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Acceleration of relativistic plasma in the magnetosphere of an axisymmetric rotator

S.V. Bogovalov

Moscow Engineering Physics Institute, Kashirskoje Shosse 31, 115409 Moscow, Russia (e-mail: bogoval@photon.mephi.ru)

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Abstract. The flow of relativistic plasma in the magnetosphere of an axisymmetrical rotator is considered analytically as well as numerically. The selfconsistent solution of the plasma outflow in a monopole-like poloidal magnetic field is obtained accurately up to terms proportional to σ/U_0^3, where σ is Michel's magnetization parameter, $U_0 = \gamma_0 v_0/c$, while γ_0 and v_0 are the initial Lorentz-factor and the velocity of the plasma. It is shown that under the condition $\sigma/U_0^3 \ll 1$ typical for real radio pulsars, the plasma is not accelerated and is not collimated in the subsonic region. Numerical simulation of the time dependent version of this problem shows that the stationary flow of relativistic plasma is apparently unstable when $\sigma/U_0 > 1$. This instability of the flow leads to the acceleration of the plasma.

Key words: MHD – pulsars; jets and outflows – galaxies; jets

1. Introduction

The discovery of radio pulsars has initiated considerable interest into the problem of the flow of relativistic plasmas in the magnetosphere of fast rotating neutron stars. This problem has become of more interest in recent years also in connection with the discovery of relativistic jets from some galactic astrophysical objects (Mirabel & Rodrigues 1996). The discovery of high energy and ultra high energy gamma - ray emission from AGN's (Wagner 1995) shows that the acceleration of plasma to ultrarelativistic energies occurs also in these objects (Ferrari et al.1996; Pelletier et al. 1996; Melia & Königl 1989). It is clear now that an analysis of the relativistic plasma flow is necessary for understanding the processes taking place in compact galactic objects and also in AGN's. However, in the present paper we concentrate our attention totally on the problem of relativistic plasma flow in the magnetosphere of radio pulsars.

In spite of systematic research in the field of the physics of radio pulsars in the last years, the structure of the magnetosphere and the mechanisms for the generation of electromagnetic emission in these objects remain vague to large extent (Lyubarsky 1995). Up to now, all that is generally agreed upon is the mechanisms of plasma production in radio pulsar magnetosphere. The basis of the theory of plasma production in radio pulsar magnetospheres was initiated by Sturrock (1971). This theory was developed in more detail for different conditions on the stellar surface by Ruderman & Sutherland (1975) and by Arons (1981). According to them, primary leptons (electrons or positrons) produced via e^+e^- pair production processes or extracted from the surface of the star are accelerated near the stellar surface by electric fields in the so called "inner gap". At Lorentz-factors $10^6 - 10^7$, the accelerated leptons generate curvature photons which are able to further produce new leptons in the curved magnetic field. The electromagnetic cascade is developed and the density of plasma increases. When the density becomes high enough to screen the electric field parallel to the magnetic field the acceleration of the plasma is stopped. Plasma with Lorentz-factors γ_0 and densities n_0 is formed. The distance from the stellar surface where this happens is of the order of several stellar radii. The flux of the kinetic energy of this plasma is much less than the total rotational losses of the fast rotating radio pulsars. Almost all energy is carried out by the electromagnetic field. The ratio of the Poynting flux to the flux of the kinetic energy of plasma is (Lyubarsky 1995)

$$\frac{\sigma}{\gamma_0} = 10\frac{H_0}{10^{12}G}(\frac{R_{star}}{10^6 cm})^3(\frac{1s}{P})^2, \sigma = \frac{H_0^2}{4\pi mc^2 n_0}(\frac{R_{star}\Omega}{c})^2 \quad (1)$$

where σ is the magnetization parameter introduced by Michel (1969), R_{star} is the radius of the star, Ω and P are the angular velocity and the period of rotation, H_0 is the magnetic field on the surface of the star.

This ratio is very large near the stellar surface for fast rotating pulsars. At the same time we know from observations of the Crab Nebula, for example, that this ratio at large distances from the pulsar ~ 0.5 (Kundt & Krotschek 1980). Moreover the theoretical analysis of the conditions in the Crab Nebula performed by Kennel & Coroniti (1984) shows that the ratio of the Poynting flux to the flux of the kinetic energy in the relativistic wind produced by the pulsar can be $\sim 10^{-3}$ at the distance from the pulsar $\sim 10^{18} cm$. It follows from the observations that there exists some mechanism for the transformation of the Poynting flux into the flux of the kinetic energy as the plasma travels from the star to infinity (Camenzind 1989). This mechanism is apparently the basic mechanism for plasma acceleration since it ensures the transformation of at least 50% of the rotational energy of the neutron star in to the kinetic energy of the plasma. Determination and investigation of this mechanism is one of the key problems in the physics of radio pulsars. The solution of this problem is inevitably necessary for understanding the mechanisms for the generation of the electromagnetic emission in radio pulsars.

The axis of rotation of real radio pulsars is not directed along the magnetic moment. In spite of this an axisymmetrically rotating star can be considered as an appropriate model for an investigation of the mechanism of the Poynting flux transformation in radio pulsar's magnetosphere. It appears that the axisymmetrically rotating star ejecting relativistic plasma has rotational energy losses comparable with the rotational losses of real pulsars. The energy of rotation is carried out near the surface of the star by the Poynting flux. Therefore the problem of the Poynting flux transformation can be considered in this model. At the same time this problem is remarkably simplifield in the model of the axisymmetrical rotator since the plasma flow can be considered as stationary and axisymmetric. This is why many attempts have been made to investigate the flow of a relativistic plasma in the model of the axisymmetrical rotator.

