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A second order Laplace-Lagrange theory applied to the uranian satellite system

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Abstract. Computer algebra is used to derive analytical expressions for the elements of a secular perturbation theory of the second order in the masses. The linear part of this theory is incorporated into the standard Laplace-Lagrange solution which is then applied to the motion of the uranian satellites. This enables the effects of mean motion near-resonances in this system to be modelled. The results are in very good agreement with the filtered output of a numerical integration by Malhotra et al (1989). The additional complexities of studying the outer solar system using a similar approach are discussed.

Key words: celestial mechanics – satellites: satellites of Uranus – methods: analytical

1. Introduction

The satellite system of the planet Uranus is an ideal testing ground for secular perturbation theories since (i) currently there are no mean motion resonances between the satellites and (ii) the satellites' masses, eccentricities and inclinations are small enough to make most non-linear contributions negligible.

Dermott & Nicholson (1986) made use of these special properties to set important constraints on the masses of the satellites and especially the outer pair, Titania and Oberon, by means of a linear theory that also took account of the effects of planetary oblateness. The Voyager 2 flyby in January 1986 provided the planetary science community with a set of reliable limits on the masses of all five major satellites.

Laskar (1986) demonstrated that the existence of approximate relationships between the mean motions of the orbiting bodies (mean motion near-resonances or MMNRs for short) in the system renders the traditional Laplace-Lagrange theory incapable of providing accurate estimates for the satellite masses. These near-resonances introduce periodic variations in the solution, the amplitudes of which are comparable to those computed by considering the effect of secular perturbations alone, at least if the treatment is restricted to the first order in the masses.

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An attempt by Laskar & Jacobson (1987) to fit the theory described by Laskar (1986) to Voyager and Earth-based observations, the latter spanning approximately one orbital period of Uranus, necessitated the introduction of empirical corrections to their derived secular eigenfrequencies, possibly due to the unmodelled effect of these near-resonances. The first analytical treatment of these phenomena was performed by Malhotra et al (1989) (hereafter referred to as MFMN) in which the effects of the 2:1 Umbriel-Titania and the 3:2 Titania-Oberon near-resonances on the Laplace-Lagrange model were quantified. The result was found to be in very good agreement with a numerical integration described in the same paper. However, there still remained a discrepancy exceeding the 1% level (a benchmark set by their results for the inclination variations) that can probably be attributed to the effect of weaker near-resonances.

The subject of this paper is the description and rigorous testing of a model that is based on a general theory of the second order in the masses. This was described in detail by Duriez (1979) and was subsequently presented in a more systematic form by Laskar (1985). The advantage of this theory, with respect to the model developed by MFMN, is that, by performing a *complete* averaging over short period variables, it allows a precise determination of the long period behaviour of the orbital elements.

The applications that appear in the existing literature, however, involve a direct numerical computation of the relevant terms in which the analytical form of the expressions used is not preserved. The advent of symbolic computation, which is already being widely used in solar system dynamics, has enabled us to make some progress in this area. In particular, we have taken advantage of Laskar's special formulation of the theory to write a program in Mathematica (Wolfram 1991) that can express the coefficients of individual terms explicitly in terms of the planetary or satellite orbital elements. In this form the individual contributions of *all* candidate near-resonances are explicitly modelled and can thus be included or excluded from a particular application. By using this program to model the uranian system we have produced results which, when compared to the approximate theory of MFMN, are in better agreement (relative error $< 1\%$) with their numerical modelling results.

In the following section we give a detailed description of this theory and explain how it can be formulated as an extension of the classical Laplace-Lagrange solution for the long term variations in the orbital elements of a system of bodies orbiting a central object. We then apply this model to the test satellite system used in MFMN and compare the results with their analytical and numerical solution. We then proceed to discuss the results of these comparisons and what they tell us about the importance of various dynamical processes in our attempt to model this system analytically.

2. Analytical model

2.1. The general formulation

The configuration of a system of n point-mass bodies orbiting a central object at a specific time is uniquely defined by the set $\{a_i, e_i, I_i, \varpi_i, \Omega_i, M_i\}_{i=1,2,\cdots,n}$ where a is the semimajor axis, e the eccentricity, I the inclination, ϖ the longitude of pericentre, Ω the longitude of the ascending node and M the mean anomaly corresponding to the ith body's osculating keplerian orbit.

The theoretical construct that we shall use, taken from Duriez (1979) and Laskar (1985), dictates a switch to a new set of variables commonly referred to as regular elliptic variables. This new set is itself based on the commonly used osculating elements of a keplerian orbit and can be written as $\{p_i, q_i, z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i\}_{i=1,2,\cdots,n}$ where

$$z_i = e_i \exp \mathrm{i}\varpi_i \qquad \text{and} \qquad \zeta_i = \sin(I_i/2) \exp \mathrm{i}\Omega_i$$

where bars denote the complex conjugates and $\mathrm{i}^2 = -1$.

In order to define p and q we introduce the concept of the mean mean motion N and mean semimajor axis A. In a general theory of the type described below these quantities are considered to be constants of the motion and represent the fundamental parameters of circular and coplanar orbits of the bodies under consideration in the vicinity of which the theoretical model is constructed. If we define p and q by

$$a_i = A_i(1 + p_i)^{-2/3} \qquad \text{and} \qquad \lambda_i = N_i t - \mathrm{i}q_i$$

where λ_i is the mean longitude of the ith body, then these new variables denote deviations in the quantities A and Nt. Note here that z_i and ζ_i are complex quantities, whereas p_i is real and q_i is imaginary.

The classical Lagrange equations that describe the variation of the new elements with time are written as in the paper by Laskar (1985) where we have dropped the body index i:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{3\mathrm{i}}{NA^2}(1+p)^{4/3}\frac{\partial R}{\partial q} \tag{1}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \mathrm{i}Np + \mathrm{i}\frac{(1+p)^{1/3}}{NA^2}\left[3(1+p)\frac{\partial R}{\partial p} + \phi\psi\left(z\frac{\partial R}{\partial z} + \bar{z}\frac{\partial R}{\partial \bar{z}}\right) + \frac{1}{2\phi}\left(\zeta\frac{\partial R}{\partial \zeta} + \bar{\zeta}\frac{\partial R}{\partial \bar{\zeta}}\right)\right] \tag{2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}\frac{(1+p)^{1/3}}{NA^2}\left[2\phi\frac{\partial R}{\partial \bar{z}} - \phi\psi z\frac{\partial R}{\partial q} + \frac{z}{2\phi}\left(\zeta\frac{\partial R}{\partial \zeta} + \bar{\zeta}\frac{\partial R}{\partial \bar{\zeta}}\right)\right] \tag{3}$$

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \mathrm{i}\frac{(1+p)^{1/3}}{2\phi NA^2}\left[\frac{\partial R}{\partial \bar{\zeta}} - \zeta\frac{\partial R}{\partial q} + \left(-z\frac{\partial R}{\partial z} + \bar{z}\frac{\partial R}{\partial \bar{z}}\right)\right] \tag{4}$$

where

$$\phi = \sqrt{1 - z\bar{z}}, \tag{5}$$

$$\psi = 1/(1+\phi) \tag{6}$$

and R is the cummulative planetary disturbing function which contains, in series form, the relevant perturbations to the body's motion as a result of gravitational interactions with the other members of the system as well as perturbations due to planetary oblateness.

