

# Mergers of galaxies in clusters: Monte Carlo simulation of mass and angular momentum distribution

D.S. Krivitsky and V.M. Kontorovich

Institute of Radio Astronomy, Krasnoznamenaya 4, 310002, Kharkov, Ukraine (e-mail: rai@ira.kharkov.ua)

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**Abstract.** A Monte Carlo simulation of mergers in clusters of galaxies is carried out. An “explosive” character of the merging process (an analog of phase transition), suggested earlier by Cavaliere et al. (1991), Kontorovich et al. (1992), is confirmed. In particular, a giant object similar to cD-galaxy is formed in a comparatively short time as a result of mergers. Mass and angular momentum distribution function for galaxies is calculated. An intermediate asymptotics of the mass function is close to a power law with the exponent  $\alpha \approx 2$ . It may correspond to recent observational data for steep faint end of the luminosity function. The angular momentum distribution formed by mergers is close to Gaussian, the rms dimensionless angular momentum  $S/(GM^3R)^{1/2}$  being approximately independent of mass, which is in accordance with observational data.

**Key words:** galaxies: clusters: general – galaxies: cD – galaxies: interactions – galaxies: mass function – galaxies: statistics

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## 1. Introduction

In recent years, evolution of galaxy clusters has attracted ever more attention. On the one hand, it has become accessible for observations, both with ground-based and space instruments. On the other hand, discovery of Butcher–Oemler effect testifies to an epoch of fast evolution in clusters, associated with galaxy interaction. Mergers of galaxies<sup>1</sup> are considered to be one of the most probable explanations for the change of colour which accompanies appearance of ellipticals and lenticulars instead of early-type spirals at  $z \sim 0.2$ – $0.4$  (see, e.g., Dressler et al. 1994 and cited there). Comparatively fast evolution of clusters and groups, caused by mergers, is also confirmed by the results of direct numerical simulation (see, e.g., Barnes 1996).

One of the effects associated with mergers is rapid evolution of the galaxy mass and angular momentum distribution function

*Send offprint requests to:* D.S. Krivitsky

<sup>1</sup> Another possible explanation is “galaxy harassment” discussed by Moore et al. (1996).

$f(M, S, t)$ , related to appearance of massive galaxies. Calculation of  $f(M, S, t)$  is of great interest both by itself and in the context of a merger model of activity suggested by Kats & Kontorovich (1990, 1991). Given  $f(M, S, t)$ , one can predict the luminosity function of active galactic nuclei in this model.

The mass function of merging galaxies  $\Phi(M, t)$  can be described in terms of the Smoluchowski kinetic equation. As Cavaliere et al. (1991, 1992) and, independently, Kontorovich et al. (1992), Kats et al. (1992) have shown, mergers result in an analog of phase transition (or “explosive evolution”), which implies fast formation of a distribution tail (corresponding to massive galaxies) and a “new phase”: cD-galaxies<sup>2</sup>.

The joint distribution function which takes into consideration both mass and angular momentum can be described by a generalized Smoluchowski equation (see Kats & Kontorovich 1990, 1992, where this equation was solved in the simplest case of a constant merger probability – an analog of phase transition does not take place in this variant – and without allowance for the orbital angular momentum). In this paper we present the results of simulation of galaxy mass and momentum evolution in clusters, with the orbital momentum and more realistic mass dependence of the merger probability taken into account.

In Sect. 2 we discuss Monte Carlo simulation of mergers. In Sect. 3 we compare the results for  $\Phi(M, t)$  with a direct numerical solution of the Smoluchowski equation. Sect. 4 contains discussion of the results.

## 2. Joint mass and angular momentum distribution: Monte Carlo simulation

### 2.1. Generalized Smoluchowski equation and merger probability

The kinetic equation which describes  $f(M, S, t)$  (the generalized Smoluchowski equation) is

$$\frac{\partial f}{\partial t} = \int W_{MS|M_1S_1, M_2S_2} f_1 f_2 dM_1 dM_2 d^3S_1 d^3S_2$$

<sup>2</sup> This transition has been known for a long time in other applications of the Smoluchowski equation (see Stockmayer 1943; Trubnikov 1971; Voloshchuk 1984; Ernst 1986).

$$-2 \int W_{M_1 S_1 | M S, M_2 S_2} f f_2 dM_1 dM_2 d^3 S_1 d^3 S_2, \quad (1)$$

where  $f_1 \equiv f(M_1, \mathbf{S}_1, t)$  etc. The function  $W_{M S | M_1 S_1, M_2 S_2}$  (the kernel of the equation) is a characteristic of the probability for the merger  $M_1 \mathbf{S}_1, M_2 \mathbf{S}_2 \rightarrow M \mathbf{S}$ . Taking into account mass and momentum conservation laws,  $W$  can be rewritten as

$$W_{M S | M_1 S_1, M_2 S_2} = U_{M S | M_1 S_1, M_2 S_2} \times \delta(M - M_1 - M_2) \delta(\mathbf{S} - \mathbf{S}_1 - \mathbf{S}_2) \quad (2)$$

(without orbital momentum). If we take into consideration the orbital momentum  $\mathbf{J}$ , the second  $\delta$ -function should be replaced by a function of a finite width  $\sim \bar{J}$ . The function  $U$  can be calculated as  $U = \bar{\sigma} \bar{v}$ , where  $\sigma$  is the merger cross-section,  $v$  is the relative velocity at infinity, the bar means an average over velocities.

An exact computation of collision dynamics and determining the merger cross-section is a very complicated problem. Nevertheless, main features of this process are known, both from analytical consideration and numerical experiments (Roos & Norman 1979; Aarseth & Fall 1980; Farouki & Shapiro 1982; Farouki et al. 1983; Chatterjee 1992) and enables one to formulate conditions necessary for a merger: the galaxies must pass at a small distance (interaction is especially intense if the outer parts overlap) and the relative velocity must be small enough. Namely, we shall assume below that a merger occurs provided that (i) the minimal distance between the two galaxies is less than the sum of their radii  $R_1 + R_2$  and (ii) the relative velocity at infinity is less than some limit value which is of the order of the escape velocity  $v_g = \sqrt{2G(M_1 + M_2)/(R_1 + R_2)}$ , i.e.,  $v \leq \zeta v_g$ ,  $\zeta \sim 1$ . Taking into account gravitational focusing, we may derive from the former condition that the impact parameter  $p_\infty \leq (R_1 + R_2) \sqrt{1 + v_g^2/v^2}$ . Thus, the merger cross-section is

$$\sigma = \begin{cases} \pi(R_1 + R_2)^2(1 + v_g^2/v^2), & v \leq \zeta v_g \\ 0, & v > \zeta v_g. \end{cases} \quad (3)$$

We shall assume that the galaxy peculiar velocity distribution is Gaussian<sup>3</sup> with the root mean square velocity  $v_{\text{rms}}$  (obviously, the relative velocity distribution (at infinity) in this case is also Gaussian, but the root mean square velocity is  $\sqrt{2}v_{\text{rms}}$ ). So,

$$U = \bar{\sigma} \bar{v} = \int_0^{\zeta v_g} \pi(R_1 + R_2)^2(1 + v_g^2/v^2) v \times (4\pi/3)^{-3/2} v_{\text{rms}}^{-3} \exp(-3v^2/4v_{\text{rms}}^2) 4\pi v^2 dv. \quad (4)$$

After integration we obtain:

$$U = 4(\pi/3)^{1/2} v_{\text{rms}}(R_1 + R_2)^2 \times \left[ A + 1 - e^{-\zeta^2 A} (\zeta^2 A + A + 1) \right], \quad A = \frac{3v_g^2}{4v_{\text{rms}}^2}. \quad (5)$$

<sup>3</sup> It is essential for the further consideration that  $v_{\text{rms}}$  does not depend on mass. Such a behaviour is typical for gravitating systems which have passed through the violent relaxation stage (Saslaw 1987).

