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On atmospheric excitation of out-of-phase nutational components

S. Molodensky[1] **and E. Groten**[2]

[1] Institute of the Physics of the Earth, 123810, Bolshaja Gruzinskaja 10, Moscow, Russia
[2] Institute of the Physical Geodesy, Petersenstrasse 13, D-64287 Darmstadt, Germany

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Abstract. A simple self-consistent model of thermal diurnal tides in the earth's atmosphere is constructed which gives the possibility to separate annual and semi-annual prograde and retrograde components. For a realistic model of the Earth the numerical estimations of atmospheric excitation of out-of-phase nutational components are obtained. It is shown that prograde annual component may be totally explained by thermal atmospheric tides; at the same time, the influence of these tides on retrograde annual component is negligible small.

Key words: reference systems

1. Introduction

Modern VLBI-data give the unexpected result, that the out-of-phase nutational components are significant not only for retrograde annual component (0.32 mas), but also for prograde annual one (0.15 mas). Whereas the first one may be explained by the closeness of its frequency to the frequency of free core nutation and by the dissipation of tidal energy in the liquid core and anelastic mantle, the possible explanations of prograde annual component may be connected only with its outer excitation (by the atmosphere, ocean, etc.).

It seems, that a sufficiently reliable enough estimation of these effects is of special importance now, because the value of the out-of-phase retrograde nutational component is fundamental for the estimations of the quality factor Q of the nearly-diurnal free wobble (Mathews et al. 1992), as well as for the estimations of dissipative core-mantle coupling and electroconductivity of the lower mantle (Buffett et al. 1993).

Of course, an exact solution of this problem may be obtained only with the help of exact meteorological data about the dynamics of the atmosphere as a whole. But, taking into account, that (1): nutational motions are forced by very short-periodical (nearly diurnal) atmospheric oscillations and (2): observed motions are connected with the mean motions of such kind (which are averaged over the whole circle of nutational motion in space τ_0 = 1 yr), one realizes, that the necessary set of data must include an enormous number of measurements, and a practical realization of such program is not possible. A significant simplification of the problem under consideration is connected with the fact, that, in accordance with with the well-known Poinsot's kinematical relations (see, for example, Zharkov et al. 1996), the excitation of the prograde annual nutational component is connected only with the mean atmospheric motion in a very narrow band of frequencies which are close to

$$\sigma_o = 2\pi/\tau_0 - \omega \quad (1)$$

where $\omega = 2\pi/\tau_1$ is the angular frequency of the Earth's diurnal rotation, and τ_1 is one sidereal day. Taking this circumstance into account, one realizes, that relatively long-periodical atmospheric oscillations (which are connected with the large-scale variations of the weather, the motions of great cyclons, etc., which play a main role in the excitation of Chandler wobble and l.o.d. variations) are of minor importance, whereas the short-periodic regular diurnal variations (which are forced by the diurnal variations of the solar heat flow) are significantly more important.

Taking into account, that, to a large extent, the rigorous consideration of the dynamics of actual atmosphere presents a very complicated problem, it seems, that a first step in this direction should be connected with the estimation of ratios of atmospheric corrections for different nutational components. Obviously, in this way there are two possible approaches, namely, (1): it is possible to analyse some sets of measurements of distribution of the pressure or (and) winds, and (2): to use some analytical (of course, simplified) models of thermal atmospheric and oceanictides. The first approach was used in (Dehant et al. 1994), where the estimations were based on the Haurwitz & Cowley (1973) data about diurnal variations of the pressure under the assumption that the Earth's surface is an ideal ellipsoid.

Side by side with the undoubtful advantages of such an approach (the absence of any hypothesis about the dynamics of the atmosphere) there are also obvious difficulties, the principal of them are connected with insufficient spatial and, especially, frequency resolutions. In the problem under consideration these difficulties are especially important, because the main terms, both in the temperature and pressure variations, are described by the first-order spherical harmonics, whereas in the approximation of an ellipsoidal Earth's surface the perturbations in the Earth's rotation are connected with comparatively small terms which are described by second-order spherical harmonics. In view of this circumstance, even very small errors in the original (empirical) data may disturb the ratios of prograde and retrograde out-of-phase components very significantly.

Another difficulty is also obvious. If the main terms in the pressure variations are not described by the second-order spherical harmonics, then even very small deviations of the Earth's figure from the ellipsoid may also disturb the torque very significantly. Taking into account, that the local tilts of the Earth's topography exceed the angle between the elements of surfaces of sphere and the ellipsoid by a few orders of magnitude, one realises, that the high-order terms in the spherical expansions of the pressure variations may play a significant role.

To exclude the influence of these effects, we consider in this report the second possibility (of a simple analytical model construction) which is based on the assumption, that the mean nearly diurnal atmospheric flows are caused by the temperature variations. Below we consider the simplest model as follows:

(1) We shall suppose, that in both, over the oceans and over the land, the diurnal variations of the temperature are directly proportional (with some known coefficient k) to the value of the solar heat flow (with some constant phase lag which takes into account the inertia of the atmosphere). The distribution of the density and the pressure with the altitude is described by the well-known barometric (Boltzman's) formula.

(2) In view of different thermal heat capacity of the land (continents) and the ocean, the numerical values of the coefficient k are different for the oceanic and continental regions (their values correspond to the actual values for the atmosphere).

(3) Flows in the atmosphere induced by these temperature variations are described by the generalized Laplace's tidal equation which takes into account the radial variations of density and compressibility of the atmosphere.

As is shown by us below, in such a statement of the problem, the effect under consideration depends strongly on the type of distribution of the ocean and the land. In fact, it depends only on the mean (averaged over all longitudes) distribution of land and oceans with the latitude. If this distribution is symmetrical with respect to the equatorial plane, then the out-of-phase annual nutational components are equal exactly to zero. The real distribution is strongly asymmetrical (approximately 70% of land area is situated in the northern hemisphere). Owing to this circumstance, it seems possible to explain the effects under consideration using rather reasonable assumptions about the dynamics of the actual atmosphere.

2. Dynamical response of a realistic Earth's model on the surface loading

To connect the temperature tides in the atmosphere and the out-of-phase nutational components, it is necessary to use the condition of the angular momentum conservation.

As is well known, the excitation of out-of-phase nutational components (as well as the excitation of nearly-diurnal free wobble) may be connected with two different mechanisms: (1) redistribution of the angular momentum between liquid core and the mantle and (2) redistribution between the mantle, atmosphere, and (or) the ocean. In (Sasao & Wahr 1981) the first mechanism of nearly diurnal free wobble (below by NDFW) excitation was considered. The physical essence of their mechanism may be described as follows:

(1) The variable atmospheric pressure results in elastic deformations of the mantle as a whole, including the core-mantle boundary.

(2) In view of this deformation, off-diagonal components of the inertia tensor of mantle and core as well as the torque between liquid core and mantle are not equal to zero; the variations of these parameters in time result in some redistribution of the angular momentum between liquid core and mantle and to periodical motion of the liquid core with respect to the mantle. As was also shown in (Sasao & Wahr 1981), this motion is of the same type as those, exactly the same mechanism may describe, in some part, the excitation of the out-of-phase nutational components. Taking into account, that in the nearly-diurnal range of periods the variations of the atmospheric pressure are caused mainly by variations which are described by well-known Poincare's formula, and, consequently, it results in the excitation of NDFW.

Obviously of the solar heat flow, it seems resonable to assume, that in mobile (rotating) reference frame the most significant effects must take place for the period which is exactly equal to one solar day; in accordance with (1), in an immobile (space-fixed) reference frame it corresponds just to the unexplained prograde annual nutational component which was mentioned by us above.

In spite of the obvious attractiveness of this hypothesis, there are some reasons which don't allow us to consider it as a basic one, because in case of atmospheric excitation of the out-of-phase nutational components it is not clear a priori, what mechanism plays more important role: redistribution of the angular momentum between liquid core and the mantle or its redistribution between atmosphere and the mantle.

We shall consider below both possibilities. As will be shown by us below (see Section 6), there are strong reasons to conclude, that in the case under consideration the second mechanism plays a more important role, than first one.