These equations may also be written in the general form:

$$\frac{d\sigma_i}{dt} = \mathrm{i}L_i^{(\sigma)}(p, q, z, \bar{z}, \zeta, \bar{\zeta}, t) + \mathrm{i}c_\sigma N_i p_i, \quad (i = 1, \cdots, n) \tag{7}$$

where $\sigma \in \{p, q, z, \bar{z}, \zeta, \bar{\zeta}\}$,

$$c_\sigma = \begin{cases} 1 & \text{if } \sigma = q \\ 0 & \text{if } \sigma \neq q \end{cases} \tag{8}$$

and

$$L_i^{(\sigma)}(p, q, z, \bar{z}, \zeta, \bar{\zeta}, t) = N_i \sum_{j\neq i} \frac{m_j}{M_{\mathrm{c}}(1 + m_i/M_{\mathrm{c}})} \sum_{n_{10}\in\mathbb{N}_{10}} \sum_{l\in\mathbb{Z}} C^{(\sigma)}_{n_{10},l}(\alpha_{ij}) \times S_{ij}(p, z, \bar{z}, \zeta, \bar{\zeta}) \exp\left[k_i q_i + k_j q_j + \mathrm{i}\left(k_i N_i + k_j N_j\right) t\right] \tag{9}$$

where $i \neq j$, $n_{10} = \{g_i, g_j, n_i, \bar{n}_i, n_j, \bar{n}_j, \nu_i, \bar{\nu}_i, \nu_j, \bar{\nu}_j\}$, $\mathbb{N}_{10}$ is the set containing all 10-fold natural number vectors, $\mathbb{Z}$ denotes the set of integer numbers, $\alpha_{ij} = \min\left(A_i, A_j\right) / \max\left(A_i, A_j\right)$ and S_{ij} is of the form $p_i^{g_i} z_i^{n_i} \bar{z}_i^{\bar{n}_i} \zeta_i^{\nu_i} \bar{\zeta}_i^{\bar{\nu}_i} p_j^{g_j} z_j^{n_j} \bar{z}_j^{\bar{n}_j} \zeta_j^{\nu_j} \bar{\zeta}_j^{\bar{\nu}_j}$.

The d'Alembert rules for Eq. (7) can be written as

$$k_i = \bar{n}_i + \bar{\nu}_i - n_i - \nu_i + l + \tau(\sigma) \tag{10}$$

$$k_j = \bar{n}_j + \bar{\nu}_j - n_j - \nu_j - l \tag{11}$$

where $l \in \mathbb{Z}$ and

$$\tau(\sigma) = \begin{cases} 0 & \text{for } \sigma \in \{p, q\} \\ 1 & \text{for } \sigma \in \{z, \zeta\} \\ -1 & \text{for } \sigma \in \{\bar{z}, \bar{\zeta}\} \end{cases} . \tag{12}$$

We should stress here that the above are *not* the usual d'Alembert rules related to the disturbing function, but are instead relationships between the powers and time coefficients of the series terms found in the right-hand side of the Lagrange equations.

In addition they are generally different for each equation due to the form of Eq. (12).

Note that the additional term Np in Eq. (2) is due to the fact that the right-hand sides of Lagrange's equations have been expressed in terms of A and N instead of the corresponding osculating variables. The osculating semimajor axis a and mean motion n contain both short and long period time dependent terms and will therefore have different values, at any given time, from A and N respectively.

We can, in fact, derive estimates for these differences by noting that, if we take into consideration only the secular perturbation components of the above equations, then, according to Poisson's theorem, we can assume p to be constant, i.e. $\mathrm{d}p/\mathrm{d}t = 0$ up to the second order in the masses. From the definitions of p and q it is evident that by computing a non-zero solution for $\{p_i\}_{i=1,2,\cdots,n}$ we will be able to use the set of mean elements as required by our model. If we express the concept of the mean mean motion by means of a simple mathematical relationship such as

$$\lim_{t\to\infty} \frac{\lambda(t)}{t} = N$$

we can the use the definition of q to write this relationship as:

$$\lim_{t\to\infty} \dot{q}(t) = 0$$

since $q(t) \simeq \dot{q}(t)\cdot t$. This result can only be satisfied if the constant part of the $\mathrm{d}q/\mathrm{d}t$ equation, i.e. the terms where $n_{10} = \{g_i, g_j, 0, 0, 0, 0, 0, 0, 0, 0\}$ with $g_i, g_j \geq 0$ and $\{k_i, k_j\} = \{0, 0\}$, is equal to zero. We are thus able to create a linear system of n equations with n unknowns which will give us non-zero values for the constants $\{p_i\}_{i=1,2,\cdots,n}$. This treatment requires, of course, a knowledge of the mean mean motions for each satellite which can be acquired, for example, by means of a numerical integration.

By truncating Eqs. (3) and (4) to the first order in the masses and the first degree in z and ζ respectively, one has two separate integrable systems of n differential equations

$$\dot{\mathbf{z}} = \mathrm{i}C\mathbf{z}\,, \quad \dot{\boldsymbol{\zeta}} = \mathrm{i}D\boldsymbol{\zeta} \tag{13}$$

where C and D are $n \times n$ matrices whose elements are given in terms of $\{m_i, N_i, A_i, p_i\}_{i=1,2,\cdots,n}$. This simple model, when equipped with a set of initial conditions, can produce a solution that describes the long term evolution of the eccentricities and inclinations for all the members of the system. The solution for each member can be written as

$$z_i(t) = \sum_{j=1}^{n} C_j^{(i)} \exp\left[\mathrm{i}(g_j t + \beta_j)\right] \tag{14}$$

$$\zeta_i(t) = \sum_{j=1}^{n} D_j^{(i)} \exp\left[\mathrm{i}(f_j t + \gamma_j)\right] \tag{15}$$

where C_j,D_j are the real scaled amplitude vectors corresponding to eigenfrequencies g_j,f_j and phases β_j,γ_j respectively. The above expressions are commonly referred to as the Laplace-Lagrange solution.

Special care should be taken concerning the actual values of p that are going to be used in such an analytical model when a non-spherical central object is involved. In this case the osculating semimajor axis is affected by oblateness; this means that additional terms will have to be included in the linear system of equations from which the p corrections are derived. The form of these terms can be deduced by inserting the truncated potential

$$R_{\mathrm{obl}} \simeq \frac{1}{2} N^2 A^2 \left(\frac{R_{\mathrm{p}}}{A}\right)^2 (1+p)^2 J_2 \left(1 + \frac{3}{2} z\bar{z} - 6\zeta\bar{\zeta}\right)$$

generated by an oblate planet of equatorial radius R_{p} and second gravitational moment J_2 into Eq. (2).

2.2. Extension to second order in the masses

The validity of the results obtained by using this simple secular perturbation model depends mostly on whether the terms of the disturbing function that are included in the model are responsible for the dominant variations in the full long period solution for the eccentricity or the inclination. In a configuration similar to the jovian or saturnian satellite systems, where a large number of mean motion resonances are at work, the corresponding resonant arguments are more important for the purposes of secular perturbation modelling than the linear secular terms of the Lagrange equations. Moreover, even when the system is free of actual resonances, the presence of near-commensurabilities is sufficient to affect the system appreciably, resulting in significant discrepancies between the Laplace-Lagrange solution and the numerical one.

In this instance, it is possible to model analytically the effect of these near-resonances on the Laplace-Lagrange model but only if we extend our secular theory to the second order in the masses. If we incorporate the $6n$ variables $\{p_i, q_i, z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i\}_{i=1,2,\cdots,n}$ into a vector $\mathbf{V}$, we can write Eq. (7) in the form:

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \Lambda(\mathbf{V}, t) \tag{16}$$

where t is the time variable.

Let us now choose a new variable $\mathbf{V_o}$ which is related to the old variable $\mathbf{V}$ by

$$\mathbf{V} = \mathbf{V_o} \;+\; \Delta\mathbf{V}(\mathbf{V_o}, t) \tag{17}$$

where the function $\Delta\mathbf{V}(\mathbf{V_o}, t)$ is to be chosen such that $\mathbf{V_o}$ satisfies *exactly* the secular system, i.e. a system for which all the terms on the left-hand side of Eq. (16) have the property $k_i = k_j = 0$ (see Eq. (9)) and, hence, all the short period variables are averaged over.