If the density of a galaxy  $\rho_{\text{gal}}$  does not depend on its mass, then the radius  $R$  and mass  $M$  are related as  $R \propto M^{1/3}$ . For Faber–Jackson and Tully–Fisher laws ( $L \propto V^4$ ), using the virial theorem ( $MV^2 \propto GM^2/R$ ) and the mass–luminosity relation of the form  $L \propto M$ , we obtain  $R \propto M^{1/2}$ . Below we shall take

$$R = CM^\beta, \quad \text{where } \beta = \frac{1}{3} - \frac{1}{2}. \quad (6)$$

Asymptotical behaviour of Eq. (5) is

$$U \approx \begin{cases} c_2(M_1 + M_2)^2, & M \ll M_b \\ c_{1+\beta}(M_1 + M_2)(M_1^\beta + M_2^\beta), & M \gg M_b, \end{cases} \quad (7)$$

the coefficients  $c_2 = 3(3\pi)^{1/2}(\zeta^2 + \zeta^4/2)G^2/v_{\text{rms}}^3$ ,  $c_{1+\beta} = 2(3\pi)^{1/2}CG/v_{\text{rms}}$ . Here mass  $M_b$  corresponds to  $v_{\text{rms}} \sim v_g$ :

$$M_b \sim (Cv_{\text{rms}}^2/G)^{1/(1-\beta)} \sim 10^9 - 10^{10} v_7^{2/(1-\beta)} M_\odot, \quad (8)$$

where  $v_7 = v_{\text{rms}}/10^7 \text{ cm s}^{-1}$ . We shall assume  $\zeta = 1$  in the further consideration.

Note that the difference of  $U$  in the small mass region from that in Cavaliere et al. 1991 is caused by the velocity restriction for mergers in Eq. (3). For  $M < M_b$  collisions without mergers are more probable than mergers ( $\sigma = \pi(R_1 + R_2)^2$ ). The latter, however, gives the “explosive” evolution in the small mass region too. Effects related to collisions without mergers may be important (see footnote 1), however, this question is beyond the scope of this paper.

As  $v_{\text{rms}}$  is, in general, a function of time, the coefficients  $c_2$  and  $c_{1+\beta}$  and mass  $M_b$  depends on time too (see, e.g., Kontorovich et al. 1992). Below we shall assume them to be constant, neglecting changes of velocities and masses both due to capture of new members or evaporation of galaxies from clusters and groups, and mergers itself, or other reasons.

In the region  $M \gg M_b$  galaxy peculiar velocities are much less than stellar velocities in the galaxy; the relationship for  $M \ll M_b$  is inverse. In this paper we consider both the asymptotical regions  $M \gg M_b$  and  $M \ll M_b$  and the intermediate case where  $M_b$  is within the range of simulated masses.

Numerical experiments of galaxy merging (Farouki & Shapiro 1982) show that the merger probability  $U$  depends both on masses and momenta, reaching a maximum when  $\mathbf{S}_1, \mathbf{S}_2$  and  $\mathbf{J}$  have the same direction. Nevertheless, this dependence is less essential than the dependence on masses ( $\propto M^u$ ,  $u = 1 + \beta$  for  $M \gg M_b$ , which leads to the “explosive” evolution). In this paper we shall use the simple model (3) which does not take into account the dependence of the merger probability on the mutual orientation of angular momenta.

It is known from the Smoluchowski equation theory (Ernst 1986; Voloshchuk 1984) that behaviour of the solution essentially depends both on the homogeneity power  $u$  in the mass dependence of  $U$  and on the asymptotics of  $U$  for very different masses which can be characterized by exponents  $u_{1,2}$ :

$$U(M_1, M_2) \approx c_u M_1^{u_1} M_2^{u_2}, \quad M_1 \ll M_2, \quad u_1 + u_2 = u. \quad (9)$$

For the case of galaxy mergers (Eq. (7)), obviously,  $u_1 = 0$  and  $u_2 = u = 2$  ( $M \ll M_b$ ) or  $1 + \beta$  ( $M \gg M_b$ ). If  $u > 1$  then an

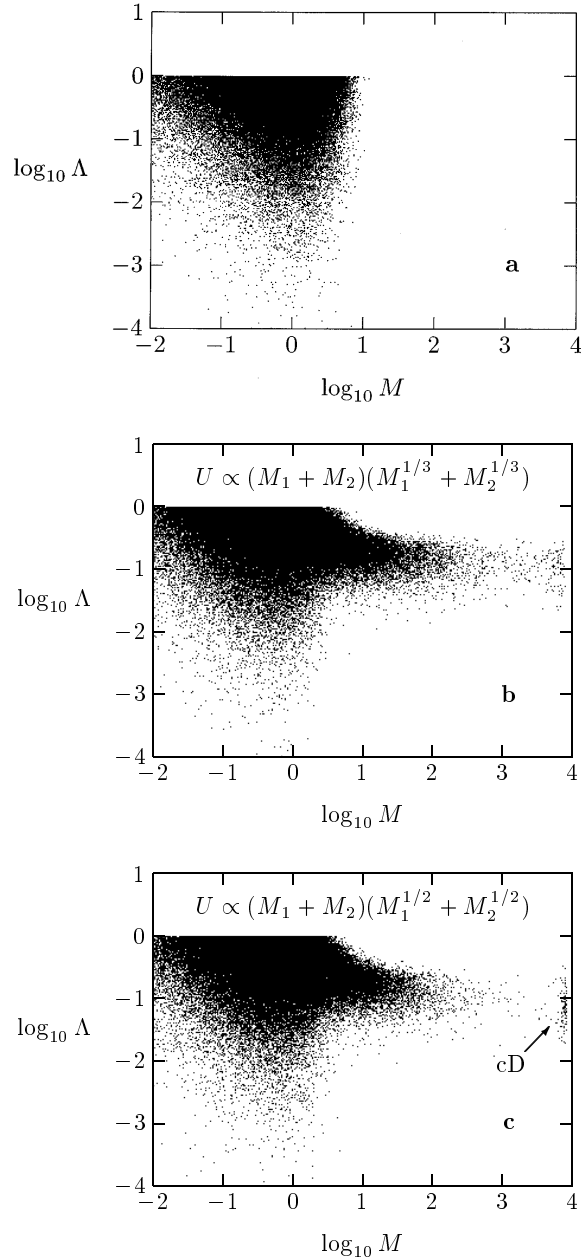
analog of phase transition takes place in the system. For an initial distribution localized in the small mass region, a slowly decreasing distribution tail is formed; in a finite time the tail reaches infinite masses (in an idealized case). The second moment of the distribution becomes infinite at  $t = t_{cr}$ . This phenomenon may, in principle, lead to fast formation of massive galaxies and quasars by mergers (Kontorovich et al. 1992), and also to formation of cD-galaxies in groups and clusters (Cavaliere et al. 1991).