To separate these two effects, consider the conditions which describe the motion of the mantle. If the motion in the liquid core is described by Poincare's formula, i. e. if the velocity vector $\boldsymbol{\nu}$ in the liquid core is described by the dynamical equations, (M.S. Molodensky 1961)

$$\dot{\boldsymbol{\nu}} + 2\boldsymbol{\omega} \times \boldsymbol{\nu} = -\nabla\Psi - \frac{2\sigma}{\omega}\frac{\partial\Phi}{\partial z}\boldsymbol{e}_z, \tag{2}$$

where

$$\Psi + \Phi(\sigma+\omega)/\omega = -\nu_t - (\boldsymbol{u}, \nabla W) - \lambda f/\rho =$$

$$-\nu_t + p_1/\rho = \psi\omega^2 r^2 \sin\vartheta\cos\vartheta\cos(\sigma t - \phi) \tag{3}$$

and

$$\Phi = -\varepsilon\omega^2 r^2 \sin\vartheta\cos\vartheta\cos(\sigma t - \phi)$$

(Φ is the variation of centrifugal potential which is connected with the nutational motion; p_1 is the variation of pressure, W and ν_t is the gravitational potential and its small variation, respectively, λ is the modulus of volume compressibility, $\boldsymbol{u}$ is the displacements vector, f = div $\boldsymbol{u}$), then the amplitude of the nutational motion ε with respect to the mobile reference frame is described by the equations of the Earth as a whole and the liquid core's angular momentum conservations as follows (see Zharkov et al. 1996, Eqs. (5.53b), (5.60)):

$$\frac{\sigma+\omega}{\omega}(A_2\varepsilon + I_2) - \varepsilon C_2 + \frac{L}{\omega^2}$$

$$+(C_1 - A_1)(\Psi - \frac{\sigma+\omega}{\omega}\varepsilon) + I_1 = 0; \tag{4}$$

$$\frac{(\sigma+\omega)}{\omega}(A\varepsilon + I_1 + I_2 + \frac{\tilde{\mu}}{\omega}) - \varepsilon C + \frac{L}{\omega^2} = 0 \tag{5}$$

where A, C, A_1, C_1, A_2, and C_2 are the principal moments of inertia of the liquid core, the mantle, and the Earth's as a whole, respectively,

$$\tilde{\mu} = C_1(\frac{\omega^3\psi}{\sigma(\sigma+2\omega)} - \varepsilon\omega) + (C_1 - A_1)\frac{\psi\omega^2}{\sigma+2\omega} \tag{6}$$

defines the angular momentum of the liquid core with respect to the mantle, I_1, I_2, and I are defined by the conditions

$$I^{(1)}_{xz} = \iiint_{r_1} \rho xz \mathrm{d}\tau_1 = \mathrm{Re}(I_1 e^{i\sigma t});$$

$$I^{(1)}_{yz} = -\iiint_{r_1} \rho yz \mathrm{d}\tau_1 = \mathrm{Im}(I_1 e^{i\sigma t});$$

$$I^{(2)}_{xz} = -\iiint_{r_2} \rho xz \mathrm{d}\tau_2 = \mathrm{Re}(I_2 e^{i\sigma t});$$

$$I^{(2)}_{yz} = -\iiint_{r_2} \rho yz \mathrm{d}\tau_2 = \mathrm{Im}(I_2 e^{i\sigma t});$$

where τ_1, τ_2 are the volumes which occupy the fluid core and the mantle, respectively, and I_1, I_2 are complex-valued parameters which totally determine the off-diaginal components of the inertia tensor of the liquid core and of the mantle, respectively, L is the out er torque which acts on the Earth. (If the nutational motion is generated by outer tidal forces, then

$$L = L^{(t)} = \frac{C-A}{a^2}\nu_t, \tag{7a}$$

where a is the mean radius of the Earth and ν_t is the amplitude of the tide-generating potential

$$V_t = \nu_t \frac{r^2}{a^2}\sin\vartheta\cos\vartheta\cos(\sigma t - \phi).$$

The Eqs. (4-6) contain four unknown parameters ε, ψ, I_1, and I_2. To get the closed system of equations, it is convenient to consider four "parameters of effective rigidity of the mantle", which were introduced first in (Sasao et al. 1980). If outer torque $\boldsymbol{L}$ is generated only by tide-generating force, then, in view of linearity of the equations of elasticity, one can write

$$I_1 = C_1(\lambda_1\psi + \lambda_2\nu_t/(ga));$$

$$I_1 + I_2 = C(\lambda_3\psi + \lambda_4\nu_t/(ga)) \tag{8}$$

where C_1 and C are the principal moments of inertia of the liquid core and of the Earth as a whole, respectively, λ_1, λ_3 are the coefficients of effective rigidity of the Earth with respect to the action of hydrodynamic pressure in the liquid core which acts on the core-mantle boundary; λ_2, λ_4 describe the response of the elastic Earth on the action of volume (tide-generating) forces. If the values $\lambda_1 ... \lambda_4$ are known, then the substitution of (8) in (4), (5) results in a simple system of two linear algebraic equations with respect to two unknown parameters ε and ψ, which define these parameters uniquely. To generalize this solution for the case of atmospheric excitation of out-off-phase nutations, it is necessary to take into account, that in this case the elastic deformations of the mantle are caused by three (not two, as in the previous case) factors: (1) the pressure from the liquid core on the core-mantle boundary; (2) the force of gravitational attraction between the atmosphere and the Earth; and (3) the variable pressure of the atmosphere on the Earth's outer surface. As will be shown by us in the following Section, in view of the small thickness of the atmosphere in comparison with the typical horizontal scale of length of atmospheric flows, the radial component of the atmosphere's hydrodynamical equations is reduced to a hydrostatical one, and, consequently, the effects (2) and (3) are not independent. To get this dependence, we shall replace the actual distribution of the masses in the atmosphere by a single layer of the density,

$$d = \int_0^\infty \rho\, dh = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} d_{mn}(t) y_n^m(\vartheta,\phi) \tag{9}$$

where ρ is the density of the atmosphere, h is the altitude, $y_n^m(\vartheta,\phi) = P_n^m(\cos\vartheta)\cos(m\phi)$ are spherical functions, $P_n^m(\cos\vartheta)$ are associated Legendre's polynomials, and $d_{mn}(t)$ are the coefficients of the expansions. Using these notations, we can write the boundary conditions on the Earth's outer surface as follows:

$$S_{rr} = -g\sum_{n=0}^{\infty}\sum_{m=-n}^{n} d_{mn}(t) y_n^m(\vartheta,\phi); \tag{10}$$

$$(\delta V)'_{mn} = \frac{2n+1}{r} V_{mn} = 4\pi G d_{mn}, \tag{11}$$

The difference in the procedure of numerical calculation of λ_2, λ_4 and $\tilde{\lambda}_2$, $\tilde{\lambda}_4$ is reduced to the replacing the boundary condition $S_{rr} = 0$ by the condition (10). The result of our numerical integration of well-known equations of spheroidal static equations where S_{rr} is the radial component of the elastic stress, V_{mn} are the coefficients of spherical harmonic expansions of the gravity potential, $(\delta V)'_{mn}$ is the difference between internal and outer radial derivatives of V_{mn}, and G is the Gravitational Constant.

Using (9), (11), it is easy to express also the gravitational torque $\boldsymbol{L}^{(a)}$ between the atmosphere and the rigid Earth. Replacing in (7a) the amplitude of tidal potential by the amplitude of second-order gravitational potential of the atmosphere V_{21}, we get

$$\boldsymbol{L} = \boldsymbol{L}^{(a)} = \frac{C - A}{\alpha^2} V_{21} \tag{7b}$$

In the static case (when the motion in the liquid core are neglected) these conditions coincide exactly with those for the well-known Load Love numbers (see, for example, Melchior 1986).

If the elastic deformations of the mantle are defined by these conditions, then the relations between I_1, I_2, ε and ψ may be presented in a form similar to (8):

$$I_1 = C_1(\lambda_1\psi + \tilde{\lambda}_2\nu_t/(ga));$$

$$I_1 + I_2 = C(\lambda_3\psi + \tilde{\lambda}_4\nu_t/(ga)) \tag{12}$$

where, as before, λ_1, λ_3 are the coefficients which describe the statical response of the elastic mantle to the action of surface force, whereas the coefficients $\tilde{\lambda}_2$, $\tilde{\lambda}_4$ describe the total static response of the Earth to the interaction with the atmosphere which is defined as the sum of the effects of gravitational attraction and pressure at the Earth's surface.

For PREM (Gilbert & Dziewonski 1981) now usually adopted model of the Earth the numerical values of the coefficients λ_i were calculated in (Mathews et al. 1992) and (Molodensky & Sasao 1995). Their values are as follows:

$$\lambda_1 = -5.309 \times 10^{-4}; \lambda_2 = -0.5622;$$
$$\lambda_3 = -2.082 \times 10^{-4}; \lambda_4 = -0.2994. \tag{13}$$

in the mantle (Zharkov et al. 1996, Eqs. (3.18)-(3.19)) under the boundary conditions (10) are as follows:

$$\tilde{\lambda}_2 = 0.5616; \tilde{\lambda}_4 = 0.3097. \tag{14}$$

(it is interesting to note, that, with a sufficient accuracy, one can put $\tilde{\lambda}_2 = -\lambda_2; \tilde{\lambda}_4 = -\lambda_4$ these relations show, that the effects caused by the loading of the Earth surface exceed twice the effects of gravitational attraction and that these effects are of opposite sign).

