimation of frame parameters and star parameters in the situation when all frames cover (practically) the same area of the sky. Eichhorn and Jefferys therefore named it the "Central Overlap Method." The singularity can lead to ambiguous results.

I have previously set up the equations of condition and solved the resulting normal equations for a less general case than the one treated here (Eichhorn, 1988). Most of the notation in the subsequent developments will be the same as in that paper and is also explained in the Appendix. In the present paper I give the developments only for the x-coordinate. Let it therefore be explicitly understood that completely analogous developments (with the customary notations) also hold for the y-coordinate.

The matrix $\boldsymbol{\xi} = (\xi_{\nu\mu})$ of the standard coordinates of the μ-th star's image on the ν-th frame is calculated from the "star parameters," the elements of the matrix $\mathbf{X} = (X_{\lambda\mu})$ by

$$\boldsymbol{\xi} = \boldsymbol{\Gamma}\mathbf{X} \tag{1}$$

or explicitly

$$\xi_{\nu\mu} = \xi_\mu(t_\nu) = \sum_{\lambda=1}^{\ell} \gamma_{\nu\lambda} X_{\lambda\mu}, \tag{2}$$

where $\boldsymbol{\Gamma} = \left(\gamma_{\nu\lambda}\right)$ is a matrix whose elements are known functions of the epoch t_ν of the ν-th frame; the most common (and in almost all situations sufficient) formulation, which we will assume throughout the rest of the paper is

$$\xi_{\nu\mu} = \xi_\mu(t_\nu) = (1 \quad t_\nu \quad P_\nu) \begin{pmatrix} \xi_{0\mu} \\ \mu_\mu \\ \varpi_\mu \end{pmatrix} = \boldsymbol{\gamma}_\nu{}^{\mathrm{T}} \boldsymbol{\xi}_\mu. \tag{3}$$

This equation defines the vectors $\boldsymbol{\gamma}_\nu$ and $\boldsymbol{\xi}_\mu$, the vector of star parameters of the μ-th star. P_ν is the parallax factor (in the x-coordinate) for the ν-th frame.[4]

The Eqs. 1 and those derived from them merely reflect geometry and are therefore independent of any measurements. The true values $\mathbf{x} = \left(x_{\nu\mu}\right)$ of the rectangular coordinates are inaccessible, their measured values $\hat{\mathbf{x}} = \left(\hat{x}_{\nu\mu}\right)$ are related to them by $\hat{\mathbf{x}} + \boldsymbol{\varepsilon} = \mathbf{x}$, where $\boldsymbol{\varepsilon}$ is the matrix of corrections. The standard coordinates are related to the true values of the measured coordinates and to the frame parameters by

$$\boldsymbol{\xi} = \mathbf{x} + \mathbf{A}\boldsymbol{\Phi}, \tag{4}$$

[4] It used to be safe to assume the parallax factor to be constant over the whole frame. This assumption must be reconsidered in view of recent improvements in measuring precision. One may estimate the change in e.g. P_α over a field of width $\mathrm{d}\alpha$ by considering that $|\mathrm{d}P_\alpha| = \left|\frac{\partial P_\alpha}{\partial \alpha}\mathrm{d}\alpha\right| \approx |-\cos(\alpha - \odot)\mathrm{d}\alpha| \leq |\mathrm{d}\alpha|$, thus we get $|\mathrm{d}P_\alpha| \leq 0.003$ for $|\mathrm{d}\alpha| = 10'$; a parallax would therefore have to be $0\overset{''}{.}3$ to be changed by one milliarcsecond. A gain of another order of magnitude in the precision of parallax measurements will, however, make it necessary to reconsider the assumption of a constant parallax factor for a given frame.

where $\mathbf{A} = \left(A_{\nu\kappa}\right)$ is the matrix of the "frame parameters," ν being the number of the frame and κ identifies the parameter in the set pertaining to the ν-th frame.

We assume (without restricting generality) that the units in which the coordinates $\hat{\mathbf{x}}_{\nu\mu}$ were measured are approximately the same as those in which the $\xi_{\nu\mu}$ are reckoned, that the x-axis is (almost) parallel to the ξ-axis, that the elements of $\mathbf{x}$ and $\boldsymbol{\xi}$, respectively, are for all practical purposes reckoned with respect to the same origins and that the origin of the ξ-η-system is the same on each frame. We then have $\boldsymbol{\xi} \approx \mathbf{x}$ (and since $||\varepsilon|| \ll 1$ also $\boldsymbol{\xi} \approx \hat{\mathbf{x}}$) and analogously, $\boldsymbol{\eta} \approx \mathbf{y}$.

The components of the matrix $\boldsymbol{\Phi} = \left(\boldsymbol{\varphi}_1 \ \ldots \ \boldsymbol{\varphi}_m\right)$ determine the *frame reduction model* and are known functions of (known) characteristics of the stars, primarily their rectangular coordinates on the frames and sometimes in addition other parameters such as the stars' magnitudes and colors. Under our assumption that all terms of the frame parameter matrix $\mathbf{A}$ are very small and that the elements of the matrix $\boldsymbol{\varepsilon}$ of the negative measuring errors are also very small, we may use the stars' measured coordinates $\hat{x}_{\nu\mu}$ as good approximations to the rectangular ones which are needed to set up $\boldsymbol{\Phi}$.

The measured $\hat{x}_{\nu\mu}$ (and of course, $\hat{y}_{\nu\mu}$) are relative, leaving the configuration of the measured images indeterminate with respect to (a small correction to) the scale, a (small) rotation and some zero-point that is unimportant in this context. (The images of) all stars on the frame (not only that of the target star) change their coordinates (on the frame, relative to each other) very slowly as a consequence of their proper motions and parallaxes (and in the case of multiple stars, orbital motion), and for this reason the system defined by the field stars (i.e. those whose astrometric parameters are not the *primary* targets of the investigation) changes in time. The measurements must therefore be reduced in such a way, that estimates of zero-epoch coordinates, proper motion components and parallax are calculated for each star within the frame.

In (Eichhorn, 1988) I have investigated in detail the rank deficiency of the system derived from only Eqs. 1 and 4 from which the star parameters and frame parameters can be estimated and shown that unambiguous values for both these groups of parameters cannot be estimated without information in addition to that provided by the measurements on the frames e.g., by subjecting either the star parameters or the frame parameters to any set of $k\ell$ independent constraints. There I have assumed these constraints to be rigorous. Although doing so corrects the rank deficiency of the system of normal equations for the estimation of frame- and star parameters, this assumption cannot strictly be true: While available general information suggests expectation values of systematic proper-motion components and parallaxes for the stars in the frames, these will differ from the true values (e.g. because of the stars' peculiar motions) and one cannot therefore expect that the true values will rigorously conform to exactly those constraints to which the expected values of the systematic proper motions and the parallaxes would subject them (e.g. that they will be correlated in a certain definite way with the coordinates of the images on the frames). Instead of enforcing the constraint equations rigorously, one may let the constrained parallaxes and proper motions – together with their covariance matrix – be regarded as observations and include the appropriate equations in the set of condition equations. Fortunately, the latter will correct the rank deficiency as well as rigorous constraints would.

The simplest constraints would be linear:

$$\mathbf{\Gamma}^{\mathrm{T}}\mathbf{A} = \mathbf{K} \tag{5}$$

or

$$\mathbf{\Phi}\mathbf{X}^{\mathrm{T}} = \mathbf{C}. \tag{6}$$

While any arbitrary matrices **K** and **C** will remedy the singulatity, their elements must be chosen such that they account realistically for the kinematic and geometric facts of the situation.

We now derive a rigorous relationship between the constraint matrices **K** and **C** which allows us to calculate one of them if the other one is given. By eliminating $\boldsymbol{\xi}$ from Eqs. 1 and 4, we get

$$\mathbf{A}\mathbf{\Phi} = \mathbf{\Gamma}\mathbf{X} - \mathbf{x}. \tag{7}$$

Leftmultiplication of this equation by $\mathbf{\Gamma}^{\mathrm{T}}$ and rightmultiplication by $\mathbf{\Phi}^{\mathrm{T}}$ while considering Eqs. 5 and 6 results in

$$\mathbf{K}\mathbf{\Phi}\mathbf{\Phi}^{\mathrm{T}} = \mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Gamma}\mathbf{C}^{\mathrm{T}} - \mathbf{\Gamma}^{\mathrm{T}}\mathbf{x}\mathbf{\Phi}^{\mathrm{T}}, \tag{8}$$

a rigorous relationship between the constraint matrices **K** and **C**. The matrices $\mathbf{\Phi}\mathbf{\Phi}^{\mathrm{T}}$ and $\mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Gamma}$ are (except in pathological cases) nonsingular because generally $k < m$, $n < m$ and $\ell < n$, so that Eq. 8 allows us to calculate **K** in terms of **C** and *vice versa*.

In (Eichhorn, 1988) I neither gave explicit instructions for computing the terms of the constraint matrices **K** (or **C**, as the case may be), nor did I specify models for the frame reduction and for the dependence of the stars' coordinates on time or give a formula for actually calculating the parameter estimates. That paper is furthermore strictly valid only for the case that all stars are imaged on all frames. (Unfortunately, lifting the restriction that the images of all stars must be measured on all frames makes it impossible to use the compact and elegant notation with Kronecker products.)

Since we allow that not all stars' images were measured on all frames, those elements of the matrices **x**, $\hat{\mathbf{x}}$ and $\boldsymbol{\varepsilon}$ which correspond to pairs (ν, μ) in which the image of the μ-th star was not measured on the ν-th frame will be empty, these positions in the matrix generate no condition equations. The Eqs. 1 and those derived from them are valid whether or not the image of the μ-th star was measured on the the ν-th frame. If it was not, the reason might have been that it was too faint, lay outside the frame or was left out deliberately. Eqs. 4 is also valid regardless of whether the μ-th star was measured on the ν-th frame; however, if there is no corresponding measurement $\hat{x}_{\nu\mu}$, the equation which would be generated by this measurement does not exist. Eq. 7 is therefore meaningful in the stated form only if each star's image was measured on each frame, and this is also necessary for Eq. 8 to be valid.