By substituting Eq. (17) in (16) we can write the derivative of the quantity $\mathbf{V}$ as a function of $\mathbf{V_o}$, that is,

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{V_o}}{\mathrm{d}t} + \frac{\partial\Delta\mathbf{V}}{\partial t}(\mathbf{V_o}, t) + \frac{\partial\Delta\mathbf{V}}{\partial\mathbf{V_o}}(\mathbf{V_o}, t)\frac{\mathrm{d}\mathbf{V_o}}{\mathrm{d}t} \tag{18}$$

Expanding the right-hand side of Eq. (16) as a Taylor series about $\mathbf{V_o}$ gives:

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \Lambda(\mathbf{V_o}, t) + \frac{\partial \Lambda}{\partial \mathbf{V_o}}(\mathbf{V_o}, t)\Delta\mathbf{V}(\mathbf{V_o}, t) + \cdots \tag{19}$$

We can now derive an expression for $\Delta\mathbf{V}$ by writing any mixed collection S of series terms, i.e. one that contains secular and non-secular terms, as:

$$S = \langle\, S\, \rangle + \{\, S\, \}$$

where $\langle\, \cdots\, \rangle$ denotes the secular part of S and $\{\, \cdots\, \}$ the non–secular part.

If we use this operator on the two expressions for $\mathrm{d}\mathbf{V}/\mathrm{d}t$ given in Eqs. (18) and (19) we can simplify the resulting equation if we consider that $\mathbf{V_o}$ must satisfy the equation

$$\begin{aligned}\frac{\mathrm{d}\mathbf{V_o}}{\mathrm{d}t} \;=\; & \langle \Lambda(\mathbf{V_o}, t)\rangle \\ & + \left\langle \frac{\partial \Lambda}{\partial \mathbf{V_o}}(\mathbf{V_o}, t)\Delta_1\mathbf{V}(\mathbf{V_o}, t)\right\rangle + \cdots .\end{aligned} \tag{20}$$

Hence

$$\begin{aligned}\frac{\partial \Delta\mathbf{V}}{\partial t}(\mathbf{V_o}, t) \;=\; & -\frac{\partial \Delta\mathbf{V}}{\partial \mathbf{V_o}}(\mathbf{V_o}, t)\frac{\mathrm{d}\mathbf{V_o}}{\mathrm{d}t} \\ & + \{\Lambda(\mathbf{V_o}, t)\} \\ & + \left\{ \frac{\partial \Lambda}{\partial \mathbf{V_o}}(\mathbf{V_o}, t)\Delta\mathbf{V}(\mathbf{V_o}, t)\right\} + \cdots .\end{aligned} \tag{21}$$

We now limit our treatment to the second order system that is given by

$$\frac{\mathrm{d}\mathbf{V_o}}{\mathrm{d}t} = \langle \Lambda(\mathbf{V_o}, t)\rangle + \left\langle \frac{\partial \Lambda}{\partial \mathbf{V_o}}(\mathbf{V_o}, t)\Delta_1\mathbf{V}(\mathbf{V_o}, t)\right\rangle \tag{22}$$

where $\Delta_1\mathbf{V}(\mathbf{V_o}, t)$ is given by the first order approximation of (21), that is

$$\frac{\partial \Delta_1\mathbf{V}}{\partial t}(\mathbf{V_o}, t) = \{\Lambda(\mathbf{V_o}, t)\} \tag{23}$$

An extensive and detailed description of this theory is given in Duriez (1979) where it is used to generate a Poisson series solution for the long period variations in the orbital elements of the major planets.

It is quite straightforward to understand how this theory takes into account the proximity of the system to mean motion resonances. From the derivation of the transformation function $\Delta_1\mathbf{V}$ (see Eq. (23)) it is evident that each term of the second order part of Eq. (22) will include a divisor of the form $(k_iN_i + k_jN_j)$ since it involves a partial integration with respect to time of a term of the form given in Eq. (9). In the vicinity of a mean motion resonance some of these divisors will become small, thus causing the coefficients of the corresponding terms to contribute substantially to the solution.

The rules which determine the relationships between the coefficients of λ_i, λ_j and the powers of $z_i, \bar{z}_i, z_j, \bar{z}_j, \zeta_i, \bar{\zeta}_i, \zeta_j, \bar{\zeta}_j$ in every term of Eq. (22) are essentially the same as in Eqs. (1)–(4), albeit in a generalised form. Laskar (1985) expressed these rules in terms of quantities that can be computed for an arbitrary order term, namely the characteristic c_{M} of the monomial part of the term, the degree d of the term and the characteristic c_{I} of the longitude-dependent part of the term. For a term of the form given in Eq. (9) these are defined as

$$c_{\mathrm{M}}(\mathrm{term}) = \bar{n}_i + \bar{n}_j + \bar{\nu}_i + \bar{\nu}_j - (n_i + n_j + \nu_i + \nu_j) \tag{24}$$

$$d(\mathrm{term}) = n_i + \bar{n}_i + n_j + \bar{n}_j + \nu_i + \bar{\nu}_i + \nu_j + \bar{\nu}_j \tag{25}$$

$$c_{\mathrm{I}}(\mathrm{term}) = k_i + k_j \tag{26}$$

where we use the same notation. This formulation enables us to simplify greatly the problem of finding all the combinations of terms in $\partial\Lambda/\partial\mathbf{V_o}(\mathbf{V_o}, t)$ and $\Delta_1\mathbf{V}(\mathbf{V_o}, t)$ that, when multiplied with each other, give rise to a specific secular term that exists in the second order part of Eq. (22). Indeed, for two terms T_1 and T_2 that are used in the product $(\partial L_{ij}^{(\sigma)}/\partial\gamma_k)\Delta_1\gamma_{kl}$ we have

$$d(T_1) + d(T_2) = d(T) \tag{27}$$

$$c_{\mathrm{m}}(T_1) + c_{\mathrm{m}}(T_2) = -\tau(\sigma) \tag{28}$$

where T is the required secular term found on the right-hand side of equation $\mathrm{d}\sigma_i/\mathrm{d}t$. Here γ can be any of the variables used and the pairs $\{i, j\}$ and $\{k, l\}$ denote the perturbed and perturbing bodies respectively in Lagrange's equations. The problem is thus reduced to computing the coefficients of all the terms in the right-hand side of Eq. (7) whose characteristics and degrees satisfy Eqs. (27) and (28).

It is this scheme that resulted in the semi-analytical treatment of the entire solar system by Laskar (1986a, 1988) and the discovery of chaotic wandering in the long period orbital behaviour of the inner planets by Laskar (1990). As a consequence of these rules, one expects to find linear second order terms which can be added to the Laplace-Lagrange model in the form of a correction matrix δC or δD. In particular, suppose that one wishes to compute the perturbation caused by a near-resonance between the kth and lth bodies with $A_k < A_l$. The second order matrix in such a case will have the form:

$$\begin{array}{c} \\ k \\ \\ l \\ \\ \end{array}\begin{array}{c} \qquad k \qquad\qquad l \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \delta a_{kk} & \cdots & \delta a_{kl} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \delta a_{lk} & \cdots & \delta a_{ll} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\end{array} \tag{29}$$

where the actual values of the four non-zero elements $\delta a_{kk}, \delta a_{kl}, \delta a_{lk}, \delta a_{ll}$ depend on the orbital parameters of this system.