The density of the relativistic plasma produced in the electromagnetic cascade is high enough to screen the electric field parallel to the magnetic field. The plasma can be considered as cold to a first approximation (Lyubarsky 1995). The magnetohydrodynamical approximation can be used for the description of the flow of this plasma. Michel (1969) was the first to use this approach for the investigation of the relativistic plasma flow in the magnetosphere of an axisymmetrical rotator. He obtained the solution to the problem for the plasma flow in a prescribed monopole like magnetic field and It was found that the acceleration of the plasma in this system is not effective. Moreover the relativistic plasma appeared subsonic in the whole space up to infinity. It is obvious that the strongest limitation of Michel's solution is the assumption that the poloidal magnetic field is not affected by the moving plasma. The problem of the acceleration of a relativistic plasma in the magnetosphere of a rotating star taking into account the effect of the plasma on the poloidal magnetic field is not solved up to now.

The solution of the full selfconsistent problem of the stationary plasma flow in the magnetosphere of the axisymmetric rotator is connected with the solution of a set of nonlinear equations of mixed type. Several attempts to solve this problem have been made by several authors. Sakurai (1985) solved this problem numerically for a nonrelativistic plasma. For the relativistic case this problem was numerically investigated by Camenzind (1986a, 1986b, 1989). For slow rotation, a solution was obtained analytically by Bogovalov (1992). Unfortunately, the solution of the selfconsistent problem for fast rotation of the star encounters severe difficulties and a few approximations are needed. Several attempts have been made to solve the problem in the force - free approximation (Michel 1973; Mestel 1979; Scharlemann & Wagoner 1973, Beskin et al. 1983; Lyubarsky 1990; Sulkanen & Lovelace 1990). Another direction is the investigation of the self - similar problem (Blandfrod & Payne 1982; Lovelace et al. 1991; Contopoulos 1994; Tsinganos et al. 1993). These investigations have clarified the problem to a large extent. However, the problem of the relativistic plasma acceleration remains largely unsolved.

The plan of this paper is as follows. In Sect. 2 the equations describing the stationary flow of a cold relativistic plasma are presented. Sect. 3 is devoted to the analytical study of the problem of the stationary plasma outflow in the monopole like magnetic field. Special attention is directed to the Poynting flux dominated plasma flow under the conditions $\sigma/U_0 \gg 1$ and $\sigma/U_0^3 \ll 1$ typical for rapidly rotating radio pulsars. In Sect. 4 the equations and the method of solution of the nonstationary problem of the relativistic plasma flow are discussed. The results of the numerical solution of the time dependent problem of relativistic plasma outflow are presented in Sect. 5. The basic results of the work are discussed in the last section.

2. Basic equations of the stationary flow

The MHD equations governing the dynamics of a steady axisymmetric magnetized wind have been widely studied in several papers (see previous Section). We outline now the general properties of the relativistic MHD system following the formalism of Bogovalov (1994).

In an axisymmetric flow the magnetic field $\mathbf{H}$ can be expressed as a sum of the poloidal component $\mathbf{H_p}$ and the toroidal component $\mathbf{H}_\varphi$. The poloidal magnetic field can be expressed as

$$\mathbf{H_p} = \frac{\nabla\psi \times \mathbf{e}_\varphi}{\rho}, \quad (2)$$

where ρ is the distance to the axis of rotation and $\mathbf{e}_\varphi$ is the unit vector corresponding to the azimuthal direction around the axis z. The function ψ is proportional to the total flux of the poloidal magnetic field through a surface at the radius ρ. In the frozen in approximation the relationship between the electric field $\mathbf{E}$ and the poloidal magnetic field is $\mathbf{E} = (\rho\Omega/c)q(\psi)\mathbf{H_p} \times \mathbf{e}_\varphi$ (Weber & Davis 1967). The function $q(\psi)$ is constant along the poloidal field lines and describes their differential rotation.

The first equation defining the dynamics of the plasma along the poloidal field lines is the equation for the conservation of the specific energy flux

$$\gamma - F(\psi)xq(\psi)H_\varphi = W(\psi). \quad (3)$$

We neglect gravitation of the neutron star since the gravitational energy of leptons is much less than their kinetic energy. The second equation is the equation for the conservation of the specific angular momentum flux

$$xU_\varphi - F(\psi)xH_\varphi = M(\psi). \quad (4)$$

Projection of the frozen-in condition on the electric field gives

$$xqH_p + U_pH_\varphi = U_\varphi H_p. \quad (5)$$

Here $\mathbf{U}_p = \gamma \mathbf{v}_p/c$ is the four-velocity of the plasma along a field line, U_φ is the toroidal component of the four-velocity of the plasma, $F = H_p/(4\pi mncv_p)$, n is the density of the plasma, m is the mass of the particles, $x = \rho\Omega/c$. The functions $W(\psi)$ and $M(\psi)$, proportional to the energy and to the angular momentum flux per one particle, are constant along the field lines. Therefore they depend only on ψ.

The relativistic relation between the components of the four-velocity is,

$$\gamma^2 = 1 + U_p^2 + U_\varphi^2. \tag{6}$$

It is easy to obtain from these algebraic equations the following relationships which will be useful below

$$[U_p - (1 - x^2q^2)FH_p]xh = W - x^2q^2M, \tag{7}$$

and

$$U_p = \sqrt{\frac{V^2 - 1 + x^2q^2}{1 + h^2 - x^2q^2}}, \tag{8}$$

where $h = \frac{H_\varphi}{H_p}$, $V = W - Mq = \gamma - xqU_\varphi$.