This raises an additional caution: We will normally arrive at a more accurate assessment of the constraints if we exercise judgment as to which stars we will allow to influence the terms of **C**. For this purpose, we will select stars whose parameters, while they were not previously measured, can still be predicted with reasonable accuracy and therefore, not all stars will be involved in in setting up the constraint matrix **C** from Eqs. 6 to remove the singularity. One may set up a "working" **C** and compute its elements as long as a minimum of k stars are constrained and still remove the singularity. This means that the columns corresponding to stars with nonconstrained parameters were replaced by nullvectors in the matrices $\mathbf{\Phi}$ and **X**; the products of the corresponding elements of $\mathbf{\Phi}$ and $\mathbf{X}^{\mathrm{T}}$ will then not be added when the corresponding element of **C** is accumulated. This does not matter, if the calculated expectations of the elements of **C** are used [through the Eqs. 6] to restrain the freedom of the parameters of only those stars that contributed to the estimation of the values of the elements of **C**. Eq. 8 will, however, be valid only if only those stars are used in the matrices $\mathbf{\Phi}$ and **x** [as they occur in Eq. 8] which were used for the estimation of the elements of **C**.

In this paper, we adopt the "star model" in the form of Eq. 3 unless the star has been identified as multiple or has such a large proper motion that the linear representation would falsify the result. This model does not rigorously describe the motion of the image of a star moving uniformly in space but is almost always sufficient within the framework of currently achievable measuring precision. We also specialize – without thereby compromising the generality of the principle – the "frame model" to be of the form of Eq. 13, cf. Sect. 3.1 below; one may always adopt a more complex model if the geometry (or physics) of the imaging – within the framework of the currently achieved accuracy and precision of the measurements – justifies it.

We therefore assume $\ell = 3$ and $k = 3$. The set of constraint Eqs. 5 then presumes that the correlations of like frame parameters (i.e. scale correction, rotation terms and zero points) with time and parallax factors are previously known and that the sums of like frame parameters are also known.

When extraneous reference data (estimates of star parameters from an extraneous source) are not available, it is reasonable to assume **K** to be a null matrix. This is so, because we may regard the instrument (here and in what follows always to be understood as the telescope *and* the recording medium e.g. photographic plate or CCD) as stable and presume that its scale will not drift in time, that we may rely on the instrument not rotating on its optical axis as time progresses, and we may finally assume that the direction in which the optical axis pointed during the exposures has not drifted with respect to the sky. These conditions may, however, not be satisfied in physical reality and enforcing them rigorously will likely impose distortions on the resulting star parameter estimates. The constraints could be relaxed i.e. the elements of the matrix **K** could be regarded as quasi-observations. In this case, their covariance matrix would have to be calculated, which is not overly simple.

In any case, the values of the elements of **K** characterize the *instrument* as a whole and therefore cannot depend on which stars were imaged on the frame and on which images were measured.

2. Establishing C and its covariance matrix

We include Eqs. 6 in the set of condition equations for making the entire system of condition equations nonsingular by establishing the estimated values of the terms of **C** and their covariance matrix and then using the Eqs. 6 themselves as equations of condition rather than as constraints. The terms of **C** essentially measure the covariances of the star parameters with (functions of) the stars’ coordinates in the frame and can be estimated even in the absence of reference material (positions, proper motions, parallaxes); *specific* reference material, when available, will be incorporated explicitly to set up yet additional equations of condition, cf. Sect. 3.2.

To set up Eqs. 6 as condition equations, we need predicted values for the parallaxes and the systematic proper motion components of the field stars as well as estimates of the variances of these quantities. These data are available as functions of the stars’ (galactic) coordinates, apparent magnitudes, colors and/or spectral characteristics, for a description of current practice cf. e.g. Gatewood et al. (1993). Note that only the secular motions (caused by the Sun’s motion toward the apex) depend on the stars’ apparent magnitudes; the systematic parts of the proper motions due to galactic shearing and galactic rotation are (for all practical purposes) independent of the stars’ distances and thus their apparent magnitudes. We obtain the expected magnitudes of the peculiar motions, from which we calculate the variances of the proper motion estimates, from the known dispersions of the stars’ velocities which themselves also depend on position, apparent magnitude and spectral type (or color). Note that the variances of these data (predicted proper motion components as well as parallaxes) decrease generally with increasing apparent magnitude, meaning that the variances of the predicted parallaxes will be the smaller, the smaller the parallaxes are themselves.

Those components of **C** that fix the sum of the zero-point corrections and the sums of the products of the zero-point corrections and the coordinates on the frame can be set equal to 0; the variance of one zero point correction may be assumed equal to the variance of a measured coordinate. In order to use the components of **C** correctly as quasi-observations in condition equations of the type Eqs. 6, we need also their covariance matrix $\mathbf{\Sigma}_{\mathrm{CC}}$.

In (Eichhorn, 1988) I assumed that all measurements – coordinates x and y – have unit variance.[5] This means that the variances of the estimates from which **C** is computed i.e. positions (in the form of standard coordinates), proper motions and parallaxes, must be in the same units. We assume that we know the variance of a measured coordinate in the frame and since we know the scale value, we can convert the variances of the reference coordinates to this unit. For the positions (i.e. the $\xi_{\nu\mu}$), we use the same variance as that of the measurements. The units for the variances of the reference proper motions must be chosen accordingly, meaning that the time must be reckoned in the same unit for the variances of the reference proper motions and the time in Eqs. 3. We have no choice but to assume (actually, wrongly) that the systematic part of the proper motions was computed from the best available information on solar motion, galactic rotation and galactic shearing without error. The variances to be used in the computation of the covariance matrix of $\boldsymbol{C}_2$ [cf. Eq. 9 below] are then the *squares* of the predicted random components of the proper motions, the so-called *peculiar motions*. These also can be computed from data on the triaxial velocity ellipsoid published in the literature.

The variances of the reference parallaxes are to be expressed in the same units as those of the measured coordinates, since the parallax estimates themselves will be computed in those units. The predicted parallaxes of the field stars are found by the same precepts as the parallaxes of the “reference” stars in the traditional reduction from “relative” to “absolute” parallaxes. As their variances, we may use those that would normally be associated with the computation of “photometric” parallaxes.

We now write **C** in terms of its columns

$$\mathbf{C} = (\,\boldsymbol{C}_1 \quad \boldsymbol{C}_2 \quad \boldsymbol{C}_3\,)\,, \tag{9}$$

whence Eqs. 6 become

$$\begin{pmatrix} \mathbf{\Phi}_\xi & 0 & 0 \\ 0 & \mathbf{\Phi}_\mu & 0 \\ 0 & 0 & \mathbf{\Phi}_\varpi \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\mu} \\ \boldsymbol{\varpi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \boldsymbol{C}_3 \end{pmatrix}. \tag{10}$$

$\mathbf{\Phi}_\xi$, $\mathbf{\Phi}_\mu$ and $\mathbf{\Phi}_\varpi$ are the matrix $\mathbf{\Phi}$, except that the columns corresponding to stars whose appropriate parameters were not subjected to restriction are replaced by $\boldsymbol{0}$. The dimensions of these matrices are therefore the same as that of $\mathbf{\Phi}$. $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varpi}$ are the vectors of the star parameters defined below after Eq. 17. We need not consider all stars for this nor do we have to use data pertaining to the same stars for different columns of **C**; when we expect a very discordant value of a parameter in a particular case (e.g. the target star will often have been chosen on the basis of its large proper motion and thus statistically be an outlier, therefore typically have a much larger parallax and larger total proper motion than a field star), and for this reason we better leave this datum out. Once we hve calculated $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$, Eqs. 10 become equations of conditions in which the elements of **C** play the role of observations, and (a selection of the) components of $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varpi}$ are the adjustment parameters.

The quasi-observations in the form of elements of **C** are correlated, and in order to perform a proper adjustment, we need to use their covariance matrix. For this purpose we denote the (diagonal) covariance matrix of the data used for setting up the quasi-observations (i.e. the relaxed constraints) by

$$\mathbf{\Sigma}_{\mathrm{c}} = \begin{pmatrix} \mathbf{\Sigma}_{\xi\xi} & 0 & 0 \\ 0 & \mathbf{\Sigma}_{\mu\mu} & 0 \\ 0 & 0 & \mathbf{\Sigma}_{\varpi\varpi} \end{pmatrix}.$$

We may put in any values for the variances of the nonrestricted data since these are going to be multiplied by zeros as we will

5 This assumption will be abandoned for the definitive reduction, cf. Sect. 6

see in Eq. 11 below. We have described above how we get these variances. The covariance matrix of the vector components of **C** is then the blockdiagonal matrix

$$\mathbf{\Sigma}_{\mathrm{CC}} = \mathrm{diag}\,(\,\mathbf{\Phi}_{\xi}\mathbf{\Sigma}_{\xi\xi}\mathbf{\Phi}_{\xi}{}^{\mathrm{T}} \quad \mathbf{\Phi}_{\mu}\mathbf{\Sigma}_{\mu\mu}\mathbf{\Phi}_{\mu}{}^{\mathrm{T}} \quad \mathbf{\Phi}_{\varpi}\mathbf{\Sigma}_{\varpi\varpi}\mathbf{\Phi}_{\varpi}{}^{\mathrm{T}}\,)\,, \qquad (11)$$

computed from the estimated variances of the predicted parallaxes and proper motions.