Substituting (8), (12) in (4), (5), we get the system of two algebraic equations

$$\frac{\sigma + \omega}{\omega}(A_2\varepsilon + C(\lambda_3\psi + \tilde{\lambda}_4\nu_t/(ga)))$$
$$-\varepsilon C_2 + \frac{L}{\omega^2} + (C_1 - A_1)(\psi - \frac{\sigma + \omega}{\omega}\varepsilon)$$
$$-\frac{\sigma C_1}{\omega}(\lambda_1\psi + \tilde{\lambda}_2\frac{\nu_t}{ga}) = 0. \tag{15}$$

$$\frac{(\sigma + \omega)}{\omega}(A\varepsilon + C(\lambda_3\psi + \tilde{\lambda}_4\nu_t/(ga)) + \frac{\tilde{\mu}}{\omega}) - \varepsilon C + \frac{L}{\omega^2} = 0 \tag{16}$$

Eqs. (15), (16) give two conditions for the three unknown parameters ε, ψ, and L. To get their solution, we shall separate the static and dynamical effects in the form:,

$$\varepsilon = k_1 \frac{d_{21}(t)}{d_{00}} + k_2 \frac{M_a}{M_t} \tag{17}$$

where ε is the amplitude of the nutational motion excited by the atmosphere in radians, $d_{21}(t)$ is the second-order coefficient of the expansions (9), (10),

$$d_{00} = \int_0^\infty \rho_0 dh \tag{18}$$

ρ_0, as before, is the unperturbed density of the atmosphere, M_a is the projection of the atmosphere's angular momentum on the equatorial plane, M_t is the total angular momentum of the Earth, and k_1, k_2 are dimensionless constants which describe the effects of the angular momentum redistribution between the mantle and liquid core (k_1) and between real Earth and the atmosphere (k_2).

The physical meaning of these coefficients may be clarified as follows: if the response of the atmosphere on temperature variations is static, i.e. if $M_a = 0$, then the excitation of nutations is only connected with Sasao & Wahr's mechanism described by us before. Putting in (17) $M_a = 0$, we get this effect in terms of k_1 only. Obviously, in special cases of a rigid Earth's model without liquid core or an Earth's model with homogeneous, incompressible liquid core and with an absolutely rigid core-mantle boundary, the total momentum of the mantle is conserved, and $k_1 = 0$.

If the atmospheric response is dynamical, then it is necessary in Eqs. (16, 17) to take into account both terms. In the same (simplest) case of the rigid Earth's model without liquid core, the total angular momentum of the Earth and the atmosphere as well as its projection on the equatorial plane are conserved, and $k_1 = 1$. Obviously, to get numerically the values k_1 and k_2 for a realistic Earth's model, it is sufficient to put in Eqs. (15-16) $L = 0$ or a value for L which is defined by the relation (7b), respectively. Resolving then the Eqs. (15) and (16) for these two cases, we obtain k_1 and k_2. These coefficients, as well as the values of the dynamical Loading Love numbers $\tilde{h}$, $\tilde{k}$ Shida number $\tilde{l}$ and parameters ψ entering the Eqs. (15-16) for different nutational components and for the real (PREM) Earth's model are presented in Table 1.

One sees, that the coefficients k_1 and k_2 are not close to their limiting values $k_1 = 0$ and $k_2 = 1$, only for the retrograde annual nutational component. This difference is connected with the closeness of the period $T = -366.3$ solar days to the period of free core nutation $T = -434$ solar days. The numerical comparison of the effects of static and dynamical atmospheric responses for different nutational components will be given by us in Sect. 6.

Table 1. Numerical values of dynamical Love and Shida numbers and coefficients k_1, k_2 entering Eq. (17) for a realistic (PREM) model of the Earth. Here $T = \omega/(\sigma + \omega)$ is the period of nutational motion in space in sidereal days (T is positive or negative for the prograde or retrograde components, respectively; shear moduli in the mantle correspond to the period of elastic oscillations T = 1s; dynamical core-mantle flattening $e = 2.63 \times 10^{-3}$; the ratio of the liquid core's moment of inertia to the moment of inertia of the Earth as a whole C_1/C = 0.1146

T	6817	-6817	366.3	-366.3	183.1	-183.1	13.7	-13.7
$\tilde{h}$	-1.323	-1.363	-1.167	1.072	-1.115	-0.769	-1.026	-1.006
$\tilde{k}$	-0.455	-0.474	-0.380	0.699	-0.355	-0.188	-0.312	-0.302
$\tilde{l}$	0.0376	0.0391	0.0318	-0.0521	0.0298	0.0168	0.0265	0.0257
k_1	0.00448	-0.00508	0.0411	0.5693	0.0535	0.1351	0.0743	0.0793
k_2	1.0121	0.986	1.111	2.525	1.145	1.360	1.228	1.187

3. Basic dynamical equations in the atmosphere

3.1. Background

The question about the reasons and causes of diurnal and semi-diurnal oscillations in the atmosphere has been discussed since the times of Lord Kelvin. In 1882, in his presidential address to the Royal Society of Edinburgh, he pointed out, that "the diurnal term, in the harmonic analysis of the variation of temperature, is undoubtedly much larger in all, or nearly all, places than the semi-diurnal. It is then very remarkable, that the semi-diurnal term of the barometric effect of the variation of temperature should be greater, and so much greater as it is, than the diurnal". Kelvin associated a possible explanation of this contradiction with the resonant amplification of the semi-diurnal oscillations in comparison with diurnal ones.

In 1890 Lord Rayleigh estimated the free periods of oscillations of the atmosphere, of diurnal and semi-diurnal type, to be respectively 23.8 and 13.7 hours. These certainly did not support Kelvin's resonance suggestion, but they were not regarded by Rayleigh as likely to be correct, because he ignored the earth's rotation.

Later, a more elaborate treatment of this question was given by Lamb (1910), Chapman (1932,1949), Haurwitz & Cowley (1969).

As was pointed out by Haurwitz (1965), "in accordance with Laplace's theory (see Lamb 1932, pp 341-2), no changes in the elevation of the free surface of an ocean of uniform depth covering the whole earth will occur if the generating force is of the form P_2^1. We may therefore surmise that the corresponding term in the diurnal temperature oscillation will not cause an applicable pressure oscillation of this form".

From our point of view, however, this statement must be considered in some strongly restricted sense, because there is some significant difference between gravitational tides in the ocean of uniform depth and thermal tides in the atmosphere, which is connected with the following factors: (1) compressibility of the atmosphere and (2): baroclinic origin of thermal tides (see, for example, Kagan & Monin 1978). In view of these differences, the problems of thermal atmospheric tides are described generally by three-dimensional hydrodynamic equations, whereas gravitational tides are described by the two-dimensional Laplace's tidal equation.

To clarify this difference in some details, consider the equations of motion of thin layers of the liquid, which are excited by gravitational forces or by thermal expansion. In accordance with Laplace's approach, in both cases the radial component of inertial force is negligible small, and the tangential components of dynamically equations may be presented as follows:

$$\rho(\ddot{u}_\vartheta - 2\omega\dot{u}_\phi \cos\vartheta) = -\frac{\partial p}{r\partial\vartheta} \tag{18a}$$

$$\rho(\ddot{u}_\phi + 2\omega\dot{u}_\vartheta \cos\vartheta) = -\frac{\partial p}{r\sin\vartheta\partial\rho} \tag{18b}$$

where ρ is density, p is pressure, ω is the angular velocity of the Earth rotation, u_ϑ and u_ϕ are θ- and ϕ-components of the displacement vector, the above dot symbol denotes the time-derivative.

If tidal motions are excited by gravitational tide-generating forces, then the pressure at the depth h is described by the relation

$$p = \rho g(\varsigma - \tilde{\varsigma}), \tag{18c}$$

where ς is the uplift of the ocean surface and $\tilde{\varsigma} = V_1/g$ is the normal displacement of the equipotential surface (V_1 is the tide-generating potential). Taking into account, that the parameters ρ and g are constants, one can conclude, that the pressure p is not dependent on the depth d. Substituting this expression in (18a,b) one sees, that θ- and ϕ-components of the vector ∇p as well as the components of displacements vector u_ϑ and u_ϕ are also independent of d, and, consequently, the problem under consideration is reduced to a two-dimensional one (for which all unknown variables depend only on two angular variables).