The transfield equation describing the balance of forces across the field lines has been investigated in canonical form by a number of authors (Ardavan 1979; Sakurai 1985; Bogovalov 1994). This is a complicated mixed-type second order equation in partial derivatives. For our purposes it is convenient to use this same equation, but in another form. We introduce a curvilinear orthogonal coordinate system formed by the poloidal magnetic and electric fields. Since the magnetic and electric field line of force pass through every point in space, we can always define such a coordinate system in the regions occupied by plasma flows. The introduction of such a coordinate system is not unique. It is convenient to choose the function ψ as one of the coordinates. We define the other coordinate to be α. When moving along a field line, ψ remains constant, but α varies. For the sake of simplicity, we consider α to increase monotonically to infinity when moving along a field line of force. A geometrical interval in these variables is equal to

$$(d\mathbf{r})^2 = g_\psi d\psi^2 + g_\alpha d\alpha^2 + x^2 d\varphi^2, \tag{9}$$

where g_ψ and g_α are components of the metric tensor. In accordance with Landau & Lifshitz (1975), the equation $\partial T^{\psi k}/\partial x^k = 0$, where T^{ik} is the energy-momentum tensor for the plasma, has the following form in these units

$$\frac{\partial}{\partial\psi}\frac{1}{8\pi}(H^2 - E^2) - \frac{\partial x}{x\partial\psi}(U_\varphi v_\varphi mcn - \frac{1}{4\pi}(H_\varphi^2 - E^2)) - \frac{\partial\sqrt{g_\alpha}}{\sqrt{g_\alpha}\partial\psi}(U_p v_p mcn - \frac{1}{4\pi}(H_p^2 - E^2)) = 0. \tag{10}$$

The choice of the coordinate α is not unique. It is possible to remedy this nonuniqueness by noting that

$$\frac{\partial\sqrt{g_\alpha}}{\sqrt{g_\alpha}\partial\psi} = \frac{1}{xR_cH_p}, \tag{11}$$

where R_c is the radius of curvature of a poloidal field line of force. It is possible to obtain expression (11) by direct calculation. R_c is positive if the radius of curvature is directed from the line of force to the rotational axis and negative if it is directed in the opposite sense. In addition

$$\frac{\partial x}{\partial\psi} = \frac{H_z}{xH_p^2}. \tag{12}$$

With these relations and the notation introduced earlier, we may obtain after some relatively strightforward manipulations the following form for the equation for the poloidal field:

$$\frac{\partial}{\partial\psi}\frac{H_p^2}{8\pi} + \frac{1}{8\pi x^2}\frac{\partial}{\partial\psi}(x^2(H_\varphi^2 - E^2)) - \frac{1}{4\pi}\frac{U_\varphi^2 H_z}{xH_pU_pF(\psi)} - \frac{(U_p - F(\psi)(1 - x^2q^2)H_p)}{4\pi xR_cF(\psi)} = 0. \tag{13}$$

3. Plasma flow in a monopole like magnetic field

We assume that a cold plasma with initial four-velocity $U_0 = \gamma_0 v_0/c \gg 1$ is ejected from the stellar surface. Here γ_0 and v_0 are the initial Lorentz-factor and the initial velocity of the plasma. The distribution of the magnetic flux on the stellar surface corresponds to that from a magnetic monopole. Therefore in the absence of stellar rotation the plasma flow is spherically symmetric. The toroidal component of the magnetic field is generated by rotation. Spherical symmetry of the flow is violated. In what follows we assume that the rotation is uniform so that $q = 1$.

The problem of the plasma flow for slow rotation under the condition $x_f \ll 1$ was solved by Bogovalov (1992) analytically. x_f is the radius of the fast mode sound surface where the velocity of the plasma equals the local fast mode velocity. It follows from this solution that the perturbation of the poloidal magnetic field produced by rotation is proportional to $\frac{x^2}{U_0^2}$ and the velocity of plasma in the subfast sonic region is constant. Below we consider the expansion of the exact solution on the parameter $\frac{1}{U_0}$ and estimate the first corrections to the leading terms of the solution nonvanishing at $U_0 \to \infty$. We do not impose the condition $x_f \ll 1$ so that the star can rotate fast.

Let us assume at the beginning that for $U_0 \to \infty$ the poloidal magnetic field goes to the field of the magnetic monopole at fixed other parameters. Below we check that this assumption is valid. In other words, we assume that the nonvanishing term in the expansion of the poloidal magnetic field in powers of $\frac{1}{U_0}$ is the field of the magnetic monopole. For this field $FH_p = FH_0x_0^2/x^2$, where the index "0" denotes values at the stellar surface.

The estimation of the first nonvanishing term in the expansion of h in powers of $1/U_0$ in the region where $h^2 - x^2 \ll 1$ follows from (7) and (8). In this region the four - velocity U_p is constant and equals U_0. Expression (7) gives the following equation for h.

$$[x^2(\gamma_0 + FH_0x_0^2) - FH_0x_0^2]\frac{h}{x} = M - x^2W. \tag{14}$$

The right and left parts of this equation must be equal to zero simultaneously at the Alfven surface. This condition gives the expression for the radius of this surface

$$x_A^2 = \frac{FH_0x_0^2}{\gamma_0 + FH_0x_0^2}. \tag{15}$$

Together with (14) it gives

$$h = -\frac{xW}{\gamma_0 + FH_0x_0^2} \tag{16}$$

It follows from definition W (3) that

$$W = \gamma_0 + FH_0x_0^2, \tag{17}$$

and for h we have

$$h = -x \tag{18}$$

In this approximation the condition $h^2 - x^2 \ll 1$ adopted above is valid everywhere. The first term in the expansion of U_φ in powers of $1/U_0$ starts with *the* first power. This term gives corrections to the poloidal magnetic field proportional to $1/U_0^3$. To obtain first corrections to h, we can neglect that correction to the poloidal field in (7). Keeping in Eq. (7) terms proportional to $1/U_0^2$ we can obtain the following equation for h

$$\{x^2((U_0 + FH_0x_0^2) + \frac{U_0}{2}(\frac{x^2}{U_0^2} - (h^2 - x^2))) - FH_0x_0^2\}\frac{h}{x} = M - x^2W. \tag{19}$$