In addition to using the Eqs. 6 as condition equations after having calculated the terms of the constraint matrix **C**, which in turn was computed from the predicted values of the parallaxes and the proper motions, we also use whatever extraneous specific reference data[6] are available (i.e. previously measured positions, proper motion components and parallaxes). If the reference data only were used as observations for removing the rank deficiency from the system, we ignore the fact that the constraints also hold, at least in their relaxed forms and thus provide additional information beyond that furnished by the reference data, even though these would very often alone suffice to render the resulting normal equations nonsingular. For this reason, a complete reduction will utilize the related constraints in addition to whatever reference data are available.

3. Formulating the condition equations

The condition equations are linear and solved with respect to exactly one (albeit partially correlated) observation, so that we have the (almost) simplest form of a least-squares problem (there are up to 9 correlated quasi-observations at the end of the vector $\boldsymbol{x}$ of the observables). If the condition equations are

$$\boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}) = \mathbf{B}\boldsymbol{a} + \boldsymbol{x} = \boldsymbol{0},$$

the adjustment parameters (unknowns), denoted by the vector $\boldsymbol{a}$, are the frame- and star-parameters

$$\boldsymbol{a}^{\mathrm{T}} = \big(A_1\ B_1\ \ C_1\ A_2\ \ldots\ C_n \\ \zeta_1\ \zeta_2\ \ldots\ \zeta_m\ \mu_1\ \ldots\ \mu_m\ \varpi_1\ \ldots\ \varpi_m\big).$$

Let $\mathbf{\Sigma}$ be the covariance matrix of the elements of $\boldsymbol{x}$. The well known expression for the least-squares estimate $\hat{\boldsymbol{a}}$ of $\boldsymbol{a}$ is then

$$\hat{\boldsymbol{a}} = -\big(\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{B}\big)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\boldsymbol{F}(\boldsymbol{0},\boldsymbol{x}) \qquad (12)$$

There are in our problem three groups of condition equations according to the types of observations occurring in them, those generated by measured coordinates on a frame (frame measurements), those by reference parameters and those by the relaxed parameter constraints. We shall now discuss each of these groups in turn.

[6] Of those stars imaged on the frames, the one whose parallax is the main target of the investigation is the *target star*, the others are the field stars. By *reference stars* we mean those field (and when appropriate, even target) stars for which independent extraneous estimates of at least one of their parameters (positions, proper motions, parallaxes) are available and used in the computations.

3.1. The coordinate measurements on the frames

The measured coordinates of the stars' images on the frames produce the first set of condition equations. The standard coordinates in the usual sense, but (as emphasized before) *reckoned in the same units as the measured coordinates* are connected to the plate parameters by the transformation Eq. 4 which we specified, by putting $k = 3$, to be the "six constant model"

$$\xi_{\nu\mu} = x_{\nu\mu} + A_\nu x_\mu + B_\nu y_\mu + C_\nu, \qquad (13)$$

and to the star parameters by the Eq. 3 which we modify, by putting

$$\zeta_\mu = \xi_\mu - x_\mu,$$

to read

$$\xi_{\nu\mu} \equiv \xi_\mu(t_\nu) = x_\mu + \zeta_\mu + t_\nu\mu_\mu + P_\nu\varpi_\mu \qquad (14)$$

whence we get, after substituting

$$z_{\nu\mu} = x_{\nu\mu} - x_\mu$$

by eliminating $\xi_{\nu\mu}$ from Eqs. 13 and 14

$$A_\nu x_\mu + B_\nu y_\mu + C_\nu - \zeta_\mu - t_\nu\mu_\mu - P_\nu\varpi_\mu + z_{\nu\mu} = 0. \qquad (15)$$

In Eqs. 14 and 15, x_μ are approximations to $x_{\nu\mu}$ as well as to $\xi_{\nu\mu}$ which we choose arbitrarily at the beginning of our calculations e.g. by $x_\mu = \frac{1}{n_\mu}\sum_\nu x_{\nu\mu}$, where the sums go over all ν which belong to frames on which the star numbered μ was imaged. (y_μ is defined analogously.)[7] The appropriate starting values of all the parameters are thus 0 (i.e. $\boldsymbol{a}_0 = \boldsymbol{0}$) and we expect their estimates to be small. The part of $\boldsymbol{F}(\boldsymbol{0},\boldsymbol{x})$ that is contributed by the coordinate measurements on the frames is thus

$$\boldsymbol{F}_0{}^{\mathrm{T}} = (\,z_{11} \quad z_{12} \quad \ldots \quad z_{nm}\,)\,, \qquad (16)$$

All condition equations generated by coordinates measured in a frame are of this form. Frame- and star parameters alike occur in this equation. We take these equations in the order of increasing frame number and within the same frame number, in the order of increasing star number. Every measured coordinate $x_{\nu\mu}$ contributes one equation to the vector of equations. Note that not all stars need have been measured on all frames; we have, however, written Eq. 16 assuming that the m-th star was measured on the n-th frame.

We set up Eqs. 15 sequentially for the frames i.e., first all equations resulting from the measurements on the first frame and so on. Each Eq. 15 is essentially one element of Eqs. 7. We note that for the six-parameter-model, we have

$$\mathbf{\Phi}^{\mathrm{T}} = \begin{pmatrix} \boldsymbol{\varphi}_1{}^{\mathrm{T}} \\ \boldsymbol{\varphi}_2{}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{\varphi}_m{}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 1 \end{pmatrix}$$

[7] The introduction of $x_\mu = \langle x_{\nu\mu}\rangle$ facilitates the subsequent analytical developments somewhat. When these formulas are programmed, it may actually turn out that retaining the original $x_{\nu\mu}$ leads to less complicated programs.

[cf. Eqs. 4] and now write the equations generated by the measurements on the ν-th frame in the form

$$\mathbf{\Phi}_\nu{}^{\mathrm{T}} \boldsymbol{A}_\nu - \begin{pmatrix} \boldsymbol{\gamma}_\nu{}^{\mathrm{T}}\boldsymbol{\zeta}_1 \\ \boldsymbol{\gamma}_\nu{}^{\mathrm{T}}\boldsymbol{\zeta}_2 \\ \vdots \\ \boldsymbol{\gamma}_\nu{}^{\mathrm{T}}\boldsymbol{\zeta}_n \end{pmatrix} + \boldsymbol{z}_\nu = 0$$

or rearranged

$$\mathbf{\Phi}_\nu{}^{\mathrm{T}} \boldsymbol{A}_\nu - \mathbf{S}_\nu(\boldsymbol{\zeta} + t_\nu \boldsymbol{\mu} + P_\nu \boldsymbol{\varpi}) + \boldsymbol{z}_\nu = 0, \tag{17}$$

where $\boldsymbol{z}_\nu{}^{\mathrm{T}} = (\, z_{\nu 1} \quad z_{\nu 2} \quad \ldots \quad z_{\nu m}\,)$, $\boldsymbol{\zeta}_\mu{}^{\mathrm{T}} = (\, \zeta_\mu \quad \mu_\mu \quad \varpi_\mu\,)$ which we call the alternate vector of star parameters of the μ-th star, and $\boldsymbol{\gamma}_\nu{}^{\mathrm{T}} = (\, 1 \quad t_\nu \quad P_\nu\,)$; $\boldsymbol{\zeta} = (\, \zeta_1 \quad \zeta_2 \quad \ldots \quad \zeta_m\,)$, $\boldsymbol{\mu} = (\, \mu_1 \quad \mu_2 \quad \ldots \quad \mu_m\,)$ and $\boldsymbol{\varpi} = (\, \varpi_1 \quad \varpi_2 \quad \ldots \quad \varpi_m\,)$. The vector $\boldsymbol{\pi}$ of all star parameters, arranged in the order in which they appear in our equations is thus

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\mu} \\ \boldsymbol{\varpi} \end{pmatrix}. \tag{18}$$

We may add any number of statements to the effect that $0 \equiv 0$ to the set of condition equations without thereby influencing the solution. This consideration allows us to have always exactly m equations of the type Eq. 18 for each ν if we set $z_{\nu\mu} = 0$ for all pairs (ν, μ) when the image of the μ-th star was not measured on the ν-th frame. The matrix $\mathbf{\Phi}_\nu$ is then the matrix $\mathbf{\Phi}$ in which the columns corresponding to stars numbered $\mu_{\nu_1},\ \mu_{\nu_2}\ \ldots, \mu_{\nu_{\ell_\nu}}$ – which are the ordinal numbers of the stars whose images were not measured on the ν-th frame – are replaced by nullvectors, and the matrix $\mathbf{S}_\nu$ is obtained by doing the same with the identity matrix $\mathbf{I}_m$.

The submatrix $\mathbf{B}_{\mathrm{i}}$ of the matrix $\mathbf{B}$ of the condition equations generated by the measurements $x_{\nu\mu}$ on the frames thus has $3(m + n)$ columns, because there must be $3n + 3m$ columns for the n sets of 3 frame parameters each and the m sets of 3 star parameters each. Since the frame parameters of the ν-th frame occur only in the ν-th set of these condition equations, there must be $n - 1$ nullmatrices of dimension $m \times 3$ in this row of submatrices, so that the n first submatrices (all of dimension $m \times 3$) in the rows generated by the ν-th frame will be

$$\mathbf{0}\ \ldots\ \mathbf{0}\ \mathbf{\Phi}_\nu{}^{\mathrm{T}}\ \mathbf{0}\ \ldots\ \mathbf{0}.$$

Eqs. 17 allows us to read off the other coefficient matrices. Altogether, we see that the upper part $\mathbf{B}_{\mathrm{i}}$ of the matrix of the condition equations looks like this[8]:

$$\mathbf{B}_{\mathrm{i}} = \begin{pmatrix} \mathbf{\Phi}_1{}^{\mathrm{T}} & \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{S}_1 & -t_1\mathbf{S}_1 & -P_1\mathbf{S}_1 \\ \mathbf{0} & \mathbf{\Phi}_2{}^{\mathrm{T}} & \ldots & \mathbf{0} & -\mathbf{S}_2 & -t_2\mathbf{S}_2 & -P_2\mathbf{S}_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{\Phi}_n{}^{\mathrm{T}} & -\mathbf{S}_n & -t_n\mathbf{S}_n & -P_n\mathbf{S}_n \end{pmatrix}.$$

3.2. The extraneous reference materials

We have repeatedly stated already that the matrix $\mathbf{B}_{\mathrm{i}}{}^{\mathrm{T}}\mathbf{B}_{\mathrm{i}}$, which would be the matrix of the system of normal equations were we to estimate the system parameters $\hat{\boldsymbol{a}}$ from the frame measurements only, is rank deficient. This rank deficiency can, in principle, be compensated (so that an unambiguous solution for the adjustment-parameter estimates is obtained) not only by parameter constraints, but also by including previously and independently obtained extraneous estimates of the star parameters (or both). In this subsection, we set up the procedures for the inclusion of such extraneous star-parameter estimates. These can be estimates of positions, proper motions and parallaxes (and, in principle, also orbital parameters, but we shall ignore these here).