The situation is changed drastically in the case of thermal tides, when variations of pressure are caused not by outer gravitational forces but by the variations of density. Even in the simplest case, when the variations of temperature are not dependent on depth and unperturbed (corresponding to invariable temperature) density is constant, the pressure p (18c) depends on all three spatial variables, and, consequently, all solutions of the problem of thermal tides are three-dimensional.

In view of this, our further description will be based on the "generalized Laplace's equation" (see Molodensky & Sasao 1995; Molodensky & Groten 1996) which describe the most general case of three-dimensional small oscillations of a heterogeneous, compressible liquid or gas. We shall below generalise these equations also for the case of an arbitrary external thermal source distribution. These equations are integrated numerically for some realistic models of atmospheric temperature.

3.2. General three-dimensional equations in partial derivatives

Consider two states of static equilibrium of the atmosphere

$$\nabla p_0 = \rho_0 \boldsymbol{g} \tag{19a}$$

and

$$\nabla p_1 = \rho_1 \boldsymbol{g} \tag{19b}$$

where p_0, ρ_0, p_1 and ρ_1 are pressure and density corresponding to an initial distribution of the temperature $T_0(r)$ and to its slightly perturbed distribution $T_1(r)$, r is the radius. The variations of p and ρ are connected with the displacement vector $\boldsymbol{u}$ by the well-known relations (Kagan & Monin 1978)

$$\begin{aligned} \rho_1(\boldsymbol{r}) &= \rho_0(\boldsymbol{r}-\boldsymbol{u})(1-\nabla\cdot\boldsymbol{u}) \\ &= \rho_0(\boldsymbol{r}) - (\nabla\rho_0, \boldsymbol{u}) - \rho_0\nabla\cdot\boldsymbol{u} \\ &= \rho_o(\boldsymbol{r}) - \nabla\cdot(\rho_0\boldsymbol{u}) \end{aligned} \tag{20a}$$

$$\begin{aligned} p_1(\boldsymbol{r}) &= p_0(\boldsymbol{r}-\boldsymbol{u}) - \lambda(\nabla\cdot\boldsymbol{u} - \alpha\delta T) \\ &= p_0(\boldsymbol{r}) - (\nabla p_0, \boldsymbol{u}) - \lambda(\nabla\cdot\boldsymbol{u} - \alpha\delta T) \\ &= p_o(\boldsymbol{r}) - \rho_0(\boldsymbol{u}, \boldsymbol{g}) - \lambda(\nabla\cdot\boldsymbol{u} - \alpha\delta T), \end{aligned} \tag{20b}$$

where $\alpha \approx 1/300$ is the coefficient of thermal expansion of the air and λ is adiabatic Lame's parameter (which is equal to the adiabatic modulus of the volume compressibility), and $\delta T = T_1 - T_0$. Here the terms $-(\nabla\rho_0, \boldsymbol{u}) = \rho_o(\boldsymbol{r}-\boldsymbol{u}) - \rho_0(\boldsymbol{r})$, and $-(\nabla p_0, \boldsymbol{u}) = \rho_o(\boldsymbol{r}-\boldsymbol{u}) - \rho_0(\boldsymbol{r})$ describes the variation of the parameters ρ, p which are connected with the local displacements of the heterogeneous media without volume compression and variations of the temperature, whereas the terms $-\rho_0(\boldsymbol{r})\nabla\cdot\boldsymbol{u}$ and $-\lambda\nabla\cdot\boldsymbol{u}$ describe the variations of the same parameters due to adiabatic volume compression of the media without external sources of the heat, and the term $\lambda\alpha\ \delta T$ describes an additional part of the pressure due to non-adiabatic heating.

Substituting (20a) and (20b) in (19b) and subtracting then (19a), we get

$$-\nabla\cdot(\rho\boldsymbol{u})\boldsymbol{g} + \nabla(\rho(\boldsymbol{u},\boldsymbol{g}) + \lambda(\nabla\cdot\boldsymbol{u} - \alpha\ \delta T)) = 0. \tag{21}$$

Calculating now *curl* of left- and right-hand sides of (19a), we obtain

$$\boldsymbol{g}\times\nabla\rho = 0.$$

Adding to (21) a term identical to zero

$$\boldsymbol{u}\times(\boldsymbol{g}\times\nabla\rho) = \boldsymbol{g}(\nabla\rho,\boldsymbol{u}) - \nabla\rho(\boldsymbol{u},\boldsymbol{g}),$$

we can represent the general equation of static equilibrium (21) also in the form

$$\boldsymbol{L}(\boldsymbol{u},\delta T) = 0, \tag{22a}$$

where

$$\boldsymbol{L}(\boldsymbol{u},\delta T) = \rho(\nabla(\boldsymbol{u},\boldsymbol{g}) - \boldsymbol{g}\nabla\cdot\boldsymbol{u}) + \nabla(\lambda(\nabla\cdot\boldsymbol{u} - \alpha\ \delta T). \tag{22b}$$

Obviously, in the dynamic case (if the time interval between the states p_0, ρ_0, T_0 and p_1, ρ_1, T_1 is finite) the Eq. (22a) must be replaced by the following one:

$$\boldsymbol{L}(\boldsymbol{u},\delta T) = \rho_0(\ddot{\boldsymbol{u}} + 2\boldsymbol{\omega}\times\dot{\boldsymbol{u}}), \tag{22c}$$

where the dot above the symbol denotes again the time-derivative, $\ddot{\boldsymbol{u}}$ is the acceleration with respect to a rotating (together with the Earth) reference frame, and $2\boldsymbol{\omega}\times\dot{\boldsymbol{u}}$ is the Coriolis' acceleration.

Up to now we did not take into account, that the distributions of temperature, pressure and density in the atmosphere are neutrally-stratified. To perform this, it is necessary, first, to point out some important pecularity which renders the problem under consideration different from those usually considered in meteorology.

As is known, the general relation describing the relation between the parameters ρ, λ and g in any neutrally-stratified (chemically homogeneous) media may be obtained by means of the consideration of virtual infinitesimal displacement of the volume element which is connected with some fixed element of the mass. If this displacement is not accompanied by transfer of the heat (i.e. if this process is adiabatic), then the variations of density are described by the relations which are equal to the Eq. (20), where we must put $\delta T = 0$:

$$\delta\rho = \rho_1(\boldsymbol{r}) - \rho_0(\boldsymbol{r}) = -\nabla\cdot(\rho\boldsymbol{u}) = -\rho\nabla\cdot\boldsymbol{u} - \rho' u_r; \tag{23a}$$

$$\delta p = p_1(\boldsymbol{r}) - p_0(\boldsymbol{r}) = \rho_0 u_r g - \lambda\nabla\cdot\boldsymbol{u}, \tag{23b}$$

where u_r is the radial component of $\boldsymbol{u}$, ρ' is radial derivative of ρ, and $g = |\boldsymbol{g}|$. On the other hand, if the medium is neutrally stratified (i.e. if the buoyancy frequency is equal to zero), then the density and pressure calculated in such a way must coincide with their unperturbed values in the surrounding media, i.e. in Eqs. (8), (9) we must put $\delta\rho = 0$ and $\delta p = 0$. Excluding u_r from these two relations, we get the well-known Adams and Williamson's condition

$$\lambda\rho' = -\rho^2 g \tag{24}$$

which is usually used in the theory of the Earth's liquid core oscillations, in the physics of the ocean, and in meteorology (Melchior 1986; Kagan & Monin 1978).

Using this relation, one can formally represent the expression (22b) for the operator $\boldsymbol{L}$ as follows:

$$\boldsymbol{L}(\boldsymbol{u},\delta T) = -\rho\nabla\Psi - \nabla(\lambda\alpha\ \delta T) \tag{25}$$

where

$$-\Psi = (\boldsymbol{u}, \boldsymbol{g}) + \frac{\lambda}{\rho}(\nabla \cdot \boldsymbol{u}), \tag{26}$$

and the dynamic equation (22c) is reduced to the following one:

$$\ddot{\boldsymbol{u}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{u}} + \nabla\Psi = -\frac{1}{\rho}\nabla(\lambda\alpha\ \delta T). \tag{27}$$

4. Numerical integration of Eqs. (26), (27) for arbitrary distributions of temperature variations

To solve the systems of Eqs. (26), (27) numerically for any arbitrary distributions of temperature variations δT, it is convenient to present δT in form of spherical harmonics expansions as follows:

$$\delta \mathrm{T} = \sum_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} t_{nm} y_n^m(\vartheta, \phi) \cos(\sigma_i t - \beta_i)$$

where $y_n^m(\vartheta, \phi) = P_n^m(\cos\vartheta)\cos(\mathrm{m}\phi)$ are spherical harmonics of the order n and degree m, $P_n^m(\cos\vartheta)$ are associated Legendre's polynomials, σ_i is the frequency of oscillations which is close enough to the angular frequency of the Earth's diurnal rotation $-\omega$, β_i is the phase lag.