Table 1. Initial configuration of the uranian satellite system used in our work. Bracketed values denote "averaged" orbital elements derived by MFMN

Satellite	$m/M_{\rm p}$ ($\times 10^5$)	a (km)	e	I (deg.)	ϖ (deg.)	Ω (deg.)
Miranda	0.1	129775	0.0027 (0.002564)	4.22 (4.2805)	111 (112.69)	21 (21.00)
Ariel	1.8	190822	0.0034 (0.003330)	0.310 (0.3105)	120 (116.75)	263 (263.08)
Umbriel	1.1	265832	0.0050 (0.005342)	0.360 (0.3602)	193 (197.53)	279 (279.05)
Titania	3.2	436035	0.0022 (0.001347)	0.142 (0.1421)	147 (143.19)	311 (311.07)
Oberon	3.4	583117	0.0008 (0.001331)	0.101 (0.1009)	212 (180.80)	234 (234.00)

Physical parameters used: $R_{\rm p}$ = 26200 km, J_2 = 0.003345, J_4 = -0.0000321,
$GM_{\rm p}$ = 5.784184 $\times$ 10^6 km^{3}sec^{-2}

The use of computer algebra enables us not only to express explicitly these elements in terms of the above-mentioned orbital parameters but also allows us at the same time to recognise and choose the terms that correspond to a particular near-resonant argument. In applying this theory we have also used the algorithm by Duriez (1977) which computes the numerical coefficients of any first order term in Eqs. (1)-(4) efficiently without expanding the relevant series up to the point that the required term dictates. This is achieved by precomputing and storing several series in tables that are then combined in a unique way to produce the coefficient of a given term. By implementing this algorithm with the aid of computer algebra we have again been able to express the resulting coefficients explicitly in terms of the system parameters; this has a distinct advantage over repeatedly computing the same coefficients numerically for different values of $\{m_i, N_i, A_i\}_{i=1,2,\cdots,n}$ since, in our case, these expressions are generated only once and are then stored for use in various applications.

3. An example

In order for this novel and, admittedly, non-trivial theoretical scheme and its scope of applications to be better understood, we present an example of its use. This illustrates a smooth transition from the general theory to the actual dynamical modelling of the uranian satellite system. For this purpose we use the initial data in Table 1 taken from MFMN.

An important point to be made here is that the initial values for the osculating orbital elements cannot be used directly in the analytical secular perturbation theory since they are convolved with short period effects. MFMN overcame this problem by generating a synthetic theory from their numerical integration and evaluating their long period Fourier series at $t = 0$, generating a set of "averaged" initial values. These are the bracketed numbers shown in Table 1 and are the ones used in this work. However, MFMN did not apply this procedure to the semimajor axes and mean motions of the satellites. Instead they used a simple analytical correction to account for the effects of oblateness.

We now build up the standard Laplace-Lagrange theory by computing the elements of matrix C that appears in Eq. (13) (we restrict our modelling effort to the eccentricities). The elements of the matrix are computed by means of the coefficients presented in Appendix A, i.e.

Table 2. Corrections to the semimajor axes used in the analytical theory

a_i (km)	A_i (km)	p_i ($\times 10^5$)
129775	129742	-39.450
190822	190798	-17.855
265832	265824	-14.704
436035	436030	$-\ \ 7.698$
583117	583078	-20.977

$$c_{ii} = \sum_{j\neq i} N_i \frac{m_j}{1+m_i} c^{\star}_{ii}(A_i, A_j, p_i, p_j) \tag{30}$$

$$c_{ij} = N_i \frac{m_j}{1+m_i} c^{\star}_{ij}(A_i, A_j, p_i, p_j) \tag{31}$$

Our computer algebra code can represent those elements as in Eq. (9) with $C^{(\sigma)}_{n_{10},l}(\alpha_{ij})$ expressed in terms of the hypergeometric function of the second kind; we denote this by

$$H(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{32}$$

Due to the planet's non-zero oblateness, the following terms have to be included in the diagonal elements of C (Greenberg 1981):

$$\begin{aligned} \delta c^{\rm (obl)}_{ii} = N_i \Bigg[& \frac{3}{2} J_2 \left(\frac{R_{\rm p}}{A_i}\right)^2 (1+p_i)^{7/3} \\ & + \frac{63}{8} {J_2}^2 \left(\frac{R_{\rm p}}{A_i}\right)^4 (1+p_i)^{11/3} \\ & - \frac{15}{4} J_4 \left(\frac{R_{\rm p}}{A_i}\right)^4 (1+p_i)^{11/3} \Bigg] \end{aligned} \tag{33}$$

where J_2 and J_4 are the most important coefficients in the development of the gravitational potential of an oblate planet and $R_{\rm p}$ is the planet's equatorial radius.

Table 3. Comparison of eigenfrequencies (°/yr) obtained by numerical and Laplace-Lagrange modelling

Mode number	Num. integration	L-L theory	Rel. error
1	20.299	20.283	−0.1%
2	6.000	5.961	−0.65%
3	2.909	2.855	−1.85%
4	1.924	1.608	−16.50%
5	0.367	0.352	−4.10%

Table 4. Number of term combinations for the linear second order terms

Order	Number of "regular" terms	Number of "enhanced" terms
0	4	2
1	6	2
2	2	0
Total	12	4

Furthermore, our model has to account for the non-zero values of the p quantities, as described in the previous chapter. The p corrections will be incorporated into the model by noting that each term of the general form $c_1 z$ that contributes to the Laplace-Lagrange matrix has a corresponding "p-term" of the form $pc_2 z$.

The values for A_i were computed by fitting a slope to a sufficiently long time series of the mean longitudes λ_i of the five satellites followed by an application of Kepler's third law. The time series were produced by a numerical integration of the system with a mixed variable symplectic integrator by Levison & Duncan (1994) found in the public domain.

At this point we decided to split the system into two groups and perform two numerical integrations of different timespans for each group, namely 150 yrs for Miranda and Ariel and 1500 yrs for the outer group. This course of action was dictated by the need to improve the resolution of the sampling for the inner group while at the same time including all potentially significant contributions to the value of N. The timespan for this group was set to correspond with the approximate period of the third precession eigenfrequency of the system which should provide the most significant contributions of the third order in the masses for N_1, N_2. The results of these steps can be summarised in Table 2.

Finally, the contribution of nonlinear terms due to Miranda's relatively large inclination (see Table 1) is treated in a way similar to that used in MFMN, that is, by computing the coefficients of 3rd degree terms in Lagrange's equations of the general form

Table 5. Comparison of eigenfrequencies (°/yr) obtained by numerical and extended Laplace-Lagrange modelling for specific MMNRs, namely the 2:1 Umbriel-Titania and the 3:2 Titania-Oberon ones

Mode number	Num. integration	Extended L-L	Rel. error
1	20.299	20.283	−0.05%
2	6.000	5.962	−0.65%
3	2.909	2.872	−1.25%
4	1.924	1.893	−1.60%
5	0.367	0.365	−0.55%

$z_i \zeta_1 \bar{\zeta}_1$, $(i = 1, \cdots, n)$. We then approximate the square of Miranda's inclination variable by the quantity

$$\mathbf{I}_1^2 = \sum_{j=1}^{n} {D_j^{(1)}}^2, \tag{34}$$

where in general $D_j^{(i)}$ is the coordinate of the jth inclination eigenvector for the ith body.

Table 3 shows a comparison of the results obtained through the application of this standard linear model with the numerical analysis performed by MFMN, in terms of the resulting eigenfrequencies. The large discrepancy between the two groups of estimates, and especially the 4th eigenfrequency, where the error is $\sim$ 16%, is evident. As shown in MFMN, it is the near-commensurabilities present in the satellite system of Uranus and mainly the 2:1 Titania-Umbriel and 3:2 Oberon-Titania MMNRs that distort the simple dynamics of the standard Laplace-Lagrange model.

We shall now apply the extended scheme to these particular arguments in order to illustrate the workings of the theory in detail and demonstrate its efficiency.