Position estimates for various epochs may be extracted from existing star catalogues and will as a rule refer to epochs that have no relation to the epochs of the frames. Catalogues list position estimates as right ascensions and declinations. These must therefore be converted to standard coordinates (with respect to the tangential point of the frames) and *expressed in the same units as the* $x_{\nu\mu}$ before they can be used in the formulas in this paper. [Alternatively, the $x_{\nu\mu}$ may be expressed in the units normally used for standard coordinates, namely the focal length of the optical imaging device (i.e. the telescope) which produces the frames.]

Any number of any of the types of extraneous estimates (positions, proper motions, parallaxes) may be used in our calculations for any star; if there is more than one position estimate for a particular star for at least two different epochs, that star's proper motion is *implicitly* also estimated and the proper motion estimated from these data should not be introduced as an independent estimate.

In practice, parallaxes of reference stars (cf. footnote 6) will rarely have been estimated, although previous estimates of the parallaxes of the target star(s) will occasionally exist.

Note that the procedure suggested in this paper will (ideally) give absolute parallaxes directly; reference parallaxes, when utilized, must therefore also be absolute.

Let the (diagonal) covariance matrix $\mathbf{\Sigma}_{\mathrm{r}}$ of the reference data be

$$\mathbf{\Sigma}_{\mathrm{r}} = \mathrm{diag}\,(\,\mathbf{\Sigma}_{\mathrm{r}\xi\xi} \quad \mathbf{\Sigma}_{\mathrm{r}\mu\mu} \quad \mathbf{\Sigma}_{\mathrm{r}\varpi\varpi}\,)\,.$$

When a reference parallax exists and has been incorporated in our estimation process, the result of our computations will supersede it. Had we not used an existing extraneous parallax estimate in our computations, they would yield an independent estimate and some combination of the old and the new estimate would then be an improvement over both. We state below (cf.Sect. 5) the reasons why we obtain the optimum reestimation

[8] It is worth noting that $\mathbf{B}_{\mathrm{i}}$ could have been written

$$\mathbf{B}_{\mathrm{i}} = (\,\mathbf{I}_n \otimes \mathbf{\Phi}^{\mathrm{T}} \quad -\boldsymbol{J}_n \otimes \mathbf{I}_m \ -\boldsymbol{T}_n \otimes \mathbf{I}_m \ -\boldsymbol{P}_n \otimes \mathbf{I}_m\,)\,,$$

if an image of each star had been measured on each frame. The symbol $\otimes$ indicates Kronecker matrix-multiplication and $\boldsymbol{J}_n{}^{\mathrm{T}} = (\,1 \quad 1 \quad \ldots \quad 1\,)$, $\boldsymbol{T}_n{}^{\mathrm{T}} = (\,t_1 \quad t_2 \quad \ldots \quad t_n\,)$ and $\boldsymbol{P}_n{}^{\mathrm{T}} = (\,P_1 \quad P_2 \quad \ldots \quad P_n\,)$.

of a parallax by incorporating existing estimates in the calculation of the new estimate and not by making the new estimate independently and taking the (weighted) mean of the existing and the new results.

Let independent position estimates (as mentioned before, in the form of standard coordinates in the appropriate units) exist for the stars numbered $p_1, \ldots, p_k$, and let there be ℓ_κ different estimates (generally, but not necessarily for different epochs) for the star numbered p_κ, so that there will be (for all stars) altogether $\Lambda = \sum_{\kappa=1}^{k} \ell_\kappa$ independent position estimates. $\ell_\kappa \neq 0$ only if the κ-th star is a reference star. There may be more of these than there are stars, even though there may be none for some stars, because the positions of some of the stars may have been estimated several times, generally at different epochs.

Let $\mathbf{B}_\mathrm{r}$ be the submatrix consisting of those rows of the matrix $\mathbf{B}$ that are generated by reference material (of all types). Since none of this involves frame parameters, the first $3n$ columns of $\mathbf{B}_\mathrm{r}$ will contain zeroes. Altogether, Λ rows will be generated by reference *positions*. We write these in the form

$$\underset{\Lambda\times 3n}{\mathbf{0}} \quad -\underset{\Lambda\times m}{\mathbf{T}} \quad -\underset{\Lambda\times m}{\mathbf{L}} \quad -\underset{\Lambda\times m}{\mathbf{M}},$$

where $-\mathbf{T}$, $-\mathbf{L}$ and $-\mathbf{M}$ are the matrices whose elements are the coefficients of $\boldsymbol{\zeta}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varpi}$. *All these matrices have the same dimensions and nonzero terms in the same positions*. Columns corresponding to stars for which no reference positions are available will be null vectors. The structure of the columns corresponding to data for the stars numbered $p_1, \ldots, p_k$ depends on the sequence of the reference positions in the vector of observations: we arrange them in the order of increasing star number, and while the order of the position estimates of the same star is irrelevant, we prefer to arrange them – within any group pertaining to the same star – in the order of the epochs.

All nonzero elements of $\mathbf{T}$ will be 1. They will be in the first ℓ_1 rows in column p_1, the next ℓ_2 rows of column p_2 and so forth; the last will occupy the last ℓ_k rows of column p_k, so that exactly one element in each row is a 1, and none of these 1's can stand in a column left to that in which the previous 1 is.

The nonzero elements of the matrices $\mathbf{L}$ and $\mathbf{M}$ are in the same positions as those of $\mathbf{T}$ and are, in $\mathbf{L}$, the epochs at which the corresponding position estimates were obtained. The nonzero elements in $\mathbf{M}$ are the appropriate parallax factors. However, caution is indicated with these: The position estimates listed in a particular catalogue are frequently the averages of estimates made at different epochs and thus with different parallax factors so that strictly, the parallax factors in $\mathbf{M}$ would in each case have to be the averages of the parallax factors at which the individual estimates were made which were then averaged into the catalogued estimate. In current practice, however, this is a moot point because seldom are position estimates sufficiently precise and are reference star parallaxes large enough to make the use of accurate parallax factors in $\mathbf{M}$ a critical matter. In the absence of specific detailed information, we might as well set the parallax factors equal to 0.

The next rows are generated by *reference proper motions*.[9] Suppose these are available for stars numbered $m_1, \ldots, m_\mathrm{r}$. More than one independent proper-motion estimate may be available for any star; generally assume that ℓ_ρ independent proper-motion estimates are available for the m_ρ-th star, so that the total number P of available refereerence proper motions is $P = \sum_{\rho=1}^{r} \ell_\rho$. Those rows of $\mathbf{A}_r$ generated by these reference proper-motion estimates have the structure

$$\underset{P\times 3n}{\mathbf{0}} \quad \underset{P\times m}{\mathbf{0}} \quad \underset{P\times m}{\mathbf{L}_\mathrm{r}} \quad \underset{P\times m}{\mathbf{0}};$$

there are exactly P rows, as many as there are individual reference proper-motion estimates altogether. Obviously, the true proper motions are – within the assumptions of our model – not time dependent, so that we arrange the estimates in the order of the ordinal numbers of the stars to which they belong, but there is no preference in the order of those estimates that pertain to the same star.

When there are reference positions of the same star for several epochs, one might be tempted to calculate from them estimates of the proper motion and of the position at the central epoch and use these data. This is, however, an unnecessary complication and it would be more straightforward in this case to use the reference positions only and let them have whatever influence they exert on the final estimate of the proper motion of the star concerned.

In principle, the structure of $\mathbf{L}_\mathrm{r}$ is – *mutatis mutandis* – that of $\mathbf{T}$, although different in detail because we must allow for the fact that extraneous reference position estimates and extraneous reference proper-motion estimates will generally not be available for the identical subset of stars.

Finally, the last rows of $\mathbf{B}_\mathrm{r}$ are generated by whatever reference parallax-estimates are available and look like this:

$$\underset{\Pi\times 3n}{\mathbf{0}} \quad \underset{\Pi\times m}{\mathbf{0}} \quad \underset{\Pi\times m}{\mathbf{0}} \quad \underset{\Pi\times m}{\mathbf{M}_\mathrm{r}}.$$

What we have just said about $\mathbf{L}_\mathrm{r}$ holds – *mutatis mutandis* – also for $\mathbf{M}_\mathrm{r}$, whose nonzero elements will also all be 1. Π is, of course, the total number of available reference parallaxes.

Altogether, we get

$$\mathbf{B}_\mathrm{r} = \begin{pmatrix} \mathbf{0} & -\mathbf{T} & -\mathbf{L} & -\mathbf{M} \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_\mathrm{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{M}_\mathrm{r} \end{pmatrix}.$$

3.3. The relaxed constraints

I have shown in Sect. 2 how to implement the constraints by constructing quasi-observations and set up the required covariance matrix of these quasi-observations. Note that the constraints will "work" (also in their relaxed form) even when not *all* parameters are subjected to them; it is not even necessary that different kinds of parameters are constrained for the same sets of stars.