As is known, in the case of an incompressible liquid or gas, the dynamical equations in aspherical layer of sufficient small thickness (in comparison with the Earth's radius) are reduced to Laplace's tidal equation (Lamb 1932), which is based on dynami cal equations in θ- and φ-directions and on the equation of static equilibrium in radial direction.

To generalize these equations for the case of a heterogeneous, compressible atmosphere with variable temperature, consider the general three-dimensional equations (27) in a spherical reference frame:

$$\ddot{u}_\vartheta - 2\omega\dot{u}_\phi \cos\vartheta = -\frac{\partial\Psi}{\mathrm{r}\partial\vartheta} - \chi_\vartheta; \tag{28a}$$

$$\ddot{u}_\phi + 2\omega\dot{u}_\vartheta \cos\vartheta + 2\omega\dot{u}_\mathrm{r} \sin\vartheta = -\frac{\partial\Psi}{\mathrm{r}\sin\vartheta\ \partial\Phi} - \chi_\phi; \tag{28b}$$

$$\ddot{u}_\mathrm{r} - 2\omega\dot{u}_\phi \sin\vartheta = -\frac{\partial\Psi}{\partial\mathrm{r}} - \chi_\mathrm{r},$$

where $\boldsymbol{\chi} = \frac{1}{\rho}\nabla(\lambda\alpha\ \ \delta\mathrm{T})$, u_ϑ, u_ϕ, χ_ϑ, χ_ϕ are ϑ- and ϕ-components of $\boldsymbol{u}$ and $\boldsymbol{\chi}$, respectively.

Our further analysis will be mainly based on the approach which was presented in (Molodensky & Sasao 1995). Expressing u_ϑ, u_ϕ in terms of Ψ, $\boldsymbol{\chi}$, we get:

$$u_\vartheta = \frac{1}{2r\omega\delta\sigma}\left[i(-\frac{\partial\Psi}{\partial\phi}\cot\vartheta - r\cos\vartheta\ \chi_\phi) + \frac{\sigma}{2\omega}(\frac{\partial\Psi}{\partial\vartheta} + r\chi_\vartheta) + r\omega\sigma\sin(2\vartheta)\ \mathrm{u_r}\right] \tag{29a}$$

$$u_\phi = \frac{1}{2ir\omega\delta\sigma}\left[i(-\frac{\partial\Psi}{\sin\vartheta\ \partial\phi} + r\chi_\phi) - \cos\vartheta(\frac{\partial\Psi}{\partial\vartheta} + r\chi_\vartheta) - r\sigma^2\sin\vartheta\ \mathrm{u_r}\right] \tag{29b}$$

where $\delta = f^2 - \cos^2\vartheta$. Substituting these expressions in (26), we get

$$\nabla \cdot \boldsymbol{u} = \frac{\partial(r^2 u_r)}{\partial r} + \frac{\partial(\mathrm{u}_\vartheta \sin\vartheta)/\partial\vartheta + \partial u_\phi/\partial\phi}{\mathrm{r}\sin\vartheta}$$

$$= \frac{\partial\mathrm{u_r}}{\partial\mathrm{r}} + \frac{2\mathrm{u_r}}{\mathrm{r}} + \frac{1}{\omega^2 r^2}(\tilde{\mathrm{L}}(\Psi + \frac{\lambda\alpha}{\rho}\delta\mathrm{T})) + \frac{\mathrm{R}(\mathrm{u_r})}{r} = \frac{\rho}{\lambda}(\mathrm{gu_r} - \Psi)$$

or

$$\tilde{\mathrm{L}}(\Psi) = -\omega^2\mathrm{r}^2(\frac{\partial\mathrm{u_r}}{\partial\mathrm{r}} + \frac{2\mathrm{u_r}}{\mathrm{r}} + \frac{\mathrm{R}(\mathrm{u_r})}{r} - \frac{\rho}{\lambda}(\mathrm{gu_r} - \Psi)) = \frac{\lambda\alpha}{\rho}\tilde{\mathrm{L}}(\delta\mathrm{T}) \tag{30}$$

where

$$\tilde{\mathrm{L}}(\Psi) = \frac{1}{4\delta}\left[\frac{\partial^2\Psi}{\partial\vartheta^2} + \frac{\cot\vartheta}{\delta}(f^2 - 2 + \cos^2\vartheta)\frac{\partial\Psi}{\partial\vartheta} + \frac{f^2 + \cos^2\vartheta}{f\delta} - \frac{1}{\sin^2\vartheta}\Psi\right]$$

is the well-known Laplace's tidal operator $f = \sigma/(2\omega)$, and

$$R(u_r) = \frac{f u_r}{\delta} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\frac{\sin^2\vartheta\cos\vartheta\ u_r}{\delta}).$$

Eq. (30) generalizes the well-known Laplace's tidal equation (see, for example, Lamb 1932) for the three-dimensional case of radially heterogeneous and compressible liquid or gas with external sources of heating (a more rigorous consideration of this equation in partial case $\boldsymbol{\chi} = 0$ is given also in (Molodensky & Sasao 1995 and Molodensky & Groten 1996)). If the internal boundary is immobile and the medium is incompressible, then $u_r = 0$; $\mathrm{R}(\mathrm{u_r}) = 0$; $\rho/\lambda = 0$, and Eq. (30) coincides exactly with Laplace's tidal equation

$$\tilde{\mathrm{L}}(\Psi) = -\omega^2\mathrm{r}^2\zeta/\mathrm{H}$$

(where ζ is the radial displacement of the outer surface and H is the depth of the ocean), which describes the tides in an ocean of constant depth.

Representing u_r and $\nabla \cdot \boldsymbol{u}$ in the form of infinite spherical harmonics expansions

$$u_r = \sum_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_{mn}(r) y_n^m(\vartheta, \phi)\cos(\sigma_i t - \beta_i);$$

$$\nabla \cdot \boldsymbol{u} = \sum_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_{mn}(r) y_n^m(\vartheta, \phi)\cos(\sigma_i t - \beta_i), \tag{31}$$

one can present the general solution of Eq. (30) in the form of superposition of two sets of partial fundamental solutions as follows:

$$\Psi + \frac{\lambda\alpha}{\rho}T = \sum_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Psi_{nm}(r) y_n^m(\vartheta, \phi)\cos(\sigma_i t - \beta_i)$$

$$= \sum_i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (a_{nm}(\mathrm{r})\Psi_{nm}^{(1)}(\vartheta,\phi)$$

$$+\mathrm{b}_{nm}(\mathrm{r})\Psi_{nm}^{(2)}(\vartheta,\phi))\cos(\sigma_i t-\beta_i) \quad (32)$$

where $\Psi_{nm}^{(1)}(\vartheta,\phi)$ and $\Psi_{nm}^{(2)}(\vartheta,\phi)$ are solutions of the equations

$$\tilde{\mathrm{L}}(\Psi_{nm}^{(1)}) = \mathrm{y_n^m}(\vartheta,\phi) \quad (33)$$

and

$$\tilde{\mathrm{L}}(\Psi_{nm}^{(2)}) = \mathrm{R}(\mathrm{y_n^m}(\vartheta,\phi)), \quad (34)$$

respectively,

$$a_{nm} = \omega^2 r^2 (h'_{nm} + \frac{2h_{nm}}{r} - d_{nm}); \quad (35)$$

$$b_{nm} = \omega^2 r h_{nm}; \quad (36)$$

the above prime symbol denotes the derivative with respect to the radius r.

The spherical harmonic coefficients of the functions $\Psi_{nm}^{(1)}$ and $\Psi_{nm}^{(2)}$ were calculated in (Molodensky & Groten 1995). An important property of these coefficients is connected with resonance for the values n = 2, m = 1. As was mentioned by us above (see, for example, Lamb 1932), diurnal tides in the ocean of constant depth are not accompanied by displacements of the ocean outer surface. In our designations this means, that the functions $\Psi_{nm}^{(1)}$ and $\Psi_{nm}^{(2)}$ tend to infinity if σ tends to $-\omega$ (or if f $\longrightarrow (-1/2)$). Taking into account, that in the case of main annual and semi-annual nutational waves the resonant parameter $\kappa^{-1} = \omega/(\sigma+\omega)$ is very large (± 366 and ± 183, respectively), we shall restrict ourselves below to accuracy of the order of κ, and we shall not consider all other com ponents of the atmospheric tides except resonant ones.