It is easy to verify by direct calculation that the expression for h *which is* accurate up to terms proportional to $1/U_0^2$ is as follows

$$h = -\frac{\gamma_0 x}{U_0}. \tag{20}$$

U_φ turns out to be equal to zero in this approximation. The expressions for W and M can be obtained from their definitions. They are

$$W = \gamma_0 + \frac{\gamma_0}{U_0}FH_0x_0^2, \tag{21}$$

and

$$M = \frac{\gamma_0}{U_0}FH_0x_0^2. \tag{22}$$

The first corrections to the poloidal field proportional to x^2/U_0^2 appear in this approximation. Before we estimate these corrections let us make sure that the leading term in the expansion of the poloidal magnetic field is really the field of the magnetic monopole. Eluminating terms proportional to $\frac{1}{U_0^2}$ from Eq. (13) for the leading term we obtain the following equation

$$\frac{\partial}{\partial\psi}\frac{H_p^2}{8\pi} - \frac{(U_p - F(\psi)(1 - x^2q^2)H_p)}{4\pi x R_c F(\psi)} = 0. \tag{23}$$

For the given boundary conditions the field of the monopole is the solution of the equation above. The transfield equation accurate up to terms proportional to x^2/U_0^2 is

$$\frac{\partial}{\partial\psi}\frac{H_p^2}{8\pi} - \frac{(U_p - F(\psi)(1 - x^2q^2)H_p)}{4\pi x R_c F(\psi)} = -\frac{1}{8\pi x^2}\frac{\partial}{\partial\psi}(x^2(H_\varphi^2 - E^2)). \tag{24}$$

The first corrections to the monopole-like solution are due to the term $\frac{1}{8\pi x^2}\frac{\partial}{\partial\psi}(x^2(H_\varphi^2 - E^2))$ which plays the role of a perturbation. The ratio of this term to the leading one is of order $\sim x^2/U_0^2$. This ratio increases with distance and after some distance it exceeds 1. It is necessary to take into account however that we are discussing in this paper the flow in the subsonic region. The plasma flow in this region does not depend on the flow in the super sonic region since no MHD signal produced in the supersonic region is able to reach the subsonic region (Bogovalov 1994). Therefore it is sufficient for us to demand that the first corrections to the leading terms are small in the subsonic region. Mathematically it means

$$\frac{x_f^2}{U_0^2} \ll 1, \tag{25}$$

where x_f means the radius of the fast mode surface. For a cold plasma this radius is defined by the relationship (Bogovalov 1994)

$$U_p - (1 + h^2 - x^2)FH_p = 0. \tag{26}$$

For the monopole-like magnetic field and under the condition $x \ll U_0$ the fast mode surface radius is

$$x_f^2 = \frac{FH_0x_0^2}{U_0}. \tag{27}$$

Expression (27) together with condition (25) show that the corrections to the monopole like magnetic field in the subsonic region are proportional to σ/U_0^3, where $\sigma = \frac{H_0^2}{4\pi mc^2 n}(\frac{a\Omega}{c})^2$, and a is the distance from the basis of the field line to the axis of rotation. The corresponding corrections to h and U_φ are also to be proportional to powers of σ/U_0^3. The exact solution is expanded in powers of the parameter σ/U_0^3 in the subsonic region. In the most interesting case of real pulsars this parameter is small.

So, under the condition $\sigma/U_0^3 \ll 1$ in the subsonic region the plasma moves radially with constant Lorentz-factor in the poloidal magnetic field. This field coincides with the field of the magnetic monopole with *an* accuracy σ/U_0^3. The toroidal magnetic velocity is equal to zero. The toroidal magnetic field is defined by expression (20).

At distances $x \gg U_0$ the energy of the plasma and the poloidal magnetic field begin to change. There is no doubt that a collimation of the plasma to the axis of rotation will take place. Analysis shows that finally a part of the plasma will be collimated along the axis of rotation (Heyvaerts & Norman 1989) in a jet with characteristic transversal dimension $r_{jet} \sim \gamma_\infty c/\Omega$, where γ_∞ is the Lorentz-factor of the plasma at infinity (Bogovalov 1995).

At first sight it sounds strange that the dynamics of the plasma and the poloidal magnetic field are only slightly pertubed, when the toroidal magnetic field H_φ and the electric field $E = xH_p$ are comparable and even exceed the poloidal magnetic field. The explanation is simple. Actually these terms come into the equations in the combination $H_\varphi^2 - E^2$. This combination is small even when each of the two terms H_φ and E are not small. It is easy to understand that the value $H_\varphi^2 - E^2$ is the square of the toroidal magnetic field in a coordinate system comooving with the plasma. The poloidal magnetic field does not change in this system (Landau & Lifshitz 1975). This is why the condition that the perturbation of the flow by rotation is small coincides with the condition that the toroidal magnetic field is small in comparison with the poloidal magnetic field in the comooving coordinate system.

In conclusion of this section it is worth to compare obtained solution with that obtained by Michel (1973) in the massless approximation. It is seen that Michel's solution is the limit of our solution at the conditions $\sigma \to \infty$, $U_0 \to \infty$, $\sigma/U_0 \to \infty$, $\sigma/U_0^3 \to 0$.

4. Time dependent problem

We have made sure that under the condition $\sigma/U_0^3 \ll 1$ the plasma is not accelerated in the subsonic region at the stationary outflow. The assumption about stationarity of the flow strongly limits the validity of this conclusion. The flow of a relativistic plasma can be nonstationary. To verify this hypothesis the numerical simulation of the time dependent axisymmetrical problem has been performed.