## 2.2. Numerical simulation of mergers: methods

Direct analytical or numerical solving the generalized Smoluchowski equation with orbital momentum is a very complicated problem due to the great number of variables. This difficulty can be avoided by using numerical Monte Carlo simulation of merging process. In this section we present such simulation. A finite system consisting of  $N$  galaxies (referred to as “cluster” below, though it may also be a group) was considered. Pairs of these galaxies merged (with probability, proportional to  $U(M_1, M_2, \mathbf{S}_1, \mathbf{S}_2)$ ) until the number of the galaxies reduced to some  $N_f$ . For each merger mass and angular momentum conservation laws  $M = M_1 + M_2$ ,  $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{J}$  were fulfilled. Distribution of the angular momenta (intrinsic  $\mathbf{S}_{1,2}$  and orbital  $\mathbf{J}$ ) over directions was taken isotropic. It was assumed that the merger probability  $U$  depends only on masses according to Eqs. (5) and (7) and does not depend on momenta. The absolute value of the orbital momentum was computed in accordance with Eq. (3), namely,  $J = \frac{M_1 M_2}{M_1 + M_2} v p_\infty$ ,  $v$  being a random number distributed in the range 0 to  $v_g$  with probability density  $f(v) \propto v^2 \exp(-3v^2/4v_{rms}^2)$ , and the impact parameter  $p_\infty$  being a random number distributed in the range 0 to  $(R_1 + R_2)(1 + v_g^2/v^2)^{1/2}$  with probability density  $f(p_\infty) \propto p_\infty$ . The initial galaxy mass distribution was chosen exponential  $\Phi_0(M) \propto e^{-M/M_0}$  (the results are independent of the exact expression for initial distribution, only the fact that it is localized in the region of small masses  $\sim M_0$  and decreases rapidly for large ones is essential). Instead of the momentum  $S$ , it is convenient to introduce a dimensionless momentum  $\Lambda = \frac{S}{MR(2GM/R)^{1/2}}$ , similar to Peebles’ parameter  $(SE^{1/2})/(GM^{5/2})$  (see also Doroshkevich 1967). Dimensionless momenta  $\Lambda$  of the initial galaxies were distributed uniformly in the range 0 to 1 in our simulation (as for  $\Phi_0(M)$ , the exact form of the distribution is unessential). Computations were carried out for the following parameters:  $\beta = \frac{1}{3} - \frac{1}{2}$ ,  $N = 10^3 - 10^5$ ,  $N_f = (1-4) \cdot 10^{-1} N$ .

In conclusion we describe the procedure of simulation.

1.  $N$  initial galaxies are simulated, with mass and angular momentum distributed according to  $f_0(M, \mathbf{S})$ .
2. Two random integer numbers,  $i$  and  $j$ , distributed uniformly in the range  $[1, n]$  ( $n$  is the current number of galaxies) and satisfying the condition  $i \neq j$ , are simulated.
3. Galaxies number  $i$  and  $j$  merge with probability  $p = U(M_i, M_j)/U_{max}$ . The mass and momentum of the new galaxy are calculated as described above. With probability  $(1 - p)$  the galaxies do not merge, and jump to item 2

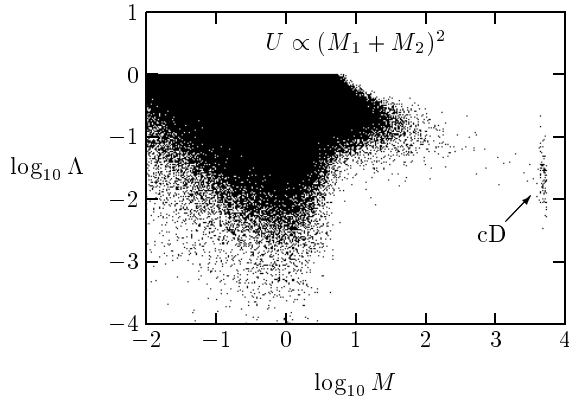


**Fig. 1a–c.** Masses  $M$  and dimensionless angular momenta  $\Lambda$  of simulated galaxies ( $M \gg M_b$ ): **a** initial, **b** and **c** formed by mergers (each dot represents one galaxy). A distribution tail, independent of the initial conditions, is formed due to mergers. The right-hand part of the figure ( $M \sim 10^4$ ) corresponds to cD-galaxies. For  $u = 3/2$  (**c**) separation of galaxies into the two phases, normal and cD, can be seen better than for  $u = 4/3$  (**b**). Each diagram shows  $10^5$  galaxies (**b** and **c** – 100 clusters of  $N_f = 1000$  galaxies each);  $N_f = 10^{-1} N$ ,  $N = 10^4$ .

is executed. Here  $U_{max} = \max_{1 \leq i, j \leq n} U(M_i, M_j)$  depends on time.

4. Items 2–3 are executed until the number of galaxies  $n$  becomes equal  $N_f$ .

The algorithm used in our simulation was somewhat different from the simplified scheme given above. The reason was that



**Fig. 2.** Masses  $M$  and dimensionless angular momenta  $\Lambda$  of simulated galaxies for the case  $M \ll M_b$ .  $N = 10^4$ ,  $N_f = 4 \cdot 10^{-1} N$ , 100 clusters ( $4 \cdot 10^5$  galaxies).

the merger probability  $p = U(M_i, M_j)/U_{\max}$  is very small for the majority of galaxies and so the simulation time is very large. In actual simulation the algorithm was modified as follows:

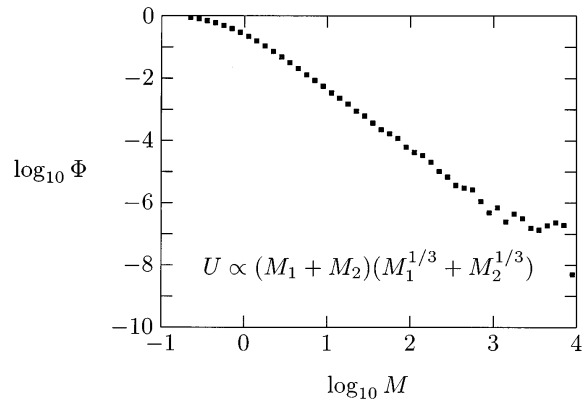
- the probability to choose the  $i$ -th galaxy in item 2 was  $k_i/N$  instead of  $1/N$ , where  $k_i$  is the number of initial galaxies which have subsequently merged into the  $i$ -th galaxy;
- to compensate this change, the function  $U'(M_i, M_j) = U(M_i, M_j)/(k_i k_j)$  was used instead of  $U(M_i, M_j)$  in item 3.