As was shown in (Molodensky & Sasao 1995), in the limiting case $\kappa \longrightarrow 0$ the increasing solutions of Eqs. (33), (34) are as follows:

$$\Psi_{21}^{(1)}(\vartheta,\phi) = -\frac{1-1.513\kappa}{10\kappa}\mathrm{y}_2^1(\vartheta,\phi) + 0.0933\ \mathrm{y}_4^1(\vartheta,\phi)$$

$$\Psi^{(2)})_{21}(\vartheta,\phi) = -\frac{1-0.228\kappa}{5\kappa}\mathrm{y}_2^1(\vartheta,\phi) - 0.280\ \mathrm{y}_4^1(\vartheta,\phi)$$

Substituting these values in (32) and neglecting then all higher-order terms (which do not contain the great multiplier κ^{-1}), we get the first relation for three unknown functions h_{21}, d_{21} and Ψ_{21} as follows:

$$\frac{1}{r^4}(r^4 h_{21})' - d_{21} - \frac{10\kappa}{\omega^2 r^2}\Psi_{21} = 0 \quad (37a)$$

Two other relations are direct consequences of Eq. (26) and of the radial component of dynamic equations (27). These equations are as follows:

$$\rho' h_{21} + \rho d_{21} - \frac{\rho'}{g}(\Psi_{21} - \frac{\lambda\alpha}{\rho}t_{21}) = 0 \quad (37b)$$

and

$$\boldsymbol{\Psi}'_{21}(r) - \frac{2}{r}\Psi_{21}(r) = \gamma(r)$$

where

$$\gamma(r) = (\lambda\alpha t_{21})'/\rho \quad (37c)$$

In our further consideration we shall take into account, that, in accordance with well-known barometric (Boltzmann's) formula, the distribution of the density in realistic atmosphere model may be presented in the form

$$\rho = \rho_0 \exp(-r/r_0) \quad (38)$$

where $r_0 \sim 1.0\times10^4$m is the effective depth of the atmosphere. Excluding d_2 from Eqs.(37a) and (37b) and taking into account (38), we can replace the system of three Eqs. (37) by a simpler second-order system as follows:

$$\frac{1}{r^4}(\rho r^4 h_{21})' - \Psi_{21}(\frac{\rho'}{g} + \frac{10\kappa\rho}{\omega^2 r^2}) = \frac{\rho'\lambda\alpha}{\rho g}t_{21}; \quad (39a)$$

$$r^2(\Psi_{21}/r^2)' = \gamma(r). \quad (39b)$$

Eqs. (39a) and (39b) represent the closed system of two heterogeneous ordinary differential equations with respect to two unknown functions h_{21} and Ψ_{21}. Putting here $t_{21} = 0$, (i.e. assuming the absence of outer sources of heating) we get the second-order system of equations which is exactly equivalent to the single Eq. (27) in (Molodensky & Sasao 1995). It is interesting to note, that both equations (27) of the paper cited above and our Eq. (39a,b) have two linearly independent solutions. But, in view of the very fast decreasing density in the atmosphere in comparison with its decrease in the liquid core, the character of solutions in the atmosphere and in the liquid core is significantly different: as is known; in the liquid core these dependencies are as follows:

$$(h_{21})^{(1)} \approx r$$

and

$$(h_{21})^{(2)} \approx r^{-4}$$

(the first root corresponds to Poincare' s solution, whereas the second one describes the effects of the solid inner core).

In contrast, the second solution of the homogeneous equations (39a,b) in the atmosphere is rapidly increasing. Indeed, substituting (38) in (39) and dividing then (38a) by the factor $\exp(-r/r_0)$, we get the system of two ordinary differential equations with fixed coefficients; the secular equation for this system reads as follows:

$$x(5 + x - R/r_0) = 0, \quad (40)$$

where x is defined by the conditions

$$h_{21} = c_1 r^{(1+x)};$$

$\Psi_{21} = c_2 r^{(2+x)}$,

R is the radius of the Earth and c_1, c_2 are any constants.

Eq. (40) has two roots:

$$x = 0 \qquad \text{and} \qquad x = 5 + R/r_0 \sim 6 \cdot 10^2.$$

The numerical integration of the heterogeneous equations (39) for some realistic models of the temperature variations will be given by us in Sect. 6.

5. The models of diurnal oscillations of the temperature

To solve the Eqs. (39) numerically, it is necessary to introduce some reasonable model of the mean temperature variations in the atmosphere.

Some aspects of daily variation of air temperature were considered by Haurwitz & Möller (1955), Kertz (1956 a,b), Haurwitz (1965), Chapman & Lindzen (1970). Haurwitz & Möller distinguished three types of regions: polar, temperate and tropical; for each of these they found the amplitudes and phase lags of the temperature variations. They subdivided the earth's surface into a set of areas bounded by meridians 20° and latitude circles 10° apart from each other; for each of these they determined the fraction k. Using typical amplitudes of diurnal temperature oscillations for each region, they got harmonic coefficients of the diurnal surface temperature waves which are presented also in (Chapman & Lindzen 1970).

Unfortunately, the analysis presented in these investigations is not sufficient to separate the contribution of thermal tides in different nutational components; this is why we shall repeat below it again taking into account some pecularities in time-variations of solar heat flow which give the possibility to do it.

As was pointed out above, the necessary set of data must include the full three-dimensional distributions during, at least, one period of nutational motion in space (one year) with a time resolution less than the period of nutation with respect to the rotating Earth (one day). In accordance with the considerations which were formulated in the introduction, we shall below restrict ourselves to a very simple model which takes into account (1) the time variation of solar heat flow and (2): the asymmetry in distributions of continents and oceans.

Obviously, the local values of solar heat flow S may be represented as follows:

$$S = \begin{cases} S_0 \cos\alpha & \text{if } \cos\alpha > 0 \\ 0 & \text{if } \cos\alpha < 0 \end{cases} \qquad (41)$$

where S_0 is some known constant, α is the angle between the normal $\boldsymbol{n}$ to the element of the Earth's surface and the direction $\boldsymbol{s}$ to the Sun. Neglecting the effects of eccentricity of the Earth's orbit, we can represent the motions of the vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ in space as follows:

$$\boldsymbol{n} = \boldsymbol{i}\cos(\phi - \omega t)\sin\vartheta + \boldsymbol{j}\sin(\phi - \omega t)\sin\vartheta + \boldsymbol{k}\cos\vartheta \qquad (42)$$

$$\boldsymbol{s} = \cos\Omega t(\boldsymbol{i}\cos\varepsilon + \boldsymbol{k}\sin\varepsilon) + \boldsymbol{j}\sin\Omega t \qquad (43)$$

and, respectively,

$$\cos\alpha = (\boldsymbol{n}, \boldsymbol{s}) = \sin\vartheta\cos\Omega t\cos\varepsilon\cos(\phi = \omega t)$$
$$+ \cos\Omega t\sin\varepsilon\cos\upsilon + \sin\Omega t\sin\vartheta\sin(\phi - \omega t). \qquad (44)$$

(44) Here $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is the right-hand system of unit vectors ($\boldsymbol{i}$, $\boldsymbol{j}$ are in the equatorial plane; $\boldsymbol{j}$ coincides with the point of vernal equinox); as before ϕ and ϑ are east longitude and co-latitude; Ω and ω are angular frequencies of the Earth's orbital and diurnal rotation, respectively, $\varepsilon = 23.5°$ is the obliquity of the equatorial plane with respect to the equiliptic.

The spherical harmonic expansion of the value S (41) may be represented in the most general form as follows:

$$S = S_0 \sum_{1-0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{P}_n^m(\cos\vartheta).$$
$$(a_{\ln m}^{(c)}\cos(m\phi)\cos((m\omega + 1\Omega)t)$$
$$+\ a_{\ln m}^{(s)}\sin(m\phi)\cos((m\omega + 1\Omega)t)$$
$$+\ b_{\ln n}^{(c)}\cos(m\phi)\sin((m\omega + 1\Omega)t)$$
$$+\ b_{\ln m}^{(s)}\sin(m\phi)\sin((m\omega + 1\Omega)t)) \qquad (45)$$

Taking into account that, in accordance with Poinsot's relation (1), long-periodic nutational motion in space is excited only by nearly-diurnal terms, we can exclude from our consideration all components with $m \neq 1$. Moreover, it is easy to show, that, in view of the symmetry with respect to the Earth's axes of rotation, for $m = 1$ and any l, n the coefficients $a_{\ln n}^{(c,s)}$, $b_{\ln m}^{(c,s)}$ posses the properties of symmetry as follows:

$$a_{\ln m}^{(s)} = b_{\ln m}^{(c)} \qquad (46a)$$

and

$$a_{\ln m}^{(c)} = b_{\ln m}^{(s)} \qquad (46b)$$

Putting in (45) $m = 1$ and using (46), we can rewrite (45) in a simple form as follows:

$$S = S_0 \sum_{1=0}^{\infty}\sum_{n=0}^{\infty} c_{\ln}\sin(n\vartheta)\cos(\phi - (\omega + 1\Omega)t) \qquad (47)$$

The numerical values entering this equation coefficients $c_{\ln}$ are given in Table 2.