As above the plasma flow is considered in the magnetosphere produced by a magnetic monopole. Similar simulations were performed earlier by us for a nonrelativistic plasma (Bogovalov 1996). It is convenient to consider the system of equations defining the time dependent flow of plasma in dimensionless variables. The transition to the new dimensionless variables is performed according to the following rules: $\mathbf{H} \to \mathbf{H}/H_c$, $n \to n/n_c$, where H_c and n_c are the poloidal magnetic field of the magnetic monopole and the density of the plasma on the distance from the center of the star equal to the light cylinder, $\psi \to \psi/H_c R_l^2$, $\mathbf{v} \to \mathbf{v}/c$, $\mathbf{U} = \gamma\mathbf{v}/c$, $t \to t\Omega$. $x = \rho/R_l$, $z \to z/R_l$, $R_l = c/\Omega$. Below, all notations of the dimensionless variables are accepted as usual dimension ones. The equations in these variables are as follows

$$\frac{\partial\psi}{\partial t} = -v_x\frac{\partial\psi}{\partial x} - v_z\frac{\partial\psi}{\partial z}, \quad (28)$$

$$\frac{\partial H_\varphi}{\partial t} = \frac{\partial}{\partial z}(v_\varphi H_z - v_z H_\varphi) - \frac{\partial}{\partial x}(v_x H_\varphi - v_\varphi H_x), \quad (29)$$

$$\frac{\partial}{\partial t}(nU_x + \sigma[\mathbf{E}\times\mathbf{H}]_x) = -\frac{1}{x}\frac{\partial}{\partial x}xnv_xU_x - \frac{\partial}{\partial z}nv_zU_x + \frac{nv_\varphi U_\varphi}{x} + \sigma(\frac{\partial}{\partial z}E_zE_x + H_z\frac{\partial H_x}{\partial z} - \frac{1}{2x^2}\frac{\partial}{\partial x}x^2 \times(H_\varphi^2 + E_\varphi^2 + E_z^2 + H_z^2 - E_x^2) + \frac{H_z^2 + E_z^2}{x}), \quad (30)$$

$$\frac{\partial}{\partial t}(nU_z + \sigma[\mathbf{E}\times\mathbf{H}]_z) = -\frac{1}{x}\frac{\partial}{\partial x}xnv_xU_z - \frac{\partial}{\partial z}nv_zU_z\sigma(\frac{\partial}{x\partial x}xE_zE_x + H_x\frac{\partial H_z}{\partial x} - \frac{1}{2}\frac{\partial}{\partial z}(H_\varphi^2 + E_\varphi^2 + E_x^2 + H_x^2 - E_z^2)), \quad (31)$$

$$\frac{\partial}{\partial t}(nxU_\varphi + \sigma x[\mathbf{E}\times\mathbf{H}]_\varphi) = -\frac{1}{x}\frac{\partial}{\partial x}x^2nv_xU_\varphi - \frac{\partial}{\partial z}nv_zxU_\varphi + \sigma(\frac{\partial}{x\partial x}x^2E_xE_\varphi \times\frac{\partial}{\partial z}xE_zE_\varphi + H_x\frac{\partial xH_\varphi}{\partial x} + H_z\frac{\partial xH_\varphi}{\partial z}), \quad (32)$$

$$\frac{\partial n}{\partial t} = -\frac{1}{x}\frac{\partial xnv_x}{\partial x} - \frac{\partial nv_z}{\partial z}, \quad (33)$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{H} = 0, \quad (34)$$

where $[\mathbf{E}\times\mathbf{H}]_x = H_zE_\varphi - H_\varphi E_z$, $[\mathbf{E}\times\mathbf{H}]_z = E_xH_\varphi - E_\varphi H_x$, $[\mathbf{E}\times\mathbf{H}]_\varphi = E_zH_x - E_xH_z$.

Eq. (28) and (29) express the frozen in condition. The first equation is written for the ψ function defined by (2). Here ψ depends not only on coordinates but also on time. This equation can be obtained easily from the well known equation $\frac{\partial\mathbf{H}}{\partial t} = [\nabla\times[\mathbf{v}\times\mathbf{H}]]$.

Eq. (30, 31, 32) define the dynamics of the plasma. Eq. (33) expresses the conservation of matter fux. Gravity is neglected.

5. The basic results of the numerical simulations

The two-step Lax-Wendroff differencing scheme was used for the numerical simulation (Press et al. 1988). It was performed in a quarter of the total box of simulation. It was assumed that the solution is symmetric in relation to the equator and to the axis of rotation. The dimensionless radius of the star was taken equal to 0.5. The outer boundary of the box of simulation was placed sufficiently far from the Fast Magnetosonic Surface (FMS). No signal can propagate from this boundary into the internal part of the box of simulation. We assumed continuous derivatives of all physical variables on the outer boundary. Boundary conditions on the axis of rotation and on the equator follow from the system of equations and from the symmetry of the problem in relation to the equator. On the stellar surface the boundary conditions were specified in accordance with our analysis (Bogovalov 1997). These boundary conditions are as follows

(i) The function ψ is specified on the stellar surface independently of time.

(ii) The tangential component of the electric field is continuous on the stellar surface.

(iii) The Lorentz - factor of the plasma on the stellar surface is equal γ_0.