Obviously, these modifications do not influence the result of the simulation. In the same time, the number of cycles reduces because the average value of  $U'/U'_{\max}$  is closer to 1 than the average value of  $U/U_{\max}$ .

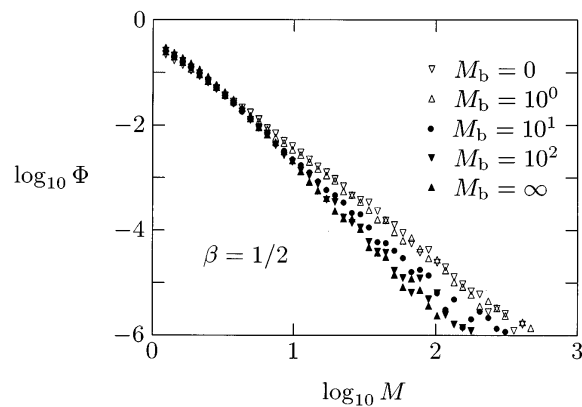
### 2.3. Numerical simulation of mergers: results

After some time, an analog of phase transition takes place in the system of merging galaxies, similarly to what occurred in the work by Cavaliere et al. (1991). The system divides into two phases: a giant galaxy which contains a major part of the total mass and many small galaxies (in the cases  $u = 3/2$  and  $u = 2$  this transition is more evident than for  $u = 4/3$ ). These giant galaxies formed by mergers can probably be identified as cD-galaxies in the centres of groups or clusters (cf. Hausman & Ostriker 1978; Cavaliere et al. 1991). The majority of low-mass galaxies are those which have never merged (Fig. 1). Such behaviour is related to strong dependence of the merger probability on masses ( $u = 1 + \beta$  or  $u = 2$ ) which, in turn, results in a steep mass function ( $\alpha \sim 2$  for  $u = 1 + \beta$ ,  $\alpha \gtrsim 2$  for  $u = 2$ , see below).

The mass function formed by mergers in the case  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$  (i.e.,  $M \gg M_b$ ) is shown in Fig. 3. In the region  $M_0 \ll M \ll M_{\max}$  it is close to a power law  $M^{-\alpha}$ ,  $\alpha \approx 2$ . The rise near the right-hand end corresponds to cD-galaxies the masses of which are comparable to the total masses of their clusters. In the case  $U \propto (M_1 + M_2)^2$  (i.e.,  $M \ll M_b$ ), and in the intermediate case of a finite  $M_b$  the mass function is steeper.



**Fig. 3.** Mass function obtained by Monte Carlo simulation ( $M \gg M_b$ ). The values of parameters are the same as in Fig. 1. Mass is given in units  $M_0$ ,  $\Phi(M)$  is normalized to unity. The part of the plot near  $M = 10^4$  corresponds to cD-galaxies.



**Fig. 4.** Mass function obtained by Monte Carlo simulation for different  $M_b$  ( $N_f = 10^{-1} N$ ). The average slope changes with  $M_b$ .

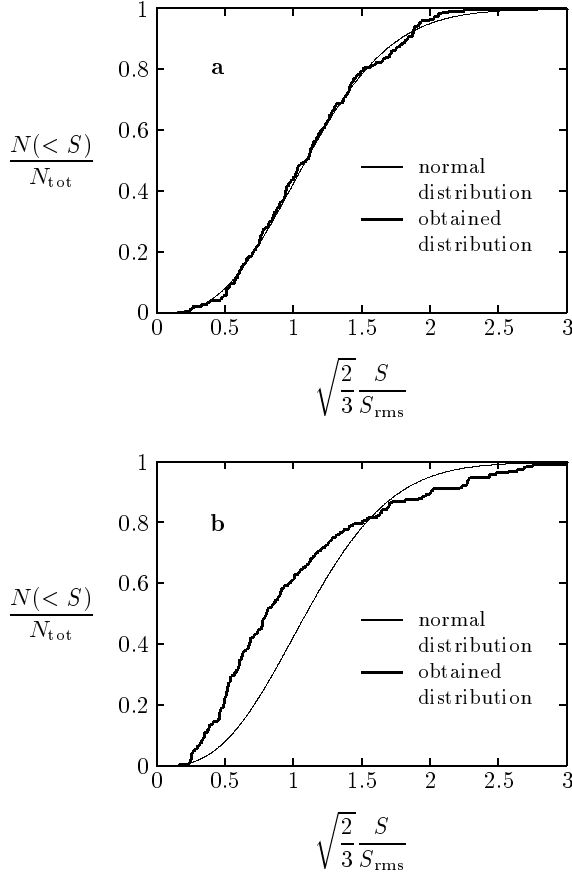
The obtained momentum distribution at fixed mass in the asymptotical region of large masses is close to the normal distribution (Fig. 5a). Thus, the distribution tail may be represented as

$$f(M, S) \approx \Phi(M) \left( \frac{2\pi}{3} \overline{S^2} \right)^{-3/2} \exp \left( -\frac{3 S^2}{2 \overline{S^2}} \right), \quad (10)$$

where  $\Phi(M)$  is the mass function,  $\overline{S^2(M)}$  is the average square value of the momentum  $S$  for given mass  $M$ . cD-galaxies in the cases  $u = 3/2$  and  $u = 2$  make an exception (Fig. 5b). Kolmogorov–Smirnov test shows evidently that their momentum distribution differs from the normal one.

For  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$  simulation shows that, irrespectively of the initial momentum, the root mean square value of the dimensionless momentum  $\Lambda_{\text{rms}}$  becomes constant at large masses:

$$\Lambda_{\text{rms}} \approx \text{const} \sim 0.1, \quad (11)$$



**Fig. 5a and b.** Comparison of the angular momentum cumulative distribution at a fixed mass with the normal distribution (Eq. (10)) for  $M \gg M_b$ . **a** The region of the distribution tail ( $\log_{10} M = 1.8$ ) for  $u = 3/2$ . The distribution is close to the normal one. **b** cD-galaxies for  $u = 3/2$ . The distribution differs from the normal one (the significance level in the Kolmogorov–Smirnov test is  $\sim 10^{-9}$ ).