The essential part of the construction of the second order theory, in the form that is presented here, is to find all the combinations of terms in $\Delta_1 \mathbf{V}(\mathbf{V_o}, t)$ and $\partial\Lambda/\partial\mathbf{V_o}(\mathbf{V_o}, t)$ that, when multiplied together, give rise to a linear secular term as their explicitly time-dependent parts are eliminated. The association of individual combinations of these terms with MMNRs is deduced by the form of the small divisor present in the solution for $\Delta_1 \mathbf{V}(\mathbf{V_o}, t)$.

Note, however, that in one instance the multiplication process becomes slightly more complicated. If we write Eq. (23) specifically for q we see that, due to the special form of Lagrange's equation for this variable we get:

$$\frac{\partial \Delta_1 q_i}{\partial t} = \left\{ L_i^{(\sigma)}(p, q, \bar{z}, \bar{\zeta}, t) \right\} + \mathrm{i} N_i \Delta_1 p_i, \; i = 1, \cdots, n. \tag{35}$$

This implies that some of the terms that appear in the solution for $\Delta_1 q$ have been derived by means of a *double* integration with respect to time that will give rise to a small divisor of the form $(k_i N_i + k_j N_j)^2$. In Table 4 we show the different types of term combinations according to the order of the near-resonance to which they are related (in general $r + s : r$ is called a sth

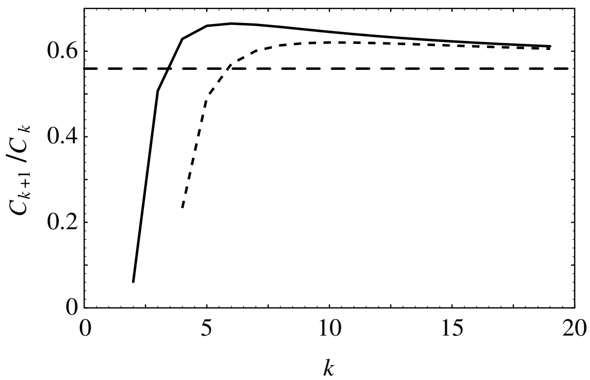

Fig. 1. Asymptotic behaviour of weak terms compared to α^2 (horizontal dashed line) for the satellite system used in MFMN. Solid and dashed lines are used to denote first and second order arguments respectively.

Table 6. Comparison of secular data obtained with two different analytical theories. Phase errors are given relative to 180°

Num. Integration	MFMN	Rel. Error	This paper	Rel. Error
Eigenfrequencies (°/yr)				
20.299	20.289	−0.05%	20.291	−0.05%
6.000	5.965	−0.60%	5.995	−0.10%
2.909	2.874	−1.20%	2.906	−0.10%
1.924	1.874	−2.60%	1.931	−0.35%
0.367	0.367	0.00%	0.367	0.00%
Amplitudes ($\times 10^6$)				
2346	2329	−0.70%	2344	−0.10%
3271	3278	0.20%	3277	0.20%
5183	5171	−0.20%	5185	0.05%
637	652	2.40%	640	0.45%
1902	1908	0.30%	1904	0.10%
Phases (degrees)				
111.91	111.9	0.00%	111.92	0.00%
99.50	100.3	0.20%	99.63	0.05%
191.06	191.6	0.15%	191.10	0.00%
113.16	115.4	0.60%	113.15	0.00%
166.96	167.0	0.00%	166.81	−0.10%

order resonance). Terms that include an "enhanced" small divisor provide contributions to 0th order (orbit proximity) and 1st order MMNRs.

It seems reasonable, therefore, to provide analytical expressions solely for these special terms and, in particular, the ones related to first order arguments. These expressions are given in Appendix B. We have evaluated these terms for the two most important MMNRs using the first order coefficients found in Appendix C. We have then incorporated these corrections in the Laplace-Lagrange system and produced the result shown in Table 5.

Table 7. Summary of arguments that make significant contributions to the second order theory for the uranian satellite system

Satellite pair	max $\lvert\delta C\rvert$ $> 10^{-1}$ °/yr	$> 10^{-2}$ °/yr	$> 10^{-3}$ °/yr
Miranda-Ariel	–	–	2:1, 3:2, 4:2, 4:3
Ariel-Umbriel	–	3:2, 5:3	–
Umbriel-Titania	–	2:1	4:2
Titania-Oberon	3:2	4:3, 5:3, 6:4	3:1–9:7, 5:4–8:7
Total	1	6	14

4. General results for the uranian system

We have extended our treatment to include all significant MMNRs (our definition of "significant" is given below) for every satellite pair and we have compared the resulting eigensystem with the results obtained by MFMN, as shown in Table 6. It is clear from this comparison that the extended Laplace-Lagrange scheme gives very good results for the uranian satellite system, perhaps even giving rise to the possibility of fitting reliable mass estimates to ground-based observations, if one succeeds in modelling short period variations sufficiently well.

The relatively large discrepancy we observe for g_4 could be related to unmodelled effects, such as, for instance, nonlinear terms (degree 3 and above) or linear terms that are generated by the interaction of three bodies. Although some of these effects can be evaluated by our algorithm, most notably the three body terms, we have chosen to keep our model as simple as possible to minimise computational cost.

During the modelling process it became clear that the magnitude of the contribution of various second order terms to the linear model is not purely a function of the amplifying effect of the small divisor. In fact, it tends to depend also on quantities such as the satellite masses, the order of the near-resonance involved, etc. To illustrate this, we have classified these contributions according to their "significance" which we have chosen to be the size of the maximum element in the matrix in Eq. (29) and the specific near-resonance that they are related to, as shown in Table 7. Every contribution smaller than a thousandth of a degree per year has been considered to be spurious, due to the limited accuracy of the numerical eigenfrequency estimates.

From this analysis, it is apparent that the modelling of second order $(r+2:r)$ MMNRs is essential if one wishes is to bring the error down to several tenths of a percent. The most notable examples of this result are the 6:4 argument between Titania and Oberon that provides $\sim$ 10% of the second order contribution for this satellite pair and, more significantly, the 5:3 argument involving satellites Ariel and Umbriel that provides $\sim$ 60% of their total.

We could not fail to notice the number of first order $(r+1:r)$ and second order $(r+2:r)$ MMNRs that contribute to the dynamics of the Titania-Oberon pair. It seems that apart from the 3:2 near-resonance that provides $\sim$ 90% of the total contribution, weaker MMNRs tend to contribute only at the thousandths of a degree per year level, but their significance decreases rather slowly with increasing r, so that the cummulative effect is quite respectable. The explanation for this effect can be found in Laskar (1985), where it is shown that the coefficient of second order terms for large values of r can be well approximated by a geometric series with ratio of order α^2, that is, if $r = \cdots, k, k+1, \cdots$ then $C_{k+1}/C_k \simeq \mathscr{O}(\alpha^2)$.

We have chosen here to include terms up to $r = 20$ beyond which the effect of neglecting this asymptotic tail is negligible. As seen in Fig. 1, the actual ratio at this point is $\simeq 0.6$ for first order and second order arguments. This ratio is not far from the approximate asymptotic value of $\alpha_{45}^2 \simeq 0.56$. If this tail were to affect the results significantly there is the option of modelling it by a geometric series of ratio 0.6 or α_{45}^2. In this way we can compensate for stopping the series at a particular value of r and strike a balance between computational cost and accuracy.

For reasons of completeness, we have also evaluated the proximity $(r:r)$ arguments for the Titania-Oberon pairs up to $r = 20$ and included them in the model, since their contribution is at the same level of significance as the asymptotic tails mentioned above.

5. Conclusions and discussion

In this paper we have described an analytical second order secular perturbation theory based on work done by Duriez (1979) and Laskar (1985) that has already found wide applications in the dynamical modelling of planetary and satellite systems (see, for example, Vienne & Duriez 1995 and Laskar 1990).