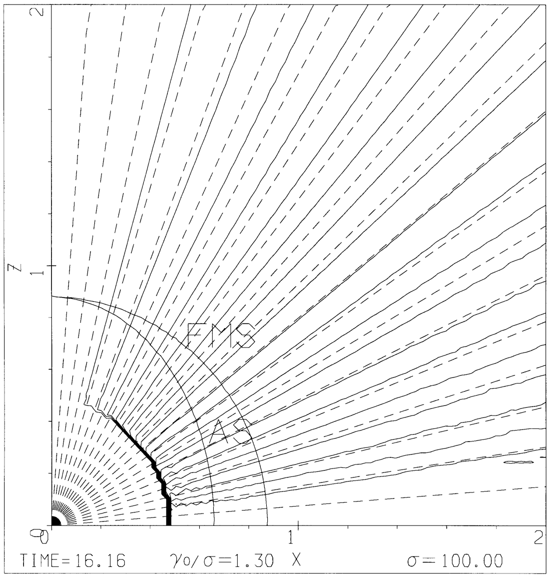

Fig. 1. The result of numerical simulation of the relativistic plasma flow at low Poynting flux. The simulation was performed at the lattice dimension 100×100. The flow is stable. There are sometimes small vorticities of the poloidal electric currents appear near the equator.

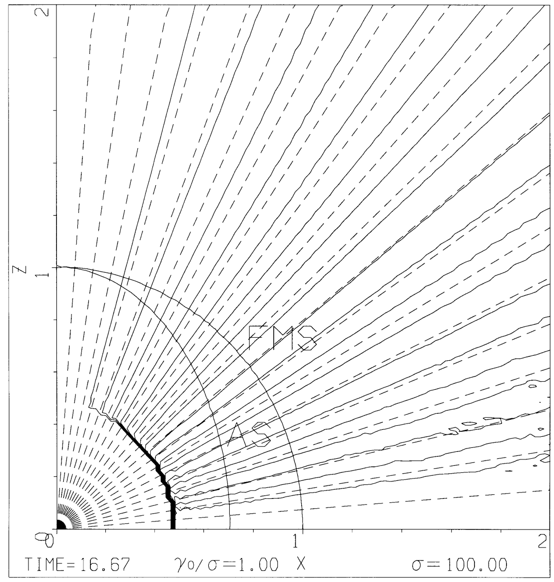

Fig. 2. The result of numerical simulation of the relativistic plasma flow when the Poynting flux is equal to the flux of the kinetic energy of plasma on the equatorial field line. The simulation was performed at the lattice dimension 100×100. The flow is stable. But the small vorticities of the poloidal electric currents appear more frequently.

(iv) The density of the matter flux on the stellar surface does not depend on time and is specified a priori.

(v) The temperature of plasma is equal to zero.

The number of the boundary conditions is equal to the number of MHD waves able to propagate from the stellar surface: one Alfven, one fast magnetosonic, two slow magnetosonic and one entropy wave. The toroidal magnetic field on the stellar surface was calculated from Eq. (29) under the assumption that the electric field in the star is known from the frozen in condition $\mathbf{E} + \mathbf{v} \times \mathbf{H} = 0$. The simulation was performed on the latticies 100×100 and 200×200.

Two initial states were used for the simulation. In a first step of computations the magnetosphere of the nonrotating star was used as the initial state. These computations showed that after several periods of rotation the flow is transformed into the flow described in Sect. (3). This is why everywhere below we use the state of the magnetosphere obtained in Sect. (3) as the initial state of the flow.

The most important result from the numerical simulations is the discovery of an instability of the stationary relativistic plasma flow at the condition $\sigma > U_0$. In the figures we use γ_0 instead U_0. They equal to each other at $\gamma_0 \rightarrow \infty$. Our main purpose was to obtain reliable proves that this instability is not due to numerical effects.

First of all the dependence of the instability of the plasma flow on the ratio U_0/σ was investigated. The simulation was performed for several values of this ratio. The results are presented in Fig. 1–4. The box of simulation is presented in these figures. The star is placed in the left lower corner of the simulation box. Dashed lines show the poloidal field lines. The solid lines show the lines of the poloidal electric currents. The Alfven (AS) and the fast magnetosonic surfaces (FMS) are presented in these figures. It follows from Fig. 1 that the plasma flow is stable when the kinetic energy of plasma dominates the Poynting flux. When these parameters become equal to each other on the equatorial field line the flow becomes nonstationary in a small region near the equator. The size of this region increases with the decrease of the ratio U_0/σ.

Another way to demonstrate the physical reality of the instability of the stationary plasma flow is to investigate the dependence of this instability on the spatial resolution of the lattice used for the numerical simulation. Fig. 5 and 6 show the results of numerical simulation on a lattice twice as large as the lattice used for the numerical simulation presented in Fig. 4. Fig. 5 shows the results of the numerical simulation for the same box as in Fig. 4. It is seen that the instability of the stationary flow develops twice faster than in similar case shown in Fig. 4. This behavior is typical for real physical instability. This instability is supressed by positive numerical viscosity and conductivity. The improvement of the spatial resolution decreases the numerical dissipation. The growth rate of perturbations increases. It is this behavior that we observe in Fig. 5. In the case of artificial numerical instability the dependence of the growth rate on the spatial resolution is opposite. The numerical instability is due to the negative numerical viscosity and conductivity. They decrease with an improvement of the spatial resolution. This is why for the numerical instability we have to expect another dependence of the growth rate of the instability on the spatial resolution. The numerical instability is supressed with improvement of the spatial resolution.