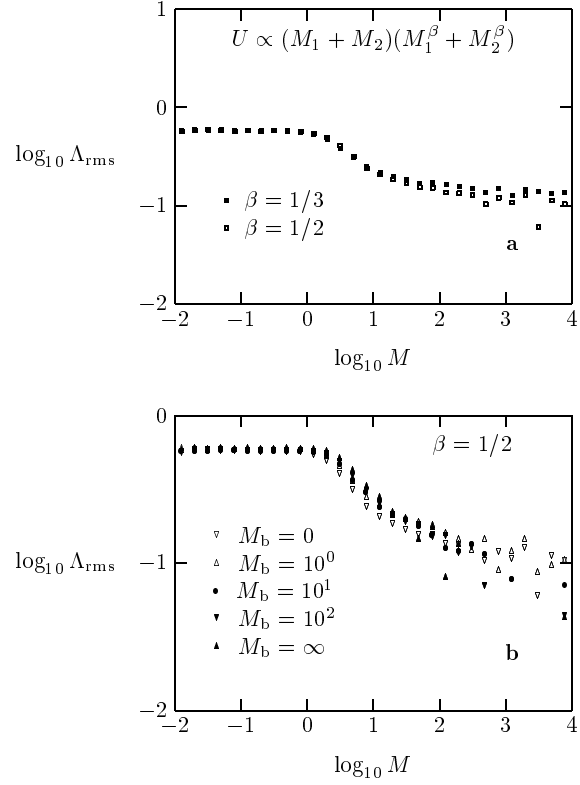
that is

$$\overline{S^2(M)} \sim 0.01 \left[ MR \sqrt{\frac{2GM}{R}} \right]^2 \quad (12)$$

(Figs. 6a and 1). The distribution at small masses depends on  $f_0(M, \mathcal{S})$ .

For  $U \propto (M_1 + M_2)^2$  the dimensionless momentum decreases with mass (Figs. 6b and 2).

The fact that  $\Lambda_{\text{rms}} \approx \text{const}$  for  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$  at large masses has a simple qualitative explanation. Consider the change of the mass and momentum due to mergers. As the mass function decreases rapidly (i.e., the number of small galaxies is very large) and  $u_1 = 0$ , it is natural to suppose that the main contribution to the change of the mass  $M$  and momentum  $S$  of a given massive galaxy is associated with accretion of small galaxies  $\sim M_0 \ll M$  (as, e.g., in Kontorovich et al. 1992). However, in this case the situation is different. The rate of changing  $M$  and  $S$  due to mergers with low mass galaxies



**Fig. 6a and b.** Dependence of the root mean square dimensionless momentum  $\Lambda_{\text{rms}}$  on mass. **a**  $M \gg M_b$ . In the left-hand part of the plot the main contribution is given by small galaxies which have never merged,  $\Lambda_{\text{rms}}$  being determined by the initial distribution for them. For large galaxies momenta are determined by mergers and do not depend on the initial conditions,  $\Lambda_{\text{rms}} \approx \text{const}$ . **b** The same for different  $M_b$  ( $N_f = 10^{-1}N$ ).  $\Lambda_{\text{rms}}$  decreases with  $M$ .

( $< M$ ) can be expressed as

$$\begin{aligned} \dot{M} &= \int_0^M U(M, M_1) M_1 \Phi(M_1) dM_1 \\ &\propto M^{u_2} \int_0^M M_1^{u_1+1} \Phi(M_1) dM_1, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\overline{S^2}} &= \int_0^M U(M, M_1) (\overline{S^2(M_1)} + \overline{J^2(M, M_1)}) \Phi(M_1) dM_1 \\ &\propto M^{u_2+1+\beta} \int_0^M M_1^{u_1+2} \Phi(M_1) dM_1, \end{aligned} \quad (14)$$

since  $\overline{S^2(M_1)} \ll \overline{J^2(M, M_1)}$ ,  $\overline{J^2(M, M_1)} \sim M_1^2 GMR$  for  $M_1 \ll M$ . In our case  $u_1 = 0$ , the slope of the mass function  $\alpha \sim 2$ . Therefore, the main contribution to integral (14) is given by large galaxies ( $\sim M$ ); contribution of large and small galaxies in Eq. (13) are of the same order.

Next we consider successive mergers of galaxies with equal large masses  $M$  and momenta  $S_{\text{rms}}(M)$ . The new galaxy has a mass  $M' = 2M$  and momentum<sup>4</sup>  $(S')^2 = 2S^2 + \overline{J^2} = 2\overline{S^2} +$

<sup>4</sup> The orbital momentum  $\overline{J^2} = (MM/(M+M))^2 v^2 p_\infty^2$ , the impact parameter  $\overline{p_\infty^2} = \frac{1}{2}(R+R)^2 v_g^2/v^2$ , so,  $\overline{J^2} = MGM R$ .

*MGM*R. Thus, the new value of the dimensionless momentum  $(\bar{\Lambda})^2 = 2^{-2-\beta}\bar{\Lambda}^2 + 2^{-4-\beta}$ . As a result of successive mergers,  $\Lambda_{\text{rms}}$  tends to the equilibrium value  $\bar{\Lambda}_{\text{rms}} = \sqrt{\frac{2^{-4-\beta}}{1-2^{-2-\beta}}} \approx 0.23-0.25$  ( $\beta = \frac{1}{3}-\frac{1}{2}$ ). So, the result  $\bar{\Lambda}^2 \approx \text{const}$  is a consequence of the fact that the main contribution to the change of mass and momentum is given by mergers between comparable mass galaxies. The obtained value for  $\bar{\Lambda}_{\text{rms}}$  is somewhat different from Eq. (11) due to simplifying assumptions made in the derivation. Note, that for the isotropic momentum distribution without allowance for the orbital momentum  $\bar{S}^2 \propto M$ ,  $\bar{\Lambda}^2 \propto M^{-2-\beta}$ ; for the anisotropic distribution (Kats & Kontorovich 1990, 1991)  $\bar{S} \propto M$ ,  $\bar{\Lambda} \propto M^{-(1+\beta)/2}$ .

In the case  $U \propto (M_1 + M_2)^2$  the main contribution to Eqs. (13), (14) is given by small masses  $\sim M_0$  (due to large  $\alpha$ ) and we can expect decreasing of  $\bar{\Lambda}_{\text{rms}}$  as  $M^{-1/2}$ . This can be demonstrated as follows. If integral  $\int_0^\infty M^{u_1+2}\Phi(M) dM$  converge at infinity then it is possible to replace the upper limit of integrals (13), (14) by infinity. Then

$$\dot{M} \propto M^{u_2}, \quad \dot{\bar{S}^2} \propto M^{u_2+1+\beta}. \quad (15)$$

Therefore,

$$\frac{d\bar{S}^2}{dM} \propto M^{1+\beta}, \quad \bar{S}^2 \propto M^{2+\beta}, \quad \bar{\Lambda}^2 \propto \frac{\bar{S}^2}{M^{3+\beta}} \propto M^{-1}. \quad (16)$$

An analysis of observational data (Kontorovich & Khodjakhikh 1993; Kontorovich et al. 1995a) confirms that  $S_{\text{rms}} \propto M^k$ , the coefficient  $k$  being rather close to  $(3 + \beta)/2$ , which is in accordance with Eq. (12).