[9] The idea that the incorporation of independently estimated proper motions (especially of the target star) will improve the accuracy of the parallaxes is due to W. J. Luyten.

The (bottom) submatrix $\mathbf{B}_\mathrm{c}$ of $\mathbf{B}$, namely that arising from the constraints on the star-parameters, will have a $9 \times 3n$ null matrix in its leftmost position and thus look like this:

$$\mathbf{B}_\mathrm{c} = \begin{pmatrix} \mathbf{0} & -\mathbf{\Phi}_\xi & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_\varpi \end{pmatrix} \tag{19}$$

The matrices $\mathbf{\Phi}_\xi$, $\mathbf{\Phi}_\mu$ and $\mathbf{\Phi}_\varpi$ are, as defined above, the matrix $\mathbf{\Phi}$ in which those columns corresponding to stars whose parameters are not covered by these constraints are replaced by null vectors. Eq. 19 is essentially a restatement of Eqs. 10.

4. The normal equations

The coefficient matrix $\mathbf{B}$ of the condition equations is thus

$$\mathbf{B} = \begin{pmatrix} \mathbf{\Phi}_1{}^\mathrm{T} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{S}_1 & -t_1\mathbf{S}_1 & -P_1\mathbf{S}_1 \\ \mathbf{0} & \mathbf{\Phi}_2{}^\mathrm{T} & \dots & \mathbf{0} & -\mathbf{S}_2 & -t_2\mathbf{S}_2 & -P_2\mathbf{S}_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{\Phi}_n{}^\mathrm{T} & -\mathbf{S}_n & -t_n\mathbf{S}_n & -P_n\mathbf{S}_n \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{T} & -\mathbf{L} & -\mathbf{M} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & -\mathbf{L}_\mathrm{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{M}_\mathrm{r} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{\Phi}_\xi & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{\Phi}_\varpi \end{pmatrix}.$$

According to Eq. 12, we need the so-called "weight matrix," which in our problem is the inverse $\mathbf{\Sigma}^{-1}$ of the covariance matrix of the observations $\boldsymbol{x}$ in order to solve for the unknowns. This matrix is largely diagonal because we assume the frame measurements (whose variance we regard initially as the unit) and the reference data to be uncorrelated. Only that part of the covariance matrix of the observations that belongs to the quasi-observations, namely the constraint terms, consists of three blocks of 3×3 dimensioned matrices on the main diagonal. We need $\mathbf{B}^\mathrm{T}\mathbf{\Sigma}^{-1}$ and $\mathbf{B}^\mathrm{T}\mathbf{\Sigma}^{-1}\mathbf{B}$. We have

$$\mathbf{B}^\mathrm{T}\mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{S}_{23} \end{pmatrix},$$

where

$$\mathbf{S}_{11} = \mathrm{diag}\big(\mathbf{\Phi}_1\ \mathbf{\Phi}_2\ \dots\ \mathbf{\Phi}_n\big),$$

$$\mathbf{S}_{21} = -\begin{pmatrix} \mathbf{S}_1{}^\mathrm{T} & \mathbf{S}_2{}^\mathrm{T} & \dots & \mathbf{S}_n{}^\mathrm{T} \\ t_1\mathbf{S}_1{}^\mathrm{T} & t_2\mathbf{S}_2{}^\mathrm{T} & \dots & t_n\mathbf{S}_n{}^\mathrm{T} \\ P_1\mathbf{S}_1{}^\mathrm{T} & P_2\mathbf{S}_2{}^\mathrm{T} & \dots & P_n\mathbf{S}_n{}^\mathrm{T} \end{pmatrix},$$

$$\mathbf{S}_{22} = -\begin{pmatrix} \mathbf{T}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1} & \mathbf{L}_\mathrm{r}{}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\mu\mu}{}^{-1} & \mathbf{0} \\ \mathbf{M}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1} & \mathbf{0} & \mathbf{M}_\mathrm{r}{}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\varpi\varpi}{}^{-1} \end{pmatrix}$$

and

$$\mathbf{S}_{23} = \mathrm{diag}\big(\mathbf{s}_\xi\ \mathbf{s}_\mu\ \mathbf{s}_\varpi\big)$$

with

$$\begin{aligned} \mathbf{s}_\xi &= \mathbf{\Phi}_\xi{}^\mathrm{T}(\mathbf{\Phi}_\xi\mathbf{\Sigma}_{\xi\xi}\mathbf{\Phi}_\xi{}^\mathrm{T})^{-1} \\ \mathbf{s}_\mu &= \mathbf{\Phi}_\mu{}^\mathrm{T}(\mathbf{\Phi}_\mu\mathbf{\Sigma}_{\mu\mu}\mathbf{\Phi}_\mu{}^\mathrm{T})^{-1} \\ \mathbf{s}_\varpi &= \mathbf{\Phi}_\varpi{}^\mathrm{T}(\mathbf{\Phi}_\varpi\mathbf{\Sigma}_{\varpi\varpi}\mathbf{\Phi}_\varpi{}^\mathrm{T})^{-1}. \end{aligned}$$

We now have the matrices we need to set up the matrix $\mathbf{N} = \mathbf{B}^\mathrm{T}\mathbf{\Sigma}^{-1}\mathbf{B}$ of the normal equations. We write

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{12}{}^\mathrm{T} & \mathbf{N}_{22} \end{pmatrix} \tag{20}$$

and see that

$$\mathbf{N}_{11} = \begin{pmatrix} \mathbf{\Phi}_1\mathbf{\Phi}_1{}^\mathrm{T} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_2\mathbf{\Phi}_2{}^\mathrm{T} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{\Phi}_n\mathbf{\Phi}_n{}^\mathrm{T} \end{pmatrix} \tag{21}$$

and furthermore, that

$$\begin{aligned} \mathbf{N}_{12} &= \begin{pmatrix} -\mathbf{\Phi}_1\mathbf{S}_1 & -t_1\mathbf{\Phi}_1\mathbf{S}_1 & -P_1\mathbf{\Phi}_1\mathbf{S}_1 \\ -\mathbf{\Phi}_2\mathbf{S}_2 & -t_2\mathbf{\Phi}_2\mathbf{S}_2 & -P_2\mathbf{\Phi}_2\mathbf{S}_2 \\ \vdots & \vdots & \vdots \\ -\mathbf{\Phi}_n\mathbf{S}_n & -t_n\mathbf{\Phi}_n\mathbf{S}_n & -P_n\mathbf{\Phi}_n\mathbf{S}_n \end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{\Phi}_1 & -t_1\mathbf{\Phi}_1 & -P_1\mathbf{\Phi}_1 \\ -\mathbf{\Phi}_2 & -t_2\mathbf{\Phi}_2 & -P_3\mathbf{\Phi}_2 \\ \vdots & \vdots & \vdots \\ -\mathbf{\Phi}_n & -t_n\mathbf{\Phi}_n & -P_n\mathbf{\Phi}_n \end{pmatrix}. \end{aligned} \tag{22}$$

In the normal equations, $\mathbf{N}_{12}$ is multiplied by $\boldsymbol{\pi}$, cf. Eq. 18. The $(3\nu-2)$-nd through 3ν-th line of the vector $\mathbf{N}_{12}\boldsymbol{\pi}$ are

$$\begin{aligned} &-\textstyle\sum_\mu x_\mu(\zeta_\mu + t_\nu\mu_\mu + P_\nu\varpi_\mu) \\ &-\textstyle\sum_\mu y_\mu(\zeta_\mu + t_\nu\mu_\mu + P_\nu\varpi_\mu) \\ &\quad-\textstyle\sum_\mu(\zeta_\mu + t_\nu\mu_\mu + P_\nu\varpi_\mu), \end{aligned}$$

where the sums are taken over those μ that are the ordinal numbers of stars that occur on the ν-th frame. If we would regard the star parameters as reference material, these 3-vectors would be exactly the product sums which we expect on the right hand side of the normal equations from which we compute the estimates $A_\nu,\ B_\nu$ and C_ν.

Before we calculate the terms of $\mathbf{N}_{22}$, we point out some properties of the matrices $\mathbf{S}_\nu$, $\mathbf{T}$, $\mathbf{L}_\mathrm{r}$, $\mathbf{M}_\mathrm{r}$, $\mathbf{L}$ and $\mathbf{M}$. The product of the transpose of any of these matrices with the original matrix is always a diagonal matrix of dimension $m \times m$. Except for $\mathbf{L}$ and $\mathbf{M}$, the nonzero terms of all these products are 1. The diagonality of these matrices is preserved even if the product has an additional diagonal factor matrix (such as $\mathbf{\Sigma}_{\mathrm{r}\xi\xi}$) in the middle.

The zero terms on the diagonal of $\mathbf{S}_\nu{}^\mathrm{T}\mathbf{S}_\nu$ correspond to the ordinal number of stars whose images were not measured on the ν-th frame (i.e. $\nu_1,\ \dots\ ,\nu_{\ell_\nu}$).

The nonzero elements of $\mathbf{T}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}\mathbf{T}$ occur at the positions corresponding to those stars whose reference positions were included in our calculations and are the sums of the reciprocals of the corresponding variances, all of which are on the diagonal of $\mathbf{\Sigma}_{\mathrm{r}\xi\xi}$. If for example two reference positions with variances σ_1 and σ_2 were incorporated for star 1, the (1,1) term of this matrix would be $\frac{1}{\sigma_1} + \frac{1}{\sigma_2}$. If no reference position was included for star 2, the (2,2) term of this matrix would be $\mathbf{0}$.

Since the structure of the matrices $\mathbf{L}$ and $\mathbf{M}$ is the same as that of $\mathbf{T}$ and that of the products $\mathbf{T}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{L}$ and $\mathbf{T}^\mathrm{T}\mathbf{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{M}$ is the same as that of $\mathbf{T}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{T}$, except that the reciprocals of the variances are multiplied by the appropriate t and P, respectively. Note that these epochs t and parallax factors P have nothing to do with epochs and parallax factors belonging to one of the frames.