The use of computer algebra has enabled us to express the basic elements of this theory explicitly in terms of the masses, semimajor axes and mean motions of the members of the system, thus reducing the computational complexity of the model. We have also shown how this construct, restricted to the linear terms, can be formulated as an extension to the standard Laplace-Lagrange scheme and how it can correct for the effects of mean motion near-resonances on the long period behaviour of the orbital elements.

In the process of testing this scheme, we have produced a solution for the uranian satellite system using the initial data given in Malhotra et al (1989). Comparison of this solution with their numerical integration results reveals a marked improvement with respect to the eigensystem produced by a different second order theory described in the same paper.

This experiment has revealed a significant dependence of the second order contributions not only on the "strength" of the near-resonance between two satellites but on other factors such as, for example, the masses of the interacting pair.

In view of the above, it is not suprising that the analytical modelling of a system comprising the outer planets requires the computation of a great number of coefficients related to mean motion arguments ($r \simeq 40$) since the masses of the outer planets are at least an order of magnitude larger than the masses of the uranian satellites relative to the primary. More importantly, the most significant MMNR that exists in this system is the 5:2 relationship (the so-called Great Disparity) between the mean motions of Jupiter and Saturn. Since this is a third order near-resonance, its effect on secular pertubations only appears in the non-linear regime (third degree and above) as shown by Duriez (1979).

The above implies that the construction of an accurate secular perturbation theory for the outer planets necessitates the solving of an autonomous system of n nonlinear, first order O.D.E.s. Analytical treatments of this system, as opposed to a number of successful numerical (Sussman & Wisdom 1988) and semi-analytical (Nobili et al 1989, Laskar 1990) experiments have so far given rise to solutions in the form of secular variation ephemerides, as in Bretagnon & Simon (1990).

A solution or a model derived by strictly analytical means holds great potential for the study of long term planetary dynamics since it would be, at least in principle, possible to express the elements of this model as explicit functions of the fundamental parameters of the system, i.e. masses m, mean mean motions N and mean semimajor axes A. One could then proceed to study the behaviour of the solution in different regions of parameter space.

A noteworthy step in this direction is the work done by Brouwer & van Woerkom (1950), the results of which are used even to this day. There are, however, several points in the description of this theory that have remained unclear, making its subsequent reproduction using improved initial data almost impossible. Recent advances in the field of nonlinear dynamics have produced several well-established methods that can potentially provide a solution in the required form. It is hoped that this problem will be tackled successfully in the near future.

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Appendix A: auxiliary quantities for the computation of Laplace-Lagrange matrix elements

In the following expressions $H_{ij}^k(\alpha) = H(\frac{i}{2}, \frac{j}{2}, k; \alpha^2)$ where H denotes the hypergeometric function and $\alpha = \alpha_{ij} = \min(A_i, A_j) / \max(A_i, A_j)$.
We can then write:

$$c_{ii}^{(\text{ext})}(\alpha) = -\frac{3}{2}\alpha^3 H_{33}^1(\alpha) + \frac{9}{4}\alpha^3 H_{35}^2(\alpha) - \frac{9}{4}\alpha^5 H_{55}^2(\alpha) + \frac{45}{16}\alpha^5 H_{57}^3 \tag{A1}$$

$$c_{ij}^{(\text{ext})}(\alpha) = \frac{3}{2}\alpha^4 H_{35}^2(\alpha) + \frac{15}{16}\alpha^4 H_{37}^3(\alpha) - \frac{27}{8}\alpha^4 H_{55}^2(\alpha) + \frac{45}{16}\alpha^6 H_{57}^3(\alpha) + \frac{105}{64}\alpha^6 H_{59}^4(\alpha) \tag{A2}$$

$$c_{ii}^{(\mathrm{int})}(\alpha) = -\frac{3}{2}\alpha^2 H_{35}^2 + \frac{9}{4}\alpha^2 H_{55}^2(\alpha) - \frac{45}{16}\alpha^4 H_{57}^3(\alpha) \tag{A3}$$

$$c_{ij}^{(\mathrm{int})}(\alpha) = \frac{9}{4}\alpha H_{33}^1(\alpha) - \frac{9}{4}\alpha H_{35}^2(\alpha) - \frac{75}{32}\alpha^3 H_{37}^3(\alpha) + \frac{27}{8}\alpha^3 H_{55}^2(\alpha) - \frac{45}{16}\alpha^3 H_{57}^3(\alpha) - \frac{105}{64}\alpha^5 H_{59}^4(\alpha) \tag{A4}$$

$$c_{ii}^{\star}(A_i, A_j, p_i, p_j) = \begin{cases} c_{ii}^{(\mathrm{ext})}(\alpha_{ij}) + p_i c_{ii,i}^{(\mathrm{ext})}(\alpha_{ij}) + p_j c_{ii,j}^{(\mathrm{ext})}(\alpha_{ij}) & \text{if } A_i < A_j \\ c_{ii}^{(\mathrm{int})}(\alpha_{ij}) + p_i c_{ii,i}^{(\mathrm{int})}(\alpha_{ij}) + p_j c_{ii,j}^{(\mathrm{int})}(\alpha_{ij}) & \text{if } A_i > A_j \end{cases} \tag{A5}$$

$$c_{ij}^{\star}(A_i, A_j, p_i, p_j) = \begin{cases} c_{ij}^{(\mathrm{ext})}(\alpha_{ij}) + p_i c_{ij,i}^{(\mathrm{ext})}(\alpha_{ij}) + p_j c_{ij,j}^{(\mathrm{ext})}(\alpha_{ij}) & \text{if } A_i < A_j \\ c_{ij}^{(\mathrm{int})}(\alpha_{ij}) + p_i c_{ij,i}^{(\mathrm{int})}(\alpha_{ij}) + p_j c_{ij,j}^{(\mathrm{int})}(\alpha_{ij}) & \text{if } A_i > A_j \end{cases} \tag{A6}$$

where

$$c_{ii,i}^{(\mathrm{ext})}(\alpha) = \frac{3}{2}\alpha^3 H_{33}^1(\alpha) - \frac{9}{4}\alpha^3 H_{35}^2(\alpha) \tag{A7}$$

$$c_{ii,j}^{(\mathrm{ext})}(\alpha) = -3\alpha^3 H_{33}^1(\alpha) + \frac{9}{2}\alpha^3 H_{35}^2(\alpha) \tag{A8}$$

$$c_{ij,i}^{(\mathrm{ext})}(\alpha) = -\frac{5}{2}\alpha^4 H_{35}^2(\alpha) - \frac{25}{16}\alpha^4 H_{37}^3(\alpha) + \frac{45}{8}\alpha^4 H_{55}^2(\alpha) \tag{A9}$$

$$c_{ij,j}^{(\mathrm{ext})}(\alpha) = 4\alpha^4 H_{35}^2(\alpha) + \frac{5}{2}\alpha^4 H_{37}^3(\alpha) - 9\alpha^4 H_{55}^2(\alpha) \tag{A10}$$

$$c_{ii,i}^{(\mathrm{int})}(\alpha) = -\frac{7}{2}\alpha^2 H_{35}^2(\alpha) + \frac{21}{4}\alpha^2 H_{55}^2(\alpha) - \frac{225}{16}\alpha^4 H_{57}^3(\alpha) + \frac{75}{8}\alpha^4 H_{77}^3(\alpha) \tag{A11}$$

$$c_{ii,j}^{(\mathrm{int})}(\alpha) = 2\alpha^2 H_{35}^2(\alpha) - 3\alpha^2 H_{55}^2(\alpha) + \frac{45}{4}\alpha^4 H_{57}^3(\alpha) - \frac{75}{8}\alpha^4 H_{77}^3(\alpha) \tag{A12}$$

$$c_{ij,i}^{(\mathrm{int})}(\alpha) = \frac{15}{4}\alpha H_{33}^1(\alpha) - \frac{15}{4}\alpha H_{35}^2(\alpha) - \frac{225}{32}\alpha^3 H_{37}^3(\alpha) - \frac{225}{16}\alpha^3 H_{57}^3(\alpha) \tag{A13}$$

$$c_{ij,j}^{(\mathrm{int})}(\alpha) = -\frac{3}{2}\alpha H_{33}^1(\alpha) + \frac{3}{2}\alpha H_{35}^2(\alpha) + \frac{75}{16}\alpha^3 H_{37}^3(\alpha) + \frac{45}{4}\alpha^3 H_{57}^3(\alpha) \tag{A14}$$

Note that the in the case of the coefficients of p_i, p_j we have only included terms up to $\mathscr{O}(\alpha^4)$ since their contribution will be small.