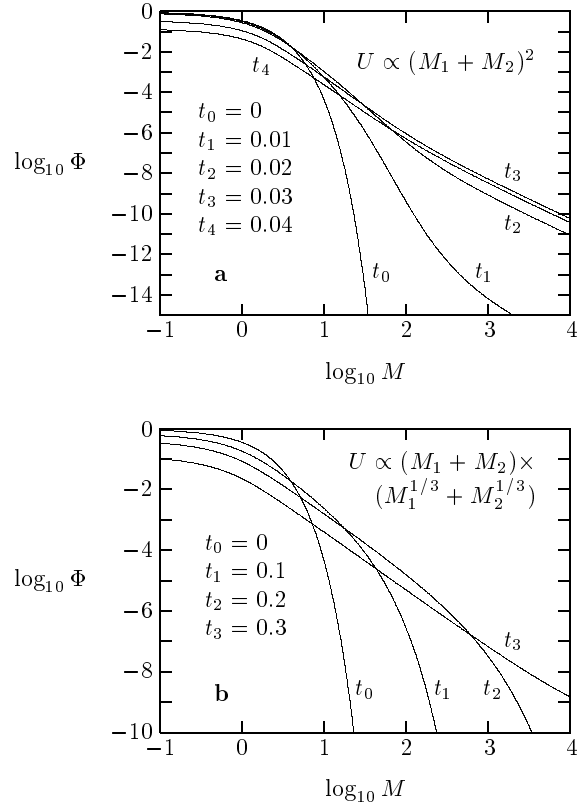
Note that allowance for dependence of the merger probability on momenta may give an increase of  $\bar{\Lambda}^2$ : numerical experiments show that a merger is more probable if all momenta have the same direction (Farouki & Shapiro 1982). On the other hand,  $\bar{\Lambda}^2$  is sensitive to the exact form of the merger cross-section (in particular, to the value of  $\zeta$  in Eq. (3)).

### 3. Comparison of simulation results with solution of the Smoluchowski equation

Integrating the generalized Smoluchowski equation (1) over momenta, one can obtain the ordinary Smoluchowski equation which describe the evolution of the galaxy mass function:

$$\begin{aligned} \frac{\partial \Phi(M, t)}{\partial t} &= \int_0^M U(M_1, M - M_1, t) \Phi(M_1, t) \Phi(M - M_1, t) dM_1 \\ &\quad - 2\Phi(M, t) \int_0^\infty U(M_1, M, t) \Phi(M_1, t) dM_1. \end{aligned} \quad (17)$$

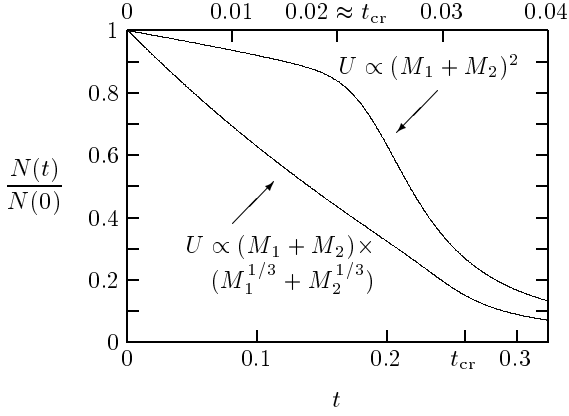
Solving this equation is another independent way to find  $\Phi(M, t)$ . In this section we compare the results obtained by Monte Carlo simulation with the obtained earlier (see Kontorovich et al. 1995b and Krivitsky 1995) results of direct numerical solving the Smoluchowski equation with kernels  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$  and  $U \propto (M_1 + M_2)^2$ .



**Fig. 7.** **a** Numerical solution of the Smoluchowski equation in the small mass region ( $M_{\text{max}} = 10^4$ ,  $M$  is measured in units  $M_0$ , time in units  $1/(c_u N_0 M_0^u)$ ,  $\Phi$  in units  $N_0/M_0$ ). **b** Solution in the large mass region, for comparison (the same case as in Cavaliere et al. (1992), but in a wider mass range and somewhat more precise).

For numerical solution of Eq. (17) and analysis of the obtained results we used methods described in Krivitsky 1995; Kontorovich et al. 1995b. Fig. 7b shows a plot of the obtained mass function for  $U = c_u(M_1 + M_2)(M_1^\beta + M_2^\beta)$ . One can see that an intermediate asymptotics, close to a power law  $M^{-\alpha}$ , is formed in the region  $M_0 \ll M \ll M_{\text{max}}$ . The exponent  $\alpha \approx 1.9$  for  $u = 4/3$  and  $\alpha \approx 2.1$  for  $u = 3/2$  (in the latter case  $\alpha$  is defined worse, see below). The value 1.9 differs from  $\alpha \approx 2.15$  given by Cavaliere et al. (1992) but, in our opinion, agrees rather well with the plot in their work. The value of  $t_{\text{cr}}$  for the two cases are<sup>5</sup>, respectively, 0.26 and 0.1 (in the same units as in Fig. 7) for the initial distribution  $\Phi_0(M) = (N_0/M_0)e^{-M/M_0}$ . The time dependence of the number of galaxies, obtained from the solution of the Smoluchowski equation, is shown in Fig. 8. The moment displayed in Fig. 1b ( $N_f = 0.1N_0$ ) approximately corresponds to  $t \approx 0.3$  (whereas  $t_{\text{cr}} \approx 0.26$ ).

<sup>5</sup> Since a characteristic time associated with mergers is of the order of  $(c_u N_0 M_0^u)^{-1}$ , such a value of  $t_{\text{cr}}$  means that the phase transition is comparatively fast. Due to considerable contribution of a comparatively small number of appearing massive galaxies, the time  $t_{\text{cr}}$  corresponding to the “phase transition” is much less than a characteristic time  $(\sigma v n)^{-1}$ , where  $\sigma$  is the merger cross-section for typical galaxies,  $n$  is the concentration of such galaxies,  $v$  is the average velocity.



**Fig. 8.** Time dependence of the number of galaxies, obtained from the solution of the Smoluchowski equation (upper time scale for  $u = 2$ , lower time scale for  $u = 4/3$ ). Though the total number of galaxies decreases, at any  $t$  the number of low-mass galaxies which have never merged is much more than the number of massive tail galaxies.

In the case  $U = c_2(M_1 + M_2)^2$  (Fig. 7a) the intermediate asymptotics is not as close to a power law as for  $U = c_u(M_1 + M_2)(M_1^\beta + M_2^\beta)$ . An effective slope  $\alpha$  in this case is 2–3. The value of  $t_{\text{cr}}$  is  $\sim 0.02$ , that is the phase transition is very fast. However, in this case the time dependence of the distribution moments is non-power and  $t_{\text{cr}}$  cannot be determined accurately (Kontorovich et al. 1995b; Krivitsky 1995).