We write

$$\mathbf{N}_{22} = \begin{pmatrix} \mathbf{n}_{11} & \mathbf{n}_{12} & \mathbf{n}_{13} \\ \mathbf{n}_{12}{}^{\mathrm{T}} & \mathbf{n}_{22} & \mathbf{n}_{23} \\ \mathbf{n}_{13}{}^{\mathrm{T}} & \mathbf{n}_{23}{}^{\mathrm{T}} & \mathbf{n}_{33} \end{pmatrix}$$

and get

$$\begin{aligned}
\mathbf{n}_{11} &= \sum_{\nu=1}^{n} \mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{T}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{T} \\
&\quad + \boldsymbol{\Phi}_{\xi}{}^{\mathrm{T}}(\boldsymbol{\Phi}_{\xi}\boldsymbol{\Sigma}_{\xi\xi}\boldsymbol{\Phi}_{\xi}{}^{\mathrm{T}})^{-1}\boldsymbol{\Phi}_{\xi}, \\
\mathbf{n}_{12} &= \sum_{\nu=1}^{n} t_{\nu}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{T}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{L}, \\
\mathbf{n}_{13} &= \sum_{\nu=1}^{n} P_{\nu}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{T}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{M}, \\
\mathbf{n}_{22} &= \sum_{\nu=1}^{n} t_{\nu}^{2}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{L}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{L} + \mathbf{L}_{\mathrm{r}}{}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\mu\mu}{}^{-1}\mathbf{L}_{\mathrm{r}} \\
&\quad + \boldsymbol{\Phi}_{\mu}{}^{\mathrm{T}}(\boldsymbol{\Phi}_{\mu}\boldsymbol{\Sigma}_{\mu\mu}{}^{-1}\boldsymbol{\Phi}_{\mu}{}^{\mathrm{T}})^{-1}\boldsymbol{\Phi}_{\mu}, \\
\mathbf{n}_{23} &= \sum_{\nu=1}^{n} t_{\nu}P_{\nu}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{L}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{M}, \\
\mathbf{n}_{33} &= \sum_{\nu=1}^{n} P_{\nu}^{2}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu} + \mathbf{M}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\xi\xi}{}^{-1}\mathbf{M} + \mathbf{M}_{\mathrm{r}}{}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{r}\varpi\varpi}{}^{-1}\mathbf{M}_{\mathrm{r}} \\
&\quad + \boldsymbol{\Phi}_{\varpi}{}^{\mathrm{T}}(\boldsymbol{\Phi}_{\varpi}\boldsymbol{\Sigma}_{\varpi\varpi}{}^{-1}\boldsymbol{\Phi}_{\varpi}{}^{\mathrm{T}})^{-1}\boldsymbol{\Phi}_{\varpi}.
\end{aligned} \tag{23}$$

These expressions are much less formidable than they look if one remembers what we have said about the structures of the matrices occurring in them. $\sum_{\nu=1}^{n}\mathbf{S}_{\nu}{}^{\mathrm{T}}\mathbf{S}_{\nu}$, for instance is the diagonal matrix whose terms are the numbers that show on how many frames the ν-th star was imaged; none of the terms can vanish because we assume that each star was imaged on at least one frame. The second matrix term in $\mathbf{n}_{11}$ is also diagonal, the terms are composed of the sums of the reciprocal variances (relative weights) of the individual reference positions and the third matrix term, unfortunately not diagonal, is related to the covariance matrices of the position constraints. $\mathbf{n}_{12}$ and $\mathbf{n}_{13}$ are again diagonal and contain the cross terms in the matrices of the normal equations from which the star parameter estimates are computed.

$\mathbf{n}_{22}$ is composed analogously. The first term consists of the diagonal matrix whose terms are the sums of the $t_{\nu}{}^{2}$ of those plates on which the star occurs, followed by the weighted sum of the t^{2} of the reference positions, to which is added (for each star) the sum of the weights of the reference proper motions, and only the last term is nondiagonal and related to the covariance matrix of the constrained proper motions. $\mathbf{n}_{23}$ is again a diagonal matrix and contains the cross terms between proper motion and parallax in the coefficient matrix of the normal equations.

$\mathbf{n}_{33}$ consists of the diagonal matrix of the sums of the squares of the parallax factors of the plates on which the star was imaged, followed by the weighted sums of the squared parallax factors at the epochs of the reference positions used, which are in turn followed by the weighted number of reference parallaxes and finally, by the nondiagonal contribution to the coefficient matrix from the correlated constraints on the parallaxes.

The following appears to be a workable strategy for solving the problem. Start with solving $\mathbf{N}_{33}\boldsymbol{\pi} = \boldsymbol{\ell}$ (where $\boldsymbol{\pi}$ are the star parameters and $\boldsymbol{\ell}$ the appropriate right hand sides) to find approximate values for the star parameters and use these in a block Gauss-Seidel scheme. There is reason to believe that this will converge rapidly. The matrix can also be inverted directly by taking advantage of the ease with which $\mathbf{N}_{11}$ is inverted; one never has to invert a matrix of dimensions larger than $3m \times 3m$.

5. The simultaneous reduction of measurements in both coordinates

For simplicity's sake, we have set up the (matrix **N** of the) normal Eqs. [Eqs. 20, 21, 22 and 23] as if they were independent in both coordinates, i.e. as if positions, proper motions *and parallaxes* were different in each coordinate. In physical reality, however, the parallaxes are not direction-specific and they must therefore have the same values regardless from which equations they were estimated. We will now show how we can remove this handicap in practice.

Suppose we have to solve a general adjustment problem in which there are three sets of unknowns (adjusment parameters): $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{y}$, and that the (linear and properly weighted) condition equations for estimating their values are

$$(\mathbf{A}_1 \quad \mathbf{0} \quad \mathbf{B}_1)\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{\ell}_1$$

$$\text{and } (\mathbf{0} \quad \mathbf{A}_2 \quad \mathbf{B}_2)\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{\ell}_2, \tag{24}$$

where the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$ as well as the vectors $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ are given.

We see that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ occur explicitly only in the first and second, respectively, set of condition equations but that $\boldsymbol{y}$ occurs in both. If we were estimating $\boldsymbol{x}_1$ and $\boldsymbol{y}$ from the first, and independently $\boldsymbol{x}_2$ and $\boldsymbol{y}$ from the second set of these equations only, we would get

$$\boldsymbol{x}_{\nu} = (\mathbf{A}_{\nu}{}^{\mathrm{T}}\mathbf{A}_{\nu})^{-1}\mathbf{A}_{\nu}(\boldsymbol{\ell}_{\nu} - \mathbf{B}_{\nu}\boldsymbol{y}); \qquad \nu \in \{1,2\} \tag{25}$$

and

$$\begin{aligned}
&\{\mathbf{B}_{\nu}{}^{\mathrm{T}}[\mathbf{I} - \mathbf{A}_{\nu}(\mathbf{A}_{\nu}{}^{\mathrm{T}}\mathbf{A}_{\nu})^{-1}\mathbf{A}_{\nu}{}^{\mathrm{T}}]\mathbf{B}_{\nu}\}\boldsymbol{y} \\
&= [\mathbf{B}_{\nu}{}^{\mathrm{T}} - (\mathbf{A}_{\nu}{}^{\mathrm{T}}\mathbf{A}_{\nu})^{-1}\mathbf{A}_{\nu}{}^{\mathrm{T}}]\boldsymbol{\ell}_{\nu}; \quad \nu \in \{1,2\}.
\end{aligned} \tag{26}$$

This equation will yield (only slightly, one hopes) different estimates for $\boldsymbol{y}$ depending on whether $\nu = 1$ or $\nu = 2$.

We can make sure that there will be only one unambiguous estimate for $\boldsymbol{y}$ if we combine the two sets of condition Eqs. 24, which then would look like this:

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\ell}_1 \\ \boldsymbol{\ell}_2 \end{pmatrix},$$

whence

$$\sum_{\nu=1}^{2} \{\mathbf{B}_\nu{}^{\mathrm{T}}[\mathbf{I} - \mathbf{A}_\nu(\mathbf{A}_\nu{}^{\mathrm{T}}\mathbf{A}_\nu)^{-1}\mathbf{A}_\nu{}^{\mathrm{T}}]\mathbf{B}_\nu\}\boldsymbol{y} = \sum_{\nu=1}^{2} [\mathbf{B}_\nu{}^{\mathrm{T}} - (\mathbf{A}_\nu{}^{\mathrm{T}}\mathbf{A}_\nu)^{-1}\mathbf{A}_\nu{}^{\mathrm{T}}]\boldsymbol{\ell}_\nu, \tag{27}$$

while the Eqs. 25 for computing $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (once $\boldsymbol{y}$ is known) remain the same. Eqs. 27 are exactly the same equations we would have obtained had we added Eqs. 26 for $\nu = 1$ and $\nu = 2$. This means that we can set up the condition Eqs. 24 for the x- and the y-direction separately and eliminate everything except $\boldsymbol{y}$, add the reduced equations and get the unambiguous estimates by solving the sum of these equations for $\boldsymbol{y}$. For calculating $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we must use the $\boldsymbol{y}$ computed from Eqs. 27 and not those obtained by solving the two sets of Eqs. 26.

This shows also the two reasons why one should not determine separate and thus different "best" estimates for $\boldsymbol{y}$ from each set of condition Eqs. 24 and then accept the weighted mean as the definitive value: First, only if the coefficient matrix of $\boldsymbol{y}$ in Eqs. 27 were diagonal would the weighted mean be identical with the solution of Eqs. 27 and second, because $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ must properly be computed with the unambiguous $\boldsymbol{y}$ obtained by solving Eqs. 27 and not from the several different $\boldsymbol{y}$ obtained by solving Eqs. 26.