Appendix B: expressions for the dominant terms in the nonzero elements of the second order correction matrix for first order MMNRs of the general form $r+1:r$

$$\begin{aligned}\delta c_{kk} = &-\frac{(r-1)N_k^2 m_l^2}{(1-r)N_k + rN_l} \times C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[(r-1)N_k - rN_l\right]\right) \\ &\times\left(N_k \frac{C_p^{(\mathrm{ext})}\left(\bar{z}_k \exp\left[(1-r)N_k + rN_l\right]\right)}{(1-r)N_k + rN_l} + C_q^{(\mathrm{ext})}\left(\bar{z}_k \exp\left[(1-r)N_k + rN_l\right]\right)\right) \\ &+\frac{rN_k N_l m_k m_l}{(1-r)N_k + rN_l} \times C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[(r-1)N_k - rN_l\right]\right) \\ &\times\left(N_l \frac{C_p^{(\mathrm{int})}\left(\bar{z}_k \exp\left[rN_l + (1-r)N_k\right]\right)}{(1-r)N_k + rN_l} + C_q^{(\mathrm{ext})}\left(\bar{z}_k \exp\left[rN_l + (1-r)N_k\right]\right)\right)\end{aligned} \tag{B1}$$

$$\delta c_{ll} = -\frac{(r-1)N_l^2 m_k^2}{(1-r)N_l + rN_k} \times C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[(r-1)N_l - rN_k\right]\right) \tag{B2}$$

$$\times\left(N_l\frac{C_p^{(\mathrm{int})}\left(\bar{z}_l\exp\left[(1-r)N_l+rN_k\right]\right)}{(1-r)N_l+rN_k}\right.$$
$$\left.+C_q^{(\mathrm{int})}\left(\bar{z}_l\exp\left[(1-r)N_l+rN_k\right]\right)\right)$$
$$+\frac{rN_kN_lm_km_l}{(1-r)N_l+rN_k}$$
$$\times C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[(r-1)N_l-rN_k\right]\right)\times$$
$$\times\left(N_k\frac{C_p^{(\mathrm{ext})}\left(\bar{z}_l\exp\left[rN_k+(1-r)N_l\right]\right)}{(1-r)N_l+rN_k}\right.$$
$$\left.+C_q^{(\mathrm{ext})}\left(\bar{z}_l\exp\left[rN_k+(1-r)N_l\right]\right)\right)$$

$$\delta c_{kl} = -\frac{(r-1)N_k^2m_l^2}{(1-r)N_k+rN_l} \tag{B3}$$
$$\times C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[(r-1)N_k-rN_l\right]\right)$$
$$\times\left(N_k\frac{C_p^{(\mathrm{ext})}\left(\bar{z}_l\exp\left[(1-r)N_k+rN_l\right]\right)}{(1-r)N_k+rN_l}\right.$$
$$\left.+C_q^{(\mathrm{ext})}\left(\bar{z}_l\exp\left[(1-r)N_k+rN_l\right]\right)\right)$$
$$+\frac{rN_kN_lm_km_l}{(1-r)N_k+rN_l}$$
$$\times C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[(r-1)N_k-rN_l\right]\right)$$
$$\times\left(N_l\frac{C_p^{(\mathrm{int})}\left(\bar{z}_l\exp\left[rN_l+(1-r)N_k\right]\right)}{(1-r)N_k+rN_l}\right.$$
$$\left.+C_q^{(\mathrm{int})}\left(\bar{z}_l\exp\left[rN_l+(1-r)N_k\right]\right)\right)$$

$$\delta c_{lk} = -\frac{(r-1)N_l^2m_k^2}{(1-r)N_l+rN_k} \tag{B4}$$
$$\times C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[(r-1)N_l-rN_k\right]\right)$$
$$\times\left(N_l\frac{C_p^{(\mathrm{int})}\left(\bar{z}_k\exp\left[(1-r)N_l+rN_k\right]\right)}{(1-r)N_l+rN_k}\right.$$
$$\left.+C_q^{(\mathrm{int})}\left(\bar{z}_k\exp\left[(1-r)N_l+rN_k\right]\right)\right)$$
$$+\frac{rN_kN_lm_km_l}{(1-r)N_l+rN_k}$$
$$\times C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[(r-1)N_l-rN_k\right]\right)$$
$$\times\left(N_k\frac{C_p^{(\mathrm{ext})}\left(\bar{z}_k\exp\left[rN_k+(1-r)N_l\right]\right)}{(1-r)N_l+rN_k}\right.$$
$$\left.+C_q^{(\mathrm{ext})}\left(\bar{z}_k\exp\left[rN_k+(1-r)N_l\right]\right)\right)$$

Appendix C: coefficients for first order arguments that contribute to the second order part of the system for specific near-resonances

1. Coefficients for the 2:1 MMNR:

$$C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[N_i-2N_j\right]\right)= \frac{9}{4}\alpha^3H_{35}^2(\alpha)-\frac{15}{8}\alpha^5H_{37}^3(\alpha) -\frac{35}{32}\alpha^5H_{39}^4(\alpha) \tag{C1}$$

$$C_p^{(\mathrm{ext})}\left(\bar{z}_i\exp\left[-N_i+2N_j\right]\right)= -\frac{27}{8}\alpha^3H_{35}^2(\alpha)-\frac{45}{16}\alpha^5H_{37}^3(\alpha) +\frac{735}{64}\alpha^5H_{39}^4(\alpha)-\frac{135}{32}\alpha^5H_{57}^3(\alpha) +\frac{2835}{512}\alpha^7H_{511}^5(\alpha) \tag{C2}$$

$$C_q^{(\mathrm{ext})}\left(\bar{z}_i\exp\left[-N_i+2N_j\right]\right)= \frac{63}{16}\alpha^3H_{35}^2(\alpha)-\frac{465}{32}\alpha^5H_{37}^3(\alpha) +\frac{1155}{128}\alpha^5H_{39}^4(\alpha)+\frac{45}{16}\alpha^5H_{57}^3(\alpha) -\frac{105}{16}\alpha^7H_{59}^4(\alpha)+\frac{945}{256}\alpha^7H_{511}^5(\alpha) \tag{C3}$$

$$C_p^{(\mathrm{int})}\left(\bar{z}_i\exp\left[2N_j-N_i\right]\right)= \frac{27}{4}\alpha^2H_{35}^2(\alpha)-\frac{105}{8}\alpha^4H_{39}^4(\alpha) +\frac{135}{32}\alpha^4H_{57}^3(\alpha)-\frac{2835}{512}\alpha^6H_{511}^5(\alpha) \tag{C4}$$

$$C_q^{(\mathrm{int})}\left(\bar{z}_i\exp\left[2N_j-N_i\right]\right)= \frac{9}{2}\alpha^2H_{35}^2(\alpha)-\frac{45}{4}\alpha^2H_{37}^3(\alpha) +\frac{35}{4}\alpha^4H_{39}^4(\alpha)+\frac{45}{16}\alpha^4H_{57}^3(\alpha) -\frac{105}{16}\alpha^4H_{59}^4(\alpha)+\frac{945}{256}\alpha^6H_{511}^5(\alpha) \tag{C5}$$