In numerical solving Eq. (17), a finite limit mass  $M_{\text{max}}$  was introduced: the integral from 0 to  $\infty$  in the right-hand part was replaced by the integral from 0 to  $M_{\text{max}}$ . Physically, such a substitution corresponds to a sink at large masses. As it was shown by Krivitsky 1995, Kontorovich et al. 1995b, consequences of this replacement are different for kernels with  $u_2 \leq 1$  and  $u_2 > 1$ . In the case of kernel (7) which belongs to the class  $u_2 > 1$ , existence of  $M_{\text{max}}$  and its value essentially influence the solution, in particular, the number of galaxies  $N = \int_0^\infty \Phi dM$  and distribution moments  $\mathcal{M}^{(p)} = \int_0^\infty \Phi M^p dM$  as functions of time, the value of  $t_{\text{cr}}$  etc. Moreover, van Dongen (1987) showed that the limit  $M_{\text{max}} \rightarrow \infty$  does not exist at all for  $u_2 > u_1 + 1$  (this is the case for Eq. (7)). The influence of  $M_{\text{max}}$  increases as  $u_2$  becomes farther from 1: for  $U \propto (M_1 + M_2) \times (M_1^\beta + M_2^\beta)$ , especially if  $\beta = 1/3$ , this influence is moderate. The farther  $u_2$  moves from 1, the greater the difference between the intermediate asymptotics and the power law becomes and the worse defined  $t_{\text{cr}}$  and  $\alpha$  are; this is the case for  $U \propto (M_1 + M_2)^2$ .

As known (see van Dongen & Ernst 1988; Voloshchuk 1984; Krivitsky 1995; Kontorovich et al. 1995b), in many cases the solution for  $u_2 \leq 1$  is self-similar:  $\Phi(M) \approx \mu^{-\tau}(t)\varphi(M/\mu(t))$  for  $M \gg M_0$ , where  $\varphi$  is a time-independent function,  $\mu(t)$  describes a “front” moving to greater masses. The numerical solution shows that  $\Phi(M, t)$  for  $U \propto (M_1 + M_2) \times (M_1^\beta + M_2^\beta)$  is closer to the self-similar form for lower  $\beta$ . The mass function for  $U \propto (M_1 + M_2)^2$  is not self-similar (nonlocal case, see discussion in Kontorovich et al. 1995b). However, for  $t > t_{\text{cr}}$  the shape of the curve becomes nearly constant (Fig. 7a): a different self-similarity appears, because mergers with the cD-

galaxy dominate and the dependence of  $U$  on the smaller mass vanishes.

Both in direct solution of the Smoluchowski equation and in numerical simulation of mergers, there exists a finite limit mass: the mass of the sink in the former case, the total mass of the system in the latter one. However, the problems which are solved in this section and in Sect. 2 are not equivalent. Nevertheless, simulation shows that, in spite of the essential influence of the limit mass ( $u_2 > 1$ ), the mass function obtained by simulation of mergers in Sect. 2 and the one obtained as a direct solution of the Smoluchowski equation have good agreement in the region  $M \ll M_{\text{max}}$ . So we can make a conjecture that  $\Phi(M)$  for  $M \ll M_{\text{max}}$  depends only on the value of the limit mass and does not depend on its nature.

#### 4. Discussion

Simulation confirms the possibility that massive galaxies may form by mergers, moreover, this process has an “explosive” character and is an analog of phase transition, cD-galaxies (with mass comparable to the total galaxy mass of the cluster) being formed as a new phase. What are the conditions which make this process possible? The expression for  $t_{\text{cr}}$  may be written as

$$t_{\text{cr}} = \frac{\xi_u}{c_u N_0 M_0^u} \quad (18)$$

(see, e.g., Voloshchuk 1984). Numerical solution of the Smoluchowski equation gives  $\xi_2 \approx 0.02$ ,  $\xi_{4/3} \approx 0.26$ ,  $\xi_{3/2} \approx 0.1$  for  $U = c_2(M_1 + M_2)^2$ ,  $U = c_{4/3}(M_1 + M_2)(M_1^{1/3} + M_2^{1/3})$ ,  $U = c_{3/2}(M_1 + M_2)(M_1^{1/2} + M_2^{1/2})$  respectively (if the initial distribution  $\Phi_0$  has a tail,  $\xi_u$  may essentially depend on  $\Phi_0$  and be much less). Assuming  $N_0 \sim \mathcal{M}/M_0$  where  $\mathcal{M}$  is an average density of the mass contained in galaxies and expressing the variables in astronomical units, we obtain the order of magnitude for  $t_{\text{cr}}$ :

$$t_{\text{cr}} \sim \begin{cases} 2 \cdot 10^{15} v_7^3 M_6^{-1} (\mathcal{M}/\rho)^{-1} \text{ years} & (u = 2), \\ 4 \cdot 10^{13} v_7 M_6^{-1/3} (\mathcal{M}/\rho)^{-1} \text{ years} & (u = 4/3), \\ 10^{14} v_7 M_6^{-1/2} (\mathcal{M}/\rho)^{-1} \text{ years} & (u = 3/2). \end{cases} \quad (19)$$

Here  $v_7 = v_{\text{rms}}/(10^7 \text{ cm s}^{-1})$ ,  $M_6 = M_0/(10^6 M_\odot)$ ,  $\mathcal{M}/\rho$  is the ratio of the local density of the mass contained in galaxies to the average density of the Universe; the coefficient  $C$  in Eq. (6) is assumed  $C \sim \frac{20 \text{ kpc}}{(2 \cdot 10^{11} M_\odot)^\beta}$ . The mass of a rich cluster is  $5 \cdot 10^{14} - 5 \cdot 10^{15} M_\odot$ , 2–7% of it is contained in galaxies (Böhringer 1995). The size (of the central part) being of the order of one megaparsec, the ratio  $\mathcal{M}/\rho$  may be several hundred to several thousand. Assuming  $\mathcal{M}/\rho \sim 10^3$  we obtain that, for a cluster with a low velocity dispersion ( $\sim 300 \text{ km s}^{-1}$ ), the critical time is less than the age of the Universe on condition that masses of the initial galaxies (which then merge)  $M_0 \sim 10^9 - 10^{10} M_\odot$  or more (for the case  $M \gg M_b$ ,  $u = 3/2$ ). A close estimate for  $M_0$  can be obtained also for the region<sup>6</sup>

<sup>6</sup> The choice of the region depends on the relation between  $M_0$  and  $M_b$ . If  $M_0 \ll M_b$  then the case  $u = 2$  is realized; if  $M_0 \gtrsim M_b$  then  $u = 1 + \beta$ .

