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SMART97: a new solution for the rotation of the rigid Earth

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Abstract. The solution SMART97 (Solution du Mouvement de l'Axe de Rotation de la Terre) is an analytical solution of the motion of the axis of rotation of the Earth, resulting from an improvement of the theory presented in Bretagnon et al. (1997). This solution is built with the analytical theories of the motion of the Moon, the Sun and the planets of the Bureau des longitudes. The VLBI observations reach an accuracy of 20 μas for the motion of the axis of the rotation of the Earth and it is necessary to obtain a theory of the rotation of the rigid Earth with a precision of about 1 μas to reveal the effects of the non-rigidity of the Earth by comparison to observation. The precision of SMART97, given by comparison with a numerical integration run over 150 yr (1900-2050), is better than 1.5 μas for ψ and φ and 0.5 μas for ω. We have also compared our solution with a numerical integration using DE403; the differences do not exceed 12 μas over 1900-2050 and 2.2 μas over 1970-2020.

Key words: astrometry – celestial mechanics – Earth

1. Introduction

Since 1980, the solutions giving the position of the equator of the Earth cannot be computed with only a theory of the rigid Earth. The IAU 1980 adopted nutation series are deduced from the Kinoshita (1977) theory of the rotation of the rigid Earth, by convolution with the non-rigid Earth's transfer functions of Wahr (1981).

By using modern techniques like Very Long Baseline Interferometry (VLBI) precession and nutations may be presently observed with a precision of about 20 μas. The differences between the theory and the observation must reveal only the effects of the non-rigidity of the Earth, so it is necessary to build a theory of the rigid Earth with a precision better than 2 μas.

In Bretagnon et al. (1997) we have built analytical solutions for each of the Earth's three Euler angles ψ, ω, φ, using truncated solution for the motion of the Moon. Over 1900-2050, the precision was 16 μas for ψ, 8 μas for ω and 15 μas for φ. For improving our solution we have computed geocentric rectangular coordinates of the Moon referred to the ecliptic and equinox J2000 from the complete solution ELP 2000 (Chapront-Touzé, Chapront, 1983). Also we have computed the right-hand sides of the equations with a better accuracy.

2. Notations and constants

The notations are the same as in Bretagnon et al. (1997). The unit of time is one thousand Julian years (tjy). The following constants are taken from the numerical integration of the JPL DE403/LE403 (Standish et al., 1995) and from the IERS Standards 1996 (IERS, 1996).
Value of the astronomical unit:

1 au = 149 597 870.691 km (DE403)

Equatorial radius of the Earth:

a_E = 6 378.13649 km (IERS1996).

We note m_S, m_E, m_M the masses of the Sun, the Earth and the Moon. From DE403/LE403 we take

$$\frac{m_S}{m_E} = 332\,946.0486$$
$$\frac{m_E}{m_M} = 81.300\,585$$
$$\frac{m_S}{m_M} = 27\,068\,708.5246$$

From the value of the Gaussian gravitational constant k = 0.017 202 098 95 and from the mass of the Earth, we compute

$$\begin{aligned} Gm_E &= 3.986\,004\,356 \times 10^{14}\ \mathrm{m^3\,s^{-2}} \\ &= 118.568\,538\,619\ \mathrm{au^3 tjy^{-2}} \end{aligned}$$

The geopotential coefficients $C_{n,k}, S_{n,k}$ are computed from the GEM-T3 model given by the IERS Standards 1992 (IERS, 1992). The longitude of the major axis of the equatorial ellipse corresponding to the principal moment of inertia A is computed from $C_{2,2}$ and $S_{2,2}$

$$\begin{aligned} \alpha &= \frac{1}{2}\tan^{-1}\left(\frac{S_{2,2}}{C_{2,2}}\right) \\ &= 14^\circ\!.9285 \quad \text{West} \end{aligned}$$

The constants depending of the precision of the solution, like principal moments of inertia, have slight modifications. So, to obtain the William's (1994) value of the constant of the general precession $\mathscr{P} = 50\,287''.7$/tjy, the equations of Sect. 3 must be solved to yield

$$\left(\frac{d\psi}{dt}\right)_{t=\mathrm{J2000}} = -50\,403''.763\,708\,8052\ /\mathrm{tjy} = -0.244\,364\,342\,254\,411\ \mathrm{rad/tjy}. \tag{1}$$