Accordingly, we partition the matrices $\mathbf{N}_{\mathrm{x}}$ and $\mathbf{N}_{\mathrm{y}}$ (i.e. one each in both coordinates) into four submatrices so, that the matrices $\mathbf{n}_{\mathrm{x}33}$ and $\mathbf{n}_{\mathrm{y}33}$ are the lower right submatrices:

$$\mathbf{N}_{\mathrm{x}} = \begin{pmatrix} \mathbf{A}_{\mathrm{x}} & \mathbf{C}_{\mathrm{x}} \\ \mathbf{C}_{\mathrm{x}}{}^{\mathrm{T}} & \mathbf{n}_{\mathrm{x}33} \end{pmatrix} \text{ and } \mathbf{N}_{\mathrm{y}} = \begin{pmatrix} \mathbf{A}_{\mathrm{y}} & \mathbf{C}_{\mathrm{y}} \\ \mathbf{C}_{\mathrm{y}}{}^{\mathrm{T}} & \mathbf{n}_{\mathrm{y}33} \end{pmatrix} \tag{28}$$

and eliminate all but $\boldsymbol{\pi}$, getting (analogously to Eq. 26

$$\mathbf{Q}_{\mathrm{x}}\boldsymbol{\pi}_{\mathrm{x}} = \boldsymbol{\ell}_{\mathrm{x}}, \qquad \mathbf{Q}_{\mathrm{y}}\boldsymbol{\pi}_{\mathrm{y}} = \boldsymbol{\ell}_{\mathrm{y}}. \tag{29}$$

According to Eq. 27, we get the equation for the computation of $\boldsymbol{\varpi}$ as

$$(\mathbf{Q}_{\mathrm{x}} + \mathbf{Q}_{\mathrm{y}})\boldsymbol{\pi} = \boldsymbol{\ell}_{\mathrm{x}} + \boldsymbol{\ell}_{\mathrm{y}}, \tag{30}$$

and we must use $\boldsymbol{\pi}$ as obtained from Eq. 30 when we resubstitute into the equations whose matrices are given in Eqs. 28 to obtain the rest of the adjustment parameters.

6. Star- and frame variances

It stands to reason [cf. (Eichhorn and Jefferys 1971)] that not all frames will yield equally precise measurements and also, that (for whatever reason) the coordinates of some of the stars can consistently be measured more precisely than those of some others. Therefore it makes sense to abandon – in the course of a second iteration – the premise that all measurements on the frames have the same variance. We still maintain, however, the (probably not strictly correct) assumption that all measurements are uncorrelated.

In the (prevailing) absence of a priori information, we cannot but estimate the appropriate variances from the residuals after a preliminary parameter estimation, carried out with undifferentiated variances for the measurements on the frames. Since the equations of condition are linear and solved with respect to the observed quantities (and – though this is irrelevant in the present context – linear in all variables), the adjustment residuals are found very easily. Under the circumstances, one might calculate the variance characteristic for a particular star (or a particular frame) by simply taking the average of the squares of the appropriate residuals. In this context we must remember, however, that the estimated parameters are not the true ones but those which minimize a particular quadratic form; *the residuals calculated by using the estimated parameters must thus be smaller than those which would have been found by using the (alas, unavailable) true ones*.

When quantities that are functions of the adjustment parameters are evaluated not with the true, but with the estimated values of these parameters, the accidental errors of the latter are propagated as systematic errors into these quantities. Eichhorn and Williams (1963) have given formulas that estimate the variances of these systematic errors. It seems therefore reasonable to assume that the actual variances would be estimated without bias by the averages of the squared actual residuals plus the appropriate parameter variances. Before the averages are taken, the squares of the residuals should be augmented by the appropriate parameter variances.

Eichhorn and Jefferys calculated the parallaxes according to the precepts outlined in their paper, they did not add the parameter variances and at first, attempted to iterate i.e. find improved values for the star- and the frame-variances during each iteration. This led, however, to patently absurd results: It turned out that three stars always emerged as the only ones whose star-variances did not grow beyond all reasonable bounds. The reason for this may be as follows. If a star, after the initial iteration, is assigned a high variance, the information provided by this star will have even less influence on the results of the second iteration than it had on the first, and this will naturally tend to increase the residuals of this material, because a low-weight star will not – if figurative speech is permitted – pull the solutions as strongly toward itself as a high-weight (i.e. low-variance) star and so, it will eventually be eliminated in the course of the iterations. Adding the parameter variances may reverse this process, but only numerical calculation on the actual measurements will clarify the appropriate procedure for estimating frame- and star-variances from the adjustment residuals. Intuition on the one hand (for what it is worth) suggests that one should be able to obtain the best estimates for the star- and frame-variances as the result of some iterative process; on the other hand, the just described train of reasoning suggests why iterations for the variances will diverge. We need numbers and experiments with actual measurements to arrive at a definitive answer.

Acknowledgements. The work on this paper was performed during July and August 1995, when I had the pleasure of working at the Osservatorio Astronomico di Torino at Pino Torinese in Italy. In particular I thank for their hospitality Attilio Ferrari and Mario Lattanzi, and especially Richard Smart for many a stimulating discussion and a thoughtful critique of the manuscript, for which I am also indebted to my colleagues Haywood Smith and Yu-lin Xu. Remarks by an unknown referee contributed significantly to the clarity of the presentation.

Appendix A: notation

Subscripts are symbols when upright, but are numbers when they are slanted or italic

$\boldsymbol{a}$: the vector of the adjustment parameters

$\underset{n\times k}{\mathbf{A}} = \begin{pmatrix} \boldsymbol{A}_1{}^{\mathrm{T}} \\ \boldsymbol{A}_2{}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{A}_n{}^{\mathrm{T}} \end{pmatrix}$: the matrix of the frame parameters

$\boldsymbol{A}_\nu{}^{\mathrm{T}} = (A_\nu \quad B_\nu \quad C_\nu)$: the frame parameters (for the x-coordinate) on the ν-th frame, based on the model $\xi = x + Ax + By + C$. (six constant model, three in each coordinate)

$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{\mathrm{i}} \\ \mathbf{B}_{\mathrm{r}} \\ \mathbf{B}_{\mathrm{c}} \end{pmatrix}$: the matrix of the condition equations

$\mathbf{B}_{\mathrm{c}}$: the lowest submatrix of $\mathbf{B}$ i.e. the coefficient matrix generated by the vector of the star-parameter constraints

$\mathbf{B}_{\mathrm{i}}$: the top submatrix of the coefficient matrix $\mathbf{B}$ of the condition eqations, generated by the measurements of the star images' coordinates on the frames

$\mathbf{B}_{\mathrm{r}}$: the submatrix of $\mathbf{B}$ that consists of the rows generated by the reference material (positions, proper motions, parallaxes)

$\underset{k\times\ell}{\mathbf{C}} = (\boldsymbol{C}_1\ \boldsymbol{C}_2\ \boldsymbol{C}_3)$: the constraint matrix $\mathbf{C}$ (for $\ell = 3$).

$\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{x}) = \boldsymbol{0}$: the set of condition equations

$\boldsymbol{F}_0$: $\boldsymbol{F}(\boldsymbol{0}, \boldsymbol{x})$

$\boldsymbol{J}_n$: an n-vector whose every component is 1

k : the number of independent frame parameters per frame (the same for all frames) in one coordinate (The developments in this paper were written for $k = 3$, k is therefore also used as the total number of stars for which reference positions are available)

$\underset{\ell\times k}{\mathbf{K}}$: the constraint matrix $\mathbf{K}$

ℓ : the number of independent star parameters per star (the same for all stars) in one coordinate

ℓ_ν : the number of stars whose images were not measured on the ν-th frame

$\boldsymbol{\ell}$: the vector on the right hand side of the reduced normal equations $\mathbf{N}\boldsymbol{\pi} = \boldsymbol{\ell}$

ℓ_κ : the number of independent extraneous reference positions available for the star numbered p_κ

ℓ_ρ : the number of independent extraneous reference proper motions available for the star numbered p_ρ

$\underset{\Lambda\times m}{\mathbf{L}}$: a matrix of the same structure as $\mathbf{T}$, but where the 1's are replaced by the epoch of the reference positions with which they are connected

$\mathbf{L}_{\mathrm{r}}$: a submatrix of $\mathbf{B}_{\mathrm{r}}$ whose nonzero terms are all 1 and which is generated by the available reference proper-motions

m : the total number of stars involved in the adjustment

$m_1, \ldots, m_{\mathrm{r}}$: the ordinal numbers of the stars for which reference proper motions are available

$\underset{\Lambda\times m}{\mathbf{M}}$: a matrix of the same structure as $\mathbf{T}$, but with the 1's replaced by the parallax factors at the epoch for which the appropriate reference positions are valid

$\mathbf{M}_{\mathrm{r}}$: a submatrix of $\mathbf{B}_{\mathrm{r}}$ whose nonzero terms are all 1 and which is generated by the reference parallaxes

n : the total number of frames involved in the adjustment

n_μ : the number of frames on which an image of the μ-th star was measured

$\mathbf{n}_{\mu\nu} \quad (\mu, \nu \in \{1, 2, 3\})$: submatrices of $\mathbf{N}_{22}$

$\mathbf{N} = \begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{12}{}^{\mathrm{T}} & \mathbf{N}_{22} \end{pmatrix}$: the coefficient matrix of the normal equations

$\mathbf{N}_{\mathrm{x}}, \mathbf{N}_{\mathrm{y}}$: the matrices $\mathbf{N}$ generated by the data in the x- and y directions, respectively

$p_1, \ldots, p_k$: The ordinal number of the stars for which there are extraneous reference positions

P : the total number of available reference proper-motions

P_ν : the parallax factor (in the x-direction) at the epoch t_ν at which the ν-th frame was taken