A remarkable property of data presented in Table 2 is connected with the values $c_{\ln}$ with $n = 2$. As was shown by us in Sect. 4, in view of resonant excitation of thermal tidal waves described by second-order spherical harmonics, these coefficients play aprincipal role in excitation of annual nutational components. One see from Table 2, that the coefficients $c_{\ln}$ for $n = 2$ and $l = 1$ or $l = -1$ are exactly equal to zero. Consequently, if we shall consider the simplest model for which mean diurnal variations of the temperature in the atmosphere are proportional to the mean solar heat flow S (18a), then atmospheric thermal tides will excite mainly semi-annual nutational components (which

Table 2. Harmonic coefficients of the solar heat flow expansion.

l	0	1	-1	2	-2	0	1	-1	2
n	1	1	1	1	1	2	2	2	2
c_{ln}	0	0.479	-2.07×10^{-2}	0	0	0	0	0	9.67×10^{-2}
l	-2	0	1	-1	2	-2			
n	2	3	3	3	3	3			
c_{ln}	-5.8×10^{-3}	0	0	0	0	0			
l	0	1	-1	2	-2				
n	4	4	4	4	4				
c_{ln}	2.25×10^{-2}	0	0	2.37×10^{-2}	-5.76×10^{-3}	0			

correspond to $n = 2$ and $l = 2$ or $n = 2$ and $l = -2$ for prograde and retrograde components, respectively). At the same time, as was pointed out by us above, VLBI-measurements give an evidence only about annual out-of-phase components.

This contradiction may be overcome by means of a consideration of a more complicated (and more real) model which takes into account not only the effects of variable solar heat flow, but also the effects of different thermal heat capacity of land and ocean.

As it was pointed out by us above, Haurwitz & Möller (1955) distinguished three types of regions: polar, temperate and tropical; for each of these they found the amplitudes and phase lags of the temperature variations. In our further description we shall not consider the polar regions separately, because, in view of small diurnal variations of the temperature and closeness of these regions to the poles, the contribution of these regions in second-order spherical harmonics terms is negligibly small.

In the open ocean with middle latitudes the values of diurnal variations of the atmospheric temperature (close to the ocean's surface) are of the order of $0.9° - 4°$ only, whereas in continental regions it is of the order of $3° - 20°$. In a sufficient approximation, these amplitudes are proportional to the solar heat flow. Thus, with a sufficient accuracy we can put

$$T = k_1 S/S_0 \quad \text{above oceans}$$

$$T = k_2 S/S_0 \quad \text{above continental regions,} \tag{48}$$

where T is the variation of temperature of the atmosphere at the Earth's surface, $k_1 \sim 1.5° - 2.5°$ and $k_2 \sim 4° - 6°$ are mean coefficients of proportionality for oceanic and continental regions, respectively.

Using relations (47) and (48), it is easy to get the spherical harmonics and spectral expansions of T for the real distribution of land and ocean. A simple numerical experiment shows, that with very high accuracy (of the order of 10^{-3}) these expansions do not depend on the distribution of land and ocean with the longitude; at the same time, its dependence on the latitude is significant. Moreover, the properties of the symmetry (46) hold not only for axially-symmetrical distributions of land and oceans, but also for asymmetrical ones. Thus, with sufficient accuracy the spectral expansion of T may be presented in the form which is analogous to (47):

$$T_S = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} t_{\text{ln}} \sin(n\vartheta) \times \cos(\phi - (\omega + 1\Omega)t) + \delta\varphi \tag{49}$$

where the coefficients t_{ln} for $n = 2$ depend only on the mean distributions of land and oceans which are averaged over the longitudes and $\delta\varphi \sim 0.6$ is the phase lag between solar heat flow and temperature.

Numerical experiments show also, that, if these mean distributions are symmetrical with respect to equatorial plane $\vartheta = \pi/2$, then, as before, the annual coefficients t for $n = 2$ and $l = +1$ are exactly equal to zero.

For the actual Earth, the distribution averaged over all the longitudes distribution of land and ocean is strongly asymmetrical (it is presented in Fig. 1). This is why the annual components are of the same order of magnitude as the semi-annual ones.

The numerical values of the coefficients t_{ln} for $n = 2$ and $l = +1, -1, +2$ and -2, (in degrees) are presented in Table 3.

Neglecting the differences between the frequencies of different nutational components and taking into account only the principal first-order terms (with $n = 1$), we get $t_{\text{ln}} = 1.1°$. For comparison, in Haurwitz's (1965) & Kertz's (1957) models these values are $1.1°$ and $0.8°$, respectively. This coincidence shows, that our simple analytical model describes the mean distribution of diurnal temperature variation with sufficient accuracy.

6. Numerical estimations of out-of-phase nutational components for the actual models of the Earth, atmosphere, and variations of temperature

Summing up the results obtained in the previous sections, we arrive at numerical calculations of out-of-phase nutational components numerical calculations as follows:

(1) Substituting numerical values t_{ln} (for $n = 2$) from Table 3 in the Eqs. (39a) and (39b) and putting in these equations $\kappa^{-1} = +366; -366; +183; -183$ for annual prograde, annual retrograde, semi-annual prograde, and semi-annual retrograde components, respectively, we get the closed system of second-order ordinary differential equations, which must be solved us-

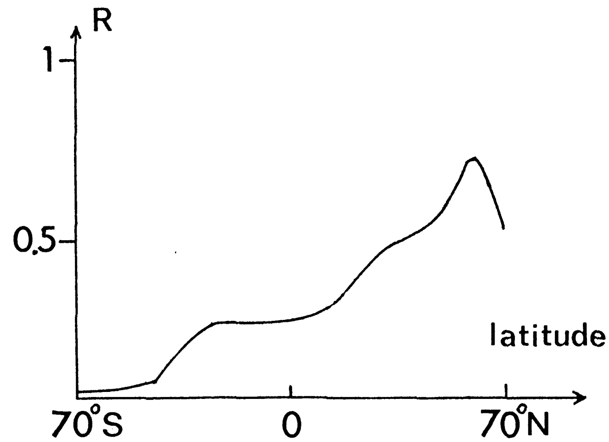

Fig. 1. Distribution of lands and oceans (averaged over all longitudes); R is the ratio of continental surface to total surface.

Table 3. The spectral expansions of the mean diurnal oscillations of the temperature of the atmosphere which are defined by the relations (48) (the values l = 0; +1; +2; +3 correspond to precession, prograde annual, semi-annual, ter-annual nutational component s; negative values l correspond to retrograde nutational components. In our numerical calculations the values $k_1 = 2.0°$ and $k_2 = 5°$ are adopted).

n =	1	1	1	1	1	1	1
l =	0	1	−1	2	−2	3	−3
t_{ln}	−0.08°	1.02°	−0.04°	−0.08°	0.01°	0	0
n =	2	2	2	2	2	2	2
l =	0	1	−1	2	−2	3	−3
t_{ln}	0.19°	−0.40°	0.017°	0.20°	−0.012°	0	0
n =	3	3	3	3	3	3	3
l =	0	1	−1	2	−2	3	−3
t_{ln}	−0.10°	0	0	−0.11°	−0.01°	0	0

ing the boundary conditions that the radial component of displacements u_r is equal to zero on the Earth's surface and that it is bounded when $r \longrightarrow \infty$.

(2) Substituting these solutions in (29a,b) we get the field of velocities in the atmosphere. Calculating the vector product $\rho \boldsymbol{r} \times \dot{\boldsymbol{u}}$ and integrating then the result over all the volume of atmosphere, it is easy to find the angular momentum of the atmosphere with respect to the mantle (it should be noted, that, in case of second-order solutions, i.e. when both Ψ and u_r are proportional to the second-order spherical harmonics, the relations (29a,b) define the velocity field in Poincare's form; in this case the variations of the angular momentum of the atmosphere is defined by the well-known relation $M_a = \psi C_a$, where C_a is the moment of inertia of the atmosphere and ψ is connected with Ψ by relation (3)).

(3) Using Table 1 and the relation (26) between the values of the parameter Ψ and variations of the pressure on the Earth's surface $\delta p|_s = -\lambda \nabla \cdot \boldsymbol{u}$, it is easy to find also the transmission of the angular momentum from the mantle to the liquid core

Table 4. Results of numerical calculations of out-of-phase nutational components for the model of the temperature variations which is given in Table 3. Here κ^{-1} is the period of nutational motion in space (positive and negative values correspond to prograde and retrograde nutational components, respectively); $(\delta\varepsilon)_1$ and $(\delta\varepsilon)_2$ are corrections (in milliarcseconds) due to the redistributions of the angular momentum with the atmosphere and liquid core, respectively.