Fig. 6 shows the dependence of the flow on the position of the outer boundaries of the box of simulation. The simulation in

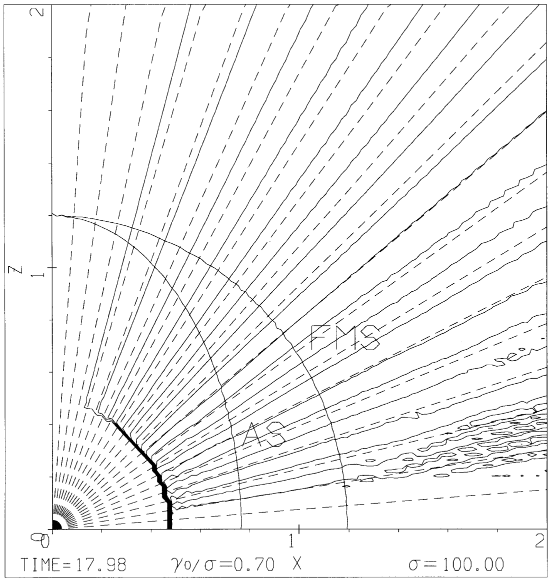

Fig. 3. The result of numerical simulation of the relativistic plasma flow when the Poynting flux slightly exceeds the flux of the kinetic energy of the plasma on the equatorial field line. The simulation was performed at the lattice dimension 100 × 100. The flow becomes nonstationary. There is a region near the equator where the stationary flow is unstable.

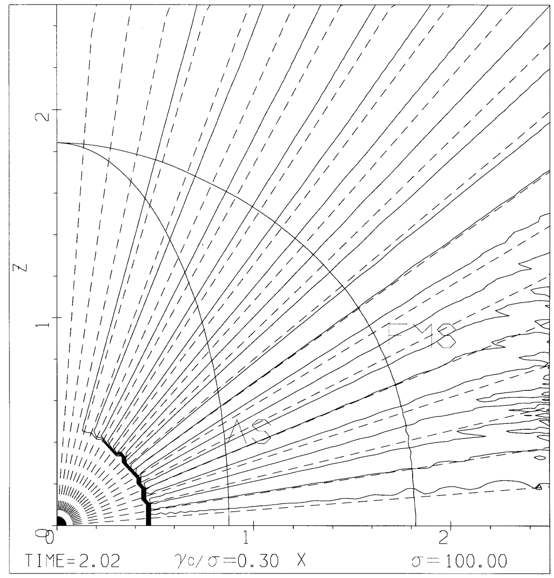

Fig. 4. The result of numerical simulation of the relativistic plasma flow when the Poynting flux remarkably exceeds the flux of the kinetic energy of plasma on the equatorial field line. The simulation was performed at the lattice dimension 100 × 100. The flow is nonstationary.

this figure is performed on the lattice with dimension 200 × 200 but in the box twice more than in Fig. 4. No evidence of the dependence of the flow on the position of the outer boundaries was found.

So, the dependence of the character of the flow on the relationship between the Poynting flux and the flux of the kinetic energy, the dependence of the growth rate of perturbations on the spatial resolution of the lattice used for the numerical simulation and independence of the flow on the position of the outer boundaries prove that the instability of the stationary flow of relativistic plasma at high σ/U_0 is likely physically real. The determination and investigation of the mechanism of the instability is beyond the scope of this paper. It will be performed in future works. Here we stress some properties of the instability which can be useful for the analytical analysis. The simulation dealt with the time dependent axisymmetric problem. It means that the stationary plasma flow is unstable with respect to the axisymmetrical perturbations. The question about the role of nonaxisymmetrical perturbations is open. It is important to note that the nonstationarity of the plasma flow takes place only in *the* superfast magnetosonic region. We have found no cases where the relativistic plasma flow was nonstationary in the subsonic region.

Nonstationarity of the plasma flow leads to the acceleration of plasma. Fig. 7 shows the plot of the Lorentz - factor of the plasma with clear evidence of the plasma acceleration.

6. Discussion

The results of our analysis presented in Sect. 3 close a long series of attempts to obtain acceleration of a relativistic plasma near the light cylinder in the model of the axisymmetrical rotator at the stationary outflow. We found that in a self - consistent solution of the problem of plasma outflow in the magnetosphere produced by a magnetic monopole plasma is not accelerated up to the distance $x = U_0$ even under the conditions $\sigma/U_0 \gg 1$ if the condition $\sigma/U_0^3 \ll 1$ is fulfilled. It is important that these are the conditions that are fulfilled in real radio pulsars. For example, for the Crab pulsar we have $\sigma \sim 10^6 - 10^7$ and $U_0 \sim 10^3 - 10^4$ (Lyubarsky 1995; Daugherty & Harding 1982). Acceleration is possible at the stationary outflow at distance larger than U_0. It will be accompanied by collimation of plasma in jet with transversal dimension of the order $\gamma_\infty c/\Omega$ (Bogovalov 1995), where γ_∞ is the Lorentz-factor of plasma at infinity.

Questions arise. How general is the conclusion about the absence of acceleration of plasma near the light cylinder at the stationary outflow? Does this conclusion remain valid for axisymmetrical dipole or for oblique rotator? And lastly, how is plasma accelerated in reality? Possible answers on these questions are discussed below.

Let us consider the value U_p given by formula (8) at the light cylinder. It is

$$U_p = \frac{V}{|h|}, \tag{35}$$

where $V \sim \gamma_0$. This value is defined totally by the bend of the field lines on the light cylinder in toroidal direction. We stress that this is valid for an arbitrary poloidal field, not only for a monopole like. It is seen from (35) that the less toroidal bend of the field lines the stronger the plasma acceleration on the light cylinder. Unfortunately it appears impossible to make the toroidal bend of field lines small even at infinitesimally small flux of plasma (we assume however that the density of plasma always strongly exceeds the Goldreich-Julian (1969) density

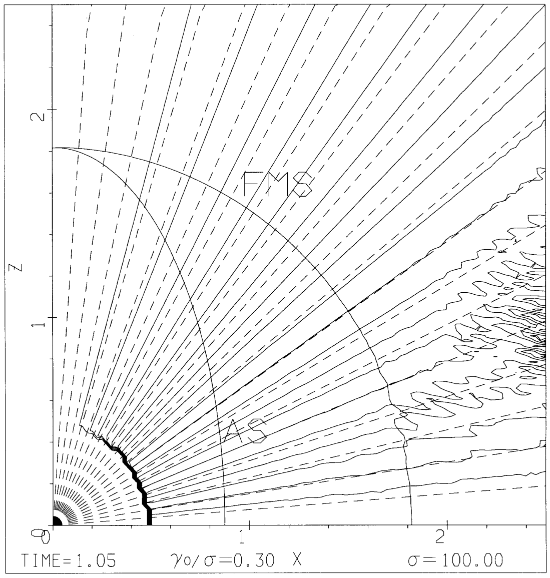

Fig. 5. The result of numerical simulation of the relativistic plasma flow at the same parameters as in the previous figure. The simulation was performed at the lattice dimension 200 × 200. The growth rate increased twice.