$M \ll M_b$ ,  $u = 2$ . On the other hand, we may consider the formation of massive galaxies and the mass function tail only if the initial mass  $M_0$  is much less than a typical mass of a large galaxy ( $\sim 10^{11} M_\odot$ ). For a cluster with a bigger velocity dispersion ( $1000 \text{ km s}^{-1}$  or more) it is much more difficult to satisfy condition (19):  $M_b$  is large, so the kernel with  $u = 2$  should be taken;  $t_{\text{cr}}$  is proportional to  $v_{\text{rms}}^3$  and can be less than the age of the Universe only for very high density ( $\mathcal{M}/\rho \sim 10^4$ ) and large enough initial masses ( $M_0 \sim 10^{10} M_\odot$ ). Thus, possible dependence of  $v_{\text{rms}}$  on time due to cluster evolution, its space nonhomogeneity etc. can essentially influence the role of mergers, especially on small masses.

The estimate for  $\mathcal{M}/\rho$  given above is based on the assumption that dark matter belongs to the whole cluster rather than to individual galaxies. If dark matter is concentrated in galaxies<sup>7</sup>, the ratio  $\mathcal{M}/\rho$  may increase by an order which results in the same decrease of  $t_{\text{cr}}$  (according to Eq. (19)).

So, the conditions necessary for the “explosive” process of mergers may be realized in many clusters.

After a cD-galaxy which contain a significant part of the total mass has formed in the cluster centre, the dynamics of the cluster is largely determined by attraction to this galaxy, and the model considered in this paper breaks down. Besides random collisions, spatial inhomogeneity and mass segregation become essential: due to dynamical friction, most massive galaxies gradually gets into the centre and are swallowed by the cD-galaxy. However, before  $t_{\text{cr}}$ , when there is no yet cD-galaxy in the cluster, galaxy mergers can be considered as random pairwise encounters with probability given by Eq. (7). It is also clear that mass segregation at a late stage of cluster evolution should be computed together with mass function evolution, using a spatially inhomogeneous Smoluchowski type kinetic equation, which is a much more complicated problem.

Note that galactic mergers due to dynamical friction were discussed by Hausman & Ostriker (1978). In spite of the difference in the model, the algorithm for simulation of mergers, used in their work, is analogous to the algorithm we use (and, thus, equivalent to the Smoluchowski equation), though the expression for the merger probability is different ( $U \propto M_1 M_2$ , which also gives the “explosive” evolution).

As shown in Sects. 2–3 (see also Kontorovich et al. 1995b; Krivitsky 1995), the mass function formed by mergers with the probability given by Eq. (7) is rather steep ( $\alpha \approx 2$  for  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$ ,  $\alpha \sim 2-3$  for  $U \propto (M_1 + M_2)^2$ ; in the latter case the asymptotics seem to be non-power, so  $\alpha$  cannot be determined accurately). It is quite possible that the obtained values of the slope correspond to the steepening of the cluster galaxy luminosity function at the faint end, which was recently discovered (de Propris et al. 1995; Kashikawa et al. 1995; Bernstein et al. 1995). According to observational data analysed in these works, the effective value of the slope for faint galaxies in clusters increases up to 2–2.2 (though, as Bernstein et al. note, there is a different interpretation of these results). Possi-

<sup>7</sup> At least cD-galaxies may contain a large quantity of dark matter (Ikebe et al. 1996).

bly, this part of the luminosity function is formed by mergers and can be described by the intermediate asymptotics for the Smoluchowski equation (if the latter can be extended to small enough masses). Recent HST data shows also an excess of faint low-mass objects for field galaxies (Cowie et al. 1995).

The appearance of relatively steep intermediate asymptotics ( $\alpha \approx 2$ ) can be easily understood from the following arguments<sup>8</sup>. Both obtained values for the index ( $\alpha \approx 1.9$  for  $u = 4/3$  and  $\alpha \approx 2.1$  for  $u = 3/2$ ) are within the range  $(u+2)/2$  to  $(u+3)/2$ . The mass function with  $\alpha = (u+3)/2$  corresponds to a constant mass flux<sup>9</sup> to infinity (i.e., to cD-galaxy, in our case). However, due to nonlocality of the distribution<sup>10</sup> ( $|u_2 - u_1| > 1$ , see Vinokurov & Kats 1980) such a solution is not realized exactly in both our cases. Nonlocality leads to an essential role of interactions between low-mass and high-mass galaxies. Then the number of massive galaxies is approximately conserved, and the constant flux of their number to infinity corresponds to  $(u+2)/2$  (Kontorovich et al. 1993). Since none of these limit cases is realized, the index is situated between these values:  $1.67 < \alpha \approx 1.9 < 2.17$  ( $u = 4/3$ ),  $1.75 < \alpha \approx 2.1 < 2.25$  ( $u = 3/2$ ), and is rather close to their arithmetic mean value (as we can see both from the simulation and the numerical solution of the Smoluchowski equation).

The density ratio  $\mathcal{M}/\rho$  in the above estimates was one of the most important parameters which control the possibility of effective merger process. The local value of this parameter may vary in a very wide range: from 1 (scales exceeding an average distance between massive field galaxies) to  $10^7$  (if we take an average density of a galaxy  $\sim 10^{-22} \text{ g cm}^{-3}$  for  $\mathcal{M}$ ). As was shown above, in clusters this parameter is large enough ( $\sim 10^3$  or even  $10^4$ ) to yield an “explosive” evolution due to mergers. Local concentrations may enable analogous phenomena for field galaxies at large  $z$  (see, e.g., Komberg & Lukash 1994; Kontorovich 1994).

Morphological changes in cluster galaxies, which is one of the results of mergers, may be related to the change of the angular momentum distribution (cf. Toomre 1977). The possibility of dependence between Hubble’s morphological type and an effective angular momentum has been discussed in the literature (see Fig. 1 in Polyachenko et al. 1971) and confirmed by an independent analysis of observational data (Kontorovich & Khodjachikh 1993; Kontorovich et al. 1995a). However, this dependence needs special consideration which is beyond the scope of this work.

The above consideration of cluster evolution takes into account only galaxy mergers in a spatially homogeneous model. It allows to obtain the “explosive” evolution, the steep part of the mass function, cD-galaxies, rapid evolution of galaxy morphological types, and a mean value of the dimensionless angular

<sup>8</sup> We consider here the case  $U \propto (M_1 + M_2)(M_1^\beta + M_2^\beta)$ , when the mass function is close to a power law.

<sup>9</sup> Solutions with a constant flux of a conserved variable are analogous to Kolmogorov spectra in the weak turbulence theory, see Zakharov et al. 1992.

<sup>10</sup> Nonlocality corresponds to divergence of the collision integral for the power-law distribution.

momentum which does not depend on the details of the initial distribution. In the same time, this approach has obvious limitations. The “explosive” evolution does not produce Schechter’s mass function with  $\alpha \approx 1.25$ . It is possible that the effective merger probability  $U$  changes at a late stage of cluster evolution (when massive galaxies have formed) in such a way that  $u_{\text{eff}}$  becomes less than one and the “explosive” process slows down<sup>11</sup>, which leads to a flatter  $\Phi(M)$ . Another possibility is that this part of the mass function may not be formed only by mergers.

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<sup>11</sup> See examples of such  $U$  changes in Kontorovich et al. 1994; van Dongen 1989.