κ^{-1}	$(\delta\varepsilon)_1$	$(\delta\varepsilon)_2$
366.3	0.17	0.007
−366.3	0.01	0.002
183.1	0.09	0.004
−183.1	0.003	0.0001

which is described by the mechanism of Sasao & Wahr (1981). Substituting numerical values of the coefficients k_1 and k_2 in the relation (17), we get the total variation of the angular momentum of the atmosphere plus that of the Earth;s liquid core which is equal to the some of the effects defined in (2) and (3).

(4) Using the condition of the total angular momentum conservation of the system (elastic mantle + atmosphere + liquid core) conservation, it is easy to find the variations of the angular momentum of the mantle and the amplitudes of out-of-phase nutational components for different frequencies which are involved in Table 3.

The results are shown in Table 4.

The corresponding in-phase corrections are small (in comparison with the VLBI measurements accuracy) for all components except prograde annual and semi-annual components (the corresponding corrections are 0.2 and 0.1 mas, respectively).

Comparing the results of this table with the results of modern VLBI-measurements (0.32 mas for the retrograde component and 0.15 mas for the prograde annual one), we can state the following conclusions:

(1) For any nutational components, direct effects of the angualr momentum redistributions between the atmosphere and mantle play a significantly more important role, than the effects of redistributions between the mantle and liquid core which are caused by the action of the variable atmospheric pressure on the elastic mantle;

(2) For our model, the theoretical out-of phase prograde annual component coincides with the results of VLBI-measurements very well (0.15 mas and 0.17 mas, respectively).

7. Conclusions and discussion

The out-of-phase nutational components give very important information about the anelestic properties of the Earth's interior and about the mechanism of the core-mantle dissipative coupling. At the same time, the correct interpretation of modern VLBI-data is possible only after a reliable elimination of oceanic and atmospheric effects. It now appears, that oceanic gravitational tides are responsible for the main part of the prograde semi-annual nutational component; at the same time, these tides are too small to give a significant input in prograde and retrograde annual components. The results obtained by us here, confirm the conclusion of Dehant et al. (1994), that these components may be explained by atmospheric thermal tides which are excited by the variable solar heat flow at the Eqrth's surface.

Summing up our results, it seems important to underline some most important statements which were formulated by us above.

(1) The diurnal variations of temperature, pressure, and winds in the atmosphere result in two different effects:
(a) The variable atmospheric pressure results in elastic deformations of the mantle as a whole, including the core-mantle boundary. In view of these deformations arise off-diagonal components of the inertia tensors of the liquid core and of the mantle and differential rotation of the liquid core with respect to the mantle. From the physical point of view, such motion may be explained by a redistribution of the angular momentum between liquid core and mantle; the torque being mainly connected with the elastic deformations of the core-mantle boundary.
(b) The variations of temperature and pressure in the atmosphere results in dynamic atmospheric effects (global winds with diurnal and semi-diurnal periods). These winds possess a certain angular momentum, and in viwe of the condition of the angular momentum conservation in the system (mantle + liquid core + atmosphere), they must be manifested in the rotation of the mantle.

Our calculations show (see Table 4), that both for prograde and retrograde annual and semi-annual nutational components, the second mechanism gives a stronger contribution in nutational motion of elastic mantle, than the first one (the first one is more significant only in the range of frequencies which are close enough to the frequency of nearly diurnal free wobble; for the frequencies of forced nutations the ratio of the first effect to the second one has its maximal value $\approx 1/5$ for the retrograde annual component).

(2) In spite of the well-known fact, that the second-order diurnal gravitational tides in a thin spherical layer of liquid of constant depth is not accompanied by tidal variations of its outer surface (see Lamb 1932), the analogous statement in case of thermal tides in the atmosphere is not correct (second-order thermal tides in a radially-heterogeneous, compressible adiabatically stratified atmosphere of constant depth are accompanied by the variations of pressure). From a physical point of view, this difference is connected with the baroclinic origin of thermal tides, in view of which the problem of thermal tides is essentialy three-dimensional. It seems, that the method of numerical integration of these equation given by us above (in Sect. 4) yields a satisfactory description of the real solutions. Formally, this method is close to those, which were adopted in (Molodensky & Sasao 1995) for the consideration of the dynamic effects of the outer liquid and inner solid Earth's cores, but, in view of the great compressibility of the atmosphere, the character of solutions in the atmosphere is quite different. Namely, in so far as in the liquid core there are two regular second-order solutions (decreasing and increasing to the center), in the atmosphere both second-order solutions are decreasing. It should be noted, that analogously to the problem of long-periodic tidal oscillations of the liquid core, the problem of thermal oscillations of the atmosphere belongs to the class of "ill-possed problems in sense of Hadamard" (see Jeffreys 1978; Melchior 1986; Molodensky & Groten 1995). This is why its complete and mathematically rigorous analysis is connected with great difficulties, and some principal and unsolved questions still remain

(3) It should also be noted, that the effects of atmospheric excitation of prograde and retrograde annual out-of-phase nutational components strongly depend on the distribution of the land and ocean. If this distribution is symmetrical with respect to equatorial plane, them diurnal variations of solar heat flow excite only semi-annual nutational component and do not excite annual ones.

(4) As we see from Table 4, reliable models of distribution between land and ocean and sufficiently reliable models of different thermo-capacities of land and oceans guide us to explain, in general, the amplitude of the prograde annual nutational component. At the same time, one can claim that, if it is correct, then the retrograde annual component can not be excited by thermal atmospheric tides, and its explanation must be connected with the anelastic properties of the Earth and with a dissipative core-mantle coupling.

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References

Chapman, S., Lindzen, R.S., 1970, Atmospheric tides. Thermal and gravitational. Reidel, Dordrecht, p. 200

Dehant, V., Biyouard, Ch., Hinderer, J., Legros, H., Lefftz, M., 1994, The effects of the atmospheric pressure on nutation. J. Systems de Reference Spatio-Tempoles Obs. de Paris

Haurwitz, B., Möller, F., 1955, The semidiurnal air-temperature variation and the solar air tide. Archiv. Meteorol. Geophys. Biokl. A8, 332-350

Hauwrity, B., 1965, The diurnal surface pressure oscillations. Archiv. Meteorol. Geophys. Biokl. A14, 361-379

Hauwrity, B., Cowley, A.D., 1973, The diurnal and semi-diurnal barometric oscillations, global distribution and annual variations. Pageoph., 102, 193-222

Groten, E., Molodensky, S.M., 1991, On the damping of the nearly diurnal free wobble. Peoc. AGU Chapman Conference of Geodetic VLBI: monitoring Global Change. NOAA Technical Report NOS 137 NGS 49, 355-375

Kagan, B.A., Monin, A.S., 1978, Tides. In Oceanologiya, v.2, Fizika okeana (Physics of the Ocean), Nauka, 255-279

Kertz, W., 1959, Partialwellen in den halb- und vierteltägigen gezeitenartigen Schwingungen der Erdatmosphäre, Arch. Meteorolog. Geophys. Biokl. A11, 48-63

Lamb, H., 1932, Hydrodynamics, 6th edn, Cambridge University Press, p. 738

Mathews, P.M., Shapiro, I.I., 1992, Nutations of the Earth, ARE&PS 20

Molodensky, M.S., 1961, The theory of nutation and diurnal Earth Tides. Comm. Obs. Roy. Belg. 18 S. Geophys., 58, 25-56

Molodensky, S.M., Sasao, T., 1995, A new approach to the theory of the Earth's nutation. I. Main equations. Phys. Solid Earth, 12, 24-38
Molodensky, S.M., Groten, E., 1996, On "pathological oscillations" of rotating fluids in the theory of nutations. Manuscripta Geodaetica (in press)
Sasao, T.S., Okubo, S., Saito, M., 1980, A simple theory of dynamically effects of stratified fluid core upon nutational motion of the Earth, in Proc. of IAU Symp. 78, Nutation and the Earth's rotation, ed. by E. Fedorov, M. Smith and P. Bender, Reidel, Hingham, Mass.
Sasao, T., Wahr, J.M., 1981, An excitation mechanism for the free "core nutation". Geophys. J. Roy. Astr. Soc. v. 64, N 3, 729-746
Siebert, M., 1961, Atmospheric tides in Advances in Geophysics, v. 7, Academic Press, New York, 105-182
Zharkov, V.N., Molodensky, S.M., Brzezinski, A., Groten, E., Varga, P., 1996, The Earth and its rotation. Low frequency geodynamics. Herbert Wichmann Verlag, Hüthig GmbH, Heidelberg

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