$\boldsymbol{P}_n$: an n-vector whose components are the parallax factors of the n frames

r : the number of stars for which reference proper-motions are available

$\mathbf{S}_\nu$: The matrix formed by replacing the rows $\nu_1, \ldots, \nu_{\ell_\nu}$ in the identity matrix $\mathbf{I}_m$ by nullvectors

$\mathbf{S}_{\mu\nu}$: the submatrices of the matrix $\mathbf{B}\boldsymbol{\Sigma}^{-1}$

t_ν : the epoch of the ν-th frame, reckoned from t_0, the zero mark adopted for reckoning time

$\boldsymbol{T}_n$: an n-vector whose components are the epochs of the n frames

$\underset{\Lambda\times m}{\mathbf{T}}$: a submatrix of $\mathbf{B}_{\mathrm{r}}$, namely the matrix of coefficients (all of which are 1) of the zero-epoch positions in the matrix of the condition equations

$\boldsymbol{x}$: the vector of observables

$\underset{n\times m}{\mathbf{x}} = (x_{\nu\mu})$: the matrix of the true values of the rectangular coordinates of the stars' images on the frames

$\underset{n\times m}{\hat{\mathbf{x}}} = (\hat{x}_{\nu\mu})$: the matrix of the measured values of the stars' images rectangular $x-$coordinates on the frames

$x_{\nu\mu}$: the measured x-coordinate of the image of the μ-th star on the ν-th frame

x_μ, y_μ : approximate coordinates of the μ-th star on all frames. We may use them because (a) of the aforementioned stability of the tangential point, which is the zero-point of the coordinate system with respect to which x andf y are reckoned and (b) almost all stars move move so slowly that they change their coordinates only little during the time interval over which the frames were taken

$\boldsymbol{x}_\nu{}^{\mathrm{T}} = (x_{\nu 1} \quad x_{\nu 2} \quad \ldots \quad x_{\nu m})$: the vector of x-coordinates measured on the ν-th frame

$\underset{\ell \times m}{\mathbf{X}} = (X_{\lambda\mu})$: The matrix of the star parameters (i.e. zero epoch position, proper motion in the x-direction, parallax *&c.*) of all stars

$\underset{n \times \ell}{\mathbf{\Gamma}} = \left(\gamma_{\nu\lambda}\right)$: the matrix of coefficients of the star parameters in the equations linking the the star parameters and the standard coordinates [Eq. 2], such as 1, t_ν (the epoch of the ν-th frame) &c

$\boldsymbol{\gamma}_\nu{}^{\mathrm{T}} = (1 \quad t_\nu \quad P_\nu)$: the vector formed by the ν-th line in the matrix of the star model coefficients

$\underset{n \times m}{\boldsymbol{\varepsilon}}$: the matrix of the corrections from the measured to the true values of the stars' rectangular coordinates on the frames

$\boldsymbol{\zeta}$: the m-vector $(\zeta_1\ \zeta_2\ \ldots\ \zeta_m)$ of the zero-point corrections in the star model; a vector of star parameters

$\boldsymbol{\zeta}_\mu{}^{\mathrm{T}} = (\zeta_\mu \quad \mu_\mu \quad \varpi_\mu)$: the vector of the modified star parameters of the μ-th star

Λ : the total number of incorporated reference positions

μ_μ : the ξ- (or rather x-) component of the μ-th star's proper motion. Note that "μ" appears here in two different meanings: as symbol for a quantity, but also (when it is a subscript) as counter for the star number. This is not likely to cause confusion.

$\mu_{\nu_1}, \ \ldots \ , \mu_{\nu_{\ell_\nu}}$: the ordinal numbers of the stars whose images were *not* measured on the ν-th frame

$\boldsymbol{\mu}$: the m-vector $(\mu_1\ \mu_2\ \ldots\ \mu_m)^{\mathrm{T}}$ of the stars' proper motions

$\nu_1,\ \nu_2.\ \ldots\ \nu_{\ell_\nu}$: see $\mu_{\nu_1}, \ \ldots \ , \mu_{\nu_{\ell_\nu}}$

$\xi_{0\mu}$: The standard coordinate ξ of the μ-th star for the epoch $t = 0$ and parallax factor 0

$\xi_{\nu\mu},\ \eta_{\nu\mu}$: the standard coordinates (in the same units as the measurements $x_{\nu\mu}$) of the μ-th star on the ν-th frame with respect to the tangential point on the ν-th frame. (Except for corrections of the first order, the spherical coordinates of the tangential point are the same on all frames. Also, we assume that each frame covers the same field)

$\boldsymbol{\xi}_\mu{}^{\mathrm{T}} = (\xi_\mu \quad \mu_\mu \quad \varpi_\mu)$: the vector of the star parameters of the μ-th star

$\underset{n \times m}{\boldsymbol{\xi}} = \left(\xi_{\nu\mu}\right)$: the matrix of the standard coordinates of the stars' images on the frames

ϖ_μ : the parallax of the μ-th star

$\boldsymbol{\varpi}^{\mathrm{T}}$: the vector $(\varpi_1\ \varpi_2\ \ldots\ \varpi_m)$ of the stars' parallaxes

$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\mu} \\ \boldsymbol{\varpi} \end{pmatrix}$: the vector of the star parameters

Π : the total number of reference parallaxes available for all stars

$\mathbf{\Sigma}$: the covariance matrix of the observations, reference material and parameter constraints, diagonal except for the part that belongs to the parameter constraints, which is blockdiagonal

$\underset{3m \times 3m}{\mathbf{\Sigma}_{\mathrm{c}}} = \mathrm{diag}\,(\mathbf{\Sigma}_{\xi\xi} \quad \mathbf{\Sigma}_{\mu\mu} \quad \mathbf{\Sigma}_{\varpi\varpi})$: the diagonal matrix of the variances of the predicted values of the constrained star parameters, with arbitrary values for the variances of the non-predicted parameters

$\mathbf{\Sigma}_{\mathrm{r}} = \mathrm{diag}\,(\mathbf{\Sigma}_{\mathrm{r}\mu\mu} \quad \mathbf{\Sigma}_{\mathrm{r}\varpi\varpi} \quad \mathbf{\Sigma}_{\mathrm{r}\xi\xi})$: the diagonal covariance matrix of the reference data

$\underset{9 \times 9}{\mathbf{\Sigma}_{\mathrm{CC}}}$: the covariance matrix of the components of the constraint matrix $\mathbf{C}$ (for $\ell = k = 3$).

$\underset{m \times m}{\mathbf{\Sigma}_{\mu\mu}}$: the diagonal covariance matrix of the predicted proper-motions

$\underset{m \times m}{\mathbf{\Sigma}_{\xi\xi}}$: the diagonal covariance matrix of the predicted positions

$\underset{m \times m}{\mathbf{\Sigma}_{\varpi\varpi}}$: the diagonal covariance matrix of the predicted parallaxes

$\mathbf{\Sigma}_{\mathrm{r}\mu\mu}$: the diagonal covariance matrix of the reference proper-motions

$\mathbf{\Sigma}_{\mathrm{r}\varpi\varpi}$: the diagonal covariance matrix of the reference parallaxes

$\mathbf{\Sigma}_{\mathrm{r}\xi\xi}$: the diagonal covariance matrix of the reference positions

$\underset{3 \times m}{\mathbf{\Phi}} = (\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2 \quad \ldots \quad \boldsymbol{\varphi}_m)$ with $\boldsymbol{\varphi}_\mu{}^{\mathrm{T}} = (x_\mu \quad y_\mu \quad 1)$: the frame-model coefficient matrix (for $k = 3$)

$\underset{3 \times m}{\mathbf{\Phi}_\mu}$: the matrix $\mathbf{\Phi}$ in which the columns corresponding to stars whose proper motions are not subjected to the parameter constraints are replaced by null vectors (for $k = 3$)

$\underset{3 \times m}{\mathbf{\Phi}_\nu}$: produced by replacing in the matrix $\mathbf{\Phi}$ by nullvectors those colums corresponding to images not measured on the ν-th frame (for $k = 3$)

$\underset{3 \times m}{\mathbf{\Phi}_\varpi}$: the matrix $\mathbf{\Phi}$ in which the columns corresponding to stars whose parallaxes are not subjected to the parameter constraints are replaced by null vectors (for $k = 3$)

$\underset{3 \times m}{\mathbf{\Phi}_\xi}$: the matrix $\mathbf{\Phi}$ in which the columns corresponding to stars whose positions are not included in the parameter constraints are replaced by null vectors (for $k = 3$)

References

Eichhorn, H.: 1975, MNRAS 173, 619

Eichhorn, H.: 1988, ApJ 334, 465

Eichhorn, H.: 1996, Celst. Mech. 64, 273

Eichhorn,H. and Jefferys, W.H.: 1971, in P.A. Ianna (ed.), *Proceedings of the Fourth Astrometric Conference*, p. 267, Leander McCormick Observatory, Charlottesville, VA, Pub. L. McCormick. Obs. vol. XVI

Eichhorn, H. and Williams, C.A.: 1963, AJ 68, 221

Gatewood, G.: 1987, AJ 94, 213

Gatewood, G.: 1992, PASP 104, 23

Gatewood, G., Kiewiet de Jonge, J., and Stephenson, B.: 1993, AJ 105, 1179

Monet, D., Dahn, C., Vrba, F. et al.: 1992, AJ 103, 638

Murray, C.A. and Corben, P.M.: 1979, MNRAS 187, 723

Schlesinger, F.: 1911, ApJ 33, 161

van Altena, W.F.: 1974, AJ 79, 826

Wielen, R., Bernstein, H.-H., Dettbarn, C. et al.: 1994, in L.V. Morrison and G.F. Gilmore (eds.), *Galactic and Solar System Optical Astrometry*, p. 74, Cambridge University Press, Cambridge