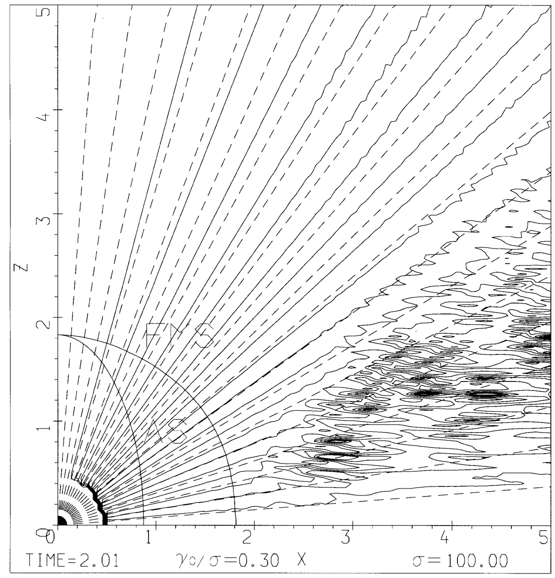

Fig. 6. The result of numerical simulation of the relativistic plasma flow at the same parameters as in the previous figure. The simulation was performed at the lattice dimension 200 × 200, but the size of the box of simulation is also increased twice. This figure shows that the instability does not depend on the position of the outer boundary.

$\Omega H/2\pi ec$). The bend of the field lines in relativistic case is defined not only by the mass flux of matter, but it is also defined by the flux of energy which is equivalent to the mass flux. This is why even in the massless approximation the toroidal bend h is equal to 1 on the light cylinder for the monopole like field. In the dipole field we have to expect the same order of magnitude of h on the light cylinder. It follows from (7) that the dependence of h on x is defined by the dependence of $(x^2H_p)_\psi$ on x at the motion along a field line. For the monopole like magnetic field $(x^2H_p)_\psi$ is constant. For dipole magnetic field $(x^2H_p)_\psi$ depends on x, but rather weakly. According to (Bogovalov 1989) this value is twice less on the light cylinder than on the surface of the star if to move along the last open field line of the dipole magnetic field. The weak dependence of $(x^2H_p)_\psi$ on x is connected with the fact that this value is approximately equal to the flux of the poloidal magnetic field through a surface limited by the field line ψ. This flux is constant. Close dependence of $(x^2H_p)_\psi$ on x for the monopole like and for the dipole magnetic field says that the value of h on the light cylinder for the dipole field will be of the same order as for the monopole like field. No acceleration is expected in dipole field at the stationary outflow.

In the light of this analysis it becomes clear the reason why Beskin et al. (1983) have obtained strong acceleration of plasma for poloidal magnetic field which fall down with distance as $1/x$. For this field the value $(x^2H_p)_\psi$ is much more larger on the light cylinder than on the surface of the star. The toroidal bend h becomes very small and according to (35) plasma is strongly accelerated. Unfortunately the law $1/x$ is not realized in reality. Therefore this mechanism of acceleration can not be realized in real pulsars.

We have not any accurate analysis of the plasma acceleration in the magnetosphere of an oblique rotator. Therefore the question about the role of the nonaxisymmetry of rotation of real pulsars for acceleration of plasma remains open. However, the numerical simulations of the time dependent plasma flow show that even in the magnetosphere of the axisymmetrical rotator the Poynting flux dominated stationary flow of relativistic plasma is likely unstable. The flow becomes nonstationary. A region of turbulent flow is formed beyond the fast mode surface. Plasma is accelerated in this region. There is no doubt that the phenomenon of instability plays a crucially important role in the magnetosphere of real pulsars.

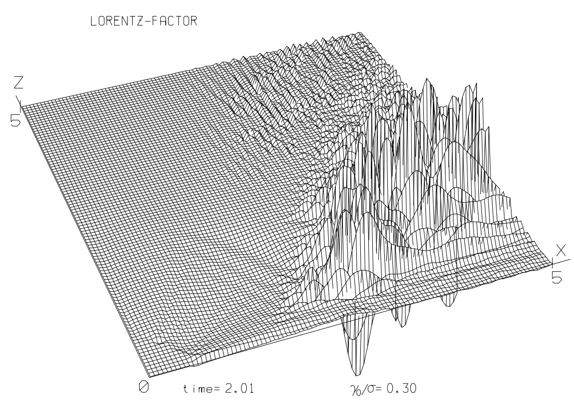

Fig. 7. The plot of Lorentz - factor of the plasma at the nonstationary flow of the Poynting flux dominated plasma. The nonstationarity of the flow leads to the acceleration of the plasma.

It is necessary to keep in mind that only axisymmetrical perturbations were admissible in the numerical simulations. The question about the stability of the flow with respect to the nonaxisymmetrical perturbations is open. We can not exclude that the flow will be nonstationary in the region beyond the Alfven surface with respect to these perturbations. Further analysis is necessary to clarify this problem.

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