We thus obtained

$$\begin{aligned} A &= 0.179\,955\,354\,561 \times 10^{-14}\ m_S \mathrm{au}^2, \\ B &= 0.179\,959\,319\,043 \times 10^{-14}\ m_S \mathrm{au}^2, \\ C &= 0.180\,548\,410\,196 \times 10^{-14}\ m_S \mathrm{au}^2. \end{aligned} \tag{2}$$

The dynamical ellipticity H_d is

$$H_d = \frac{2C - A - B}{2C} = 0.003\,273\,766\,818. \tag{3}$$

The values of the moments of inertia A, B, C and therefore H_d are slightly different from the ones of Bretagnon et al. (1997) because of the modifications of the constants used for the masses of the Earth and the Moon and for the equatorial radius of the Earth.

The linear part of the angle φ measuring the rotation of the Earth is

$$\varphi = \varphi_0 + \varphi_1\, t \tag{4}$$

with

$$\begin{aligned} \varphi_0 &= 4.894\,961\,212\,82\ \mathrm{rad} \\ \varphi_1 &= 2\,301\,216.753\,651\,536\ \mathrm{rad/tjy} \\ &= 6.300\,388\,100\,346\,438\ \mathrm{rad/d} \end{aligned} \tag{5}$$

For the obliquity ω_0 we take the angle between the rotational equator J2000 and the inertial dynamical ecliptic J2000:

$$\omega_0 = \omega(t = \mathrm{J2000}) = -23°26'21''.409 = -0.409\,092\,615\,14\ \mathrm{rad}.$$

3. Equations of motion

3.1. Basic equations

We built our analytical solution resolving the following differential equations of the second order (Bretagnon et al., 1997)

$$\begin{aligned} \ddot{\omega} + \frac{C}{A}\sin\omega_0\,\varphi_1\,\dot{\psi} &= \frac{L}{A} + F_2 + \frac{B-A}{A}F_1 \\ \sin\omega_0\,\ddot{\psi} - \frac{C}{A}\varphi_1\,\dot{\omega} &= \frac{M}{A} + G_2 + \frac{B-A}{A}G_1 \\ \ddot{\varphi} &= \frac{N}{C} + H_2 + \frac{B-A}{C}H_1 \end{aligned} \tag{6}$$

with

$$\begin{aligned} F_1 &= \frac{1}{2}\ddot{\psi}\sin 2\tilde{\varphi}\sin\omega + \dot{\psi}\dot{\varphi}\cos 2\tilde{\varphi}\sin\omega - \dot{\omega}\dot{\varphi}\sin 2\tilde{\varphi} \\ &- \frac{1}{2}\ddot{\omega}(1-\cos 2\tilde{\varphi}) + \frac{1}{4}\dot{\psi}^2(1+\cos 2\tilde{\varphi})\sin 2\omega \\ G_1 &= -\frac{1}{2}\ddot{\psi}(1+\cos 2\tilde{\varphi})\sin\omega + \dot{\psi}\dot{\varphi}\sin 2\tilde{\varphi}\sin\omega \\ &+ \dot{\omega}\dot{\varphi}\cos 2\tilde{\varphi} + \frac{1}{2}\ddot{\omega}\sin 2\tilde{\varphi} + \frac{1}{4}\dot{\psi}^2\sin 2\tilde{\varphi}\sin 2\omega \\ H_1 &= \frac{1}{2}\dot{\omega}^2\sin 2\tilde{\varphi} - \dot{\psi}\dot{\omega}\cos 2\tilde{\varphi}\sin\omega \\ &- \frac{1}{4}\dot{\psi}^2\sin 2\tilde{\varphi}(1-\cos 2\omega) \end{aligned} \tag{7}$$

and

$$\begin{aligned} F_2 &= -\frac{C}{A}\varphi_1\,\dot{\psi}\,(\sin\omega - \sin\omega_0) - \frac{C}{A}\Delta\dot{\varphi}\,\dot{\psi}\,\sin\omega \\ &-\frac{1}{2}\frac{C-A}{A}\dot{\psi}^2\,\sin 2\omega \\ G_2 &= \frac{C}{A}\Delta\dot{\varphi}\,\dot{\omega} - \ddot{\psi}\,(\sin\omega - \sin\omega_0) \\ &-\frac{A+B-C}{A}\dot{\psi}\,\dot{\omega}\,\cos\omega \\ H_2 &= -\ddot{\psi}\,\cos\omega + \dot{\psi}\,\dot{\omega}\,\sin\omega. \end{aligned} \tag{8}$$