$$C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[-2N_j+N_i\right]\right)= \frac{1}{2\alpha^2}-\frac{1}{2}\alpha H_{33}^1(\alpha)-\frac{3}{2}\alpha H_{35}^2(\alpha) +\frac{45}{16}\alpha^3H_{37}^3(\alpha) \tag{C6}$$

$$C_p^{(\mathrm{int})}\left(\bar{z}_j \exp\left[2N_j - N_i\right]\right) = \frac{3}{2\alpha^2} - \frac{15}{4}\alpha H_{33}^1(\alpha) - \frac{9}{4}\alpha H_{35}^2(\alpha) + \frac{405}{32}\alpha^3 H_{37}^3(\alpha) - \frac{27}{8}\alpha^3 H_{55}^2(\alpha) + \frac{315}{64}\alpha^5 H_{59}^4(\alpha) \tag{C7}$$

$$C_q^{(\mathrm{int})}\left(\bar{z}_j \exp\left[2N_j - N_i\right]\right) = \frac{3}{8\alpha^2} - \frac{15}{8}\alpha H_{33}^1(\alpha) + \frac{51}{8}\alpha H_{35}^2(\alpha) - \frac{405}{64}\alpha^3 H_{37}^3(\alpha) - \frac{9}{4}\alpha^3 H_{55}^2(\alpha) + \frac{45}{8}\alpha^3 H_{57}^3(\alpha) - \frac{105}{32}\alpha^5 H_{59}^5(\alpha) \tag{C8}$$

$$C_p^{(\mathrm{ext})}\left(\bar{z}_j \exp\left[-N_i + 2N_j\right]\right) = -3\alpha^2 + 3\alpha^2 H_{33}^1(\alpha) - \frac{135}{16}\alpha^4 H_{37}^3(\alpha) + \frac{27}{8}\alpha^4 H_{55}^2(\alpha) - \frac{315}{64}\alpha^6 H_{59}^4(\alpha) \tag{C9}$$

$$C_q^{(\mathrm{ext})}\left(\bar{z}_j \exp\left[-N_i + 2N_j\right]\right) = 2\alpha^2 - 2\alpha^2 H_{33}^1(\alpha) + 9\alpha^4 H_{35}^2(\alpha) - \frac{45}{8}\alpha^4 H_{37}^3(\alpha) - \frac{9}{4}\alpha^4 H_{55}^2(\alpha) + \frac{45}{8}\alpha^6 H_{57}^3(\alpha) - \frac{105}{32}\alpha^6 H_{59}^4(\alpha) \tag{C10}$$

2. Coefficients for the 3:2 MMNR:

$$C_{\bar{z}}^{(\mathrm{ext})}\left(\exp\left[2N_i - 3N_j\right]\right) = \frac{45}{16}\alpha^4 H_{37}^3(\alpha) - \frac{35}{16}\alpha^6 H_{39}^4(\alpha) - \frac{315}{256}\alpha^6 H_{311}^5(\alpha) \tag{C11}$$

$$C_p^{(\mathrm{ext})}\left(\bar{z}_i \exp\left[-2N_i + 3N_j\right]\right) = -\frac{135}{16}\alpha^4 H_{37}^3(\alpha) - \frac{105}{32}\alpha^6 H_{39}^4(\alpha) + \frac{4725}{256}\alpha^6 H_{311}^5(\alpha) - \frac{315}{64}\alpha^6 H_{59}^4(\alpha) + \frac{6237}{1024}\alpha^8 H_{513}^6(\alpha) \tag{C12}$$

$$C_q^{(\mathrm{ext})}\left(\bar{z}_i \exp\left[-2N_i + 3N_j\right]\right) = \frac{495}{64}\alpha^4 H_{37}^3(\alpha) - \frac{1505}{64}\alpha^6 H_{39}^4(\alpha) + \frac{14175}{1024}\alpha^6 H_{311}^5(\alpha) + \frac{105}{32}\alpha^6 H_{59}^4(\alpha) - \frac{945}{128}\alpha^8 H_{511}^5(\alpha) + \frac{2079}{512}\alpha^8 H_{513}^6(\alpha) \tag{C13}$$

$$C_p^{(\mathrm{int})}\left(\bar{z}_i \exp\left[3N_j - 2N_i\right]\right) = \frac{405}{32}\alpha^3 H_{37}^3(\alpha) - \frac{10395}{512}\alpha^5 H_{311}^5(\alpha) + \frac{315}{64}\alpha^5 H_{59}^4(\alpha) - \frac{6237}{1024}\alpha^7 H_{513}^6(\alpha) \tag{C14}$$

$$C_q^{(\mathrm{int})}\left(\bar{z}_i \exp\left[3N_j - 2N_i\right]\right) = \frac{135}{16}\alpha^3 H_{37}^3(\alpha) - \frac{315}{16}\alpha^3 H_{39}^4(\alpha) + \frac{3465}{256}\alpha^5 H_{311}^5(\alpha) + \frac{105}{32}\alpha^5 H_{59}^4(\alpha) - \frac{945}{128}\alpha^5 H_{511}^5(\alpha) + \frac{2079}{512}\alpha^7 H_{513}^6(\alpha) \tag{C15}$$

$$C_{\bar{z}}^{(\mathrm{int})}\left(\exp\left[-3N_j + 2N_i\right]\right) = -\frac{3}{4}\alpha^2 H_{35}^2(\alpha) - \frac{15}{8}\alpha^2 H_{37}^3(\alpha) + \frac{105}{32}\alpha^4 H_{39}^4(\alpha) \tag{C16}$$

$$C_p^{(\mathrm{int})}\left(\bar{z}_j \exp\left[3N_j - 2N_i\right]\right) = -9\alpha^2 H_{35}^2(\alpha) - \frac{45}{16}\alpha^2 H_{37}^3(\alpha) + \frac{315}{16}\alpha^4 H_{39}^4(\alpha) - \frac{135}{32}\alpha^4 H_{57}^3(\alpha) + \frac{2835}{512}\alpha^6 H_{511}^5(\alpha) \tag{C17}$$

$$C_q^{(\mathrm{int})}\left(\bar{z}_j \exp\left[3N_j - 2N_i\right]\right) = -\frac{81}{16}\alpha^2 H_{35}^2(\alpha) + \frac{435}{32}\alpha^2 H_{37}^3(\alpha) - \frac{1365}{128}\alpha^4 H_{39}^4(\alpha) - \frac{45}{16}\alpha^4 H_{57}^3(\alpha) + \frac{105}{16}\alpha^4 H_{59}^4(\alpha) - \frac{945}{256}\alpha^6 H_{511}^5(\alpha) \tag{C18}$$

$$C_p^{(\mathrm{ext})}\left(\bar{z}_j \exp\left[-2N_i + 3N_j\right]\right) = \tag{C19}$$

$$\frac{63}{8}\alpha^3 H_{35}^2(\alpha) - \frac{945}{64}\alpha^5 H_{39}^4(\alpha)$$
$$-\frac{135}{32}\alpha^5 H_{57}^3(\alpha) - \frac{2835}{512}\alpha^7 H_{511}^5(\alpha)$$

$$C_q^{(\mathrm{ext})}\left(\bar{z}_j \exp\left[-2N_i + 3N_j\right]\right) = \tag{C20}$$
$$-\frac{21}{4}\alpha^3 H_{35}^2(\alpha) + \frac{135}{8}\alpha^5 H_{37}^3(\alpha)$$
$$-\frac{315}{32}\alpha^5 H_{39}^4(\alpha) - \frac{45}{16}\alpha^5 H_{57}^3(\alpha)$$
$$+\frac{105}{16}\alpha^7 H_{59}^4(\alpha) - \frac{945}{256}\alpha^7 H_{511}^5(\alpha)$$

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