3.2. Equations for numerical integration

We used numerical integration to test our analytical development. Eqs. (6) are transformed in the form[1]

$$\begin{aligned} \ddot{\omega} = \frac{L}{A} &+ \frac{B-A}{B}\sin\tilde{\varphi}\Big(\frac{M}{A}\cos\tilde{\varphi} - \frac{L}{A}\sin\tilde{\varphi}\Big) - \dot{\psi}\dot{\varphi}\sin\omega \\ &- \frac{C-B}{A}\dot{\psi}\sin\omega\,(\dot{\varphi} + \dot{\psi}\cos\omega) \\ &+ \frac{B-A}{A}\frac{C-A-B}{B} \\ &\quad\times(\dot{\psi}\sin\omega\sin^2\tilde{\varphi} + \frac{1}{2}\dot{\omega}\sin 2\tilde{\varphi})\,(\dot{\varphi} + \dot{\psi}\cos\omega) \\ \sin\omega\,\ddot{\psi} &= \frac{M}{B} + \frac{B-A}{A}\sin\tilde{\varphi}\Big(\frac{M}{B}\sin\tilde{\varphi} + \frac{L}{B}\cos\tilde{\varphi}\Big) \\ &+ \frac{C+B-A}{B}\dot{\varphi}\dot{\omega} + \frac{C-A-B}{B}\dot{\psi}\dot{\omega}\cos\omega \\ &+ \frac{B-A}{A}\frac{C-A-B}{B} \\ &\quad\times(\dot{\omega}\sin^2\tilde{\varphi} - \frac{1}{2}\dot{\psi}\sin\omega\sin 2\tilde{\varphi})\,(\dot{\varphi} + \dot{\psi}\cos\omega) \\ \ddot{\varphi} &= \frac{N}{C} + H_2 + \frac{B-A}{C}H_1 \end{aligned} \tag{9}$$

Eqs. (9) are strictly equivalent to the basic equations.

3.3. Model used

The torques on the oblate rigid Earth due to the gravitational attraction of the Moon, the Sun and the planets from Mercury to Neptune are considered. The effects of the zonal harmonics $C_{n,0}$ with $2 \le n \le 5$ and of the tesseral harmonics $C_{2,2}$, $S_{2,2}$, $C_{3,k}$, $S_{3,k}$ with $1 \le k \le 3$, $C_{4,1}$, $S_{4,1}$ are computed.

[1] In formula (9) we correct a misprint of formula (54) in Bretagnon et al. (1997)

Table 1. Parameters of Chebychev polynomials used for the representation of ELP 2000 (Moon) and VSOP87A (Sun and planets).

body	degree	time interval (day)	accuracy (mas)
Moon	15	8	0.015
Sun	13	16	0.001
Mercury	13	8	0.005
Venus	9	16	0.001
Mars	11	32	0.001
Jupiter	7	32	0.002
Saturn	7	32	0.009
Uranus	5	32	0.080
Neptune	5	32	0.300

3.4. Construction of the analytical solution

For the motion of the Sun and the planets we use the solution VSOP87A (Bretagnon, Francou, 1988). For the motion of the Moon we use the solution ELP 2000-82B which involves the theory ELP 2000-82 (Chapront-Touzé, Chapront, 1983) and the arguments of the theory ELP 2000-85 (Chapront-Touzé, Chapront, 1988). We use also the derivatives with respect to the different constants for obtaining the same physical constants and the same tidal model as in DE403/LE403.

3.5. Numerical integration

From Eqs. (9) we run two numerical integrations. The first one uses numerical solutions of the motion of the Moon, the Sun and the planets computed from VSOP87A et ELP 2000; the second one uses DE403/LE403. We put DE403/LE403 in the inertial ecliptic and dynamical equinox by the two following rotations:

a) a rotation $R_3(-0''.05294)$ in the equator plane,

b) a rotation $R_1(23°26'21''.40928)$.

The Euler angles ψ, ω, φ are reckoned positively in positive rotation. The initial conditions of the numerical integrations, computed for t_0 = 2 451 545.0 (J2000) are

$$\begin{aligned}
\psi(t_0) &= 0.000\,067\,895\,460\,8500\,\text{rad},\\
\omega(t_0) &= -0.409\,064\,619\,071\,5125\,\text{rad},\\
\varphi(t_0) &= 4.894\,898\,930\,300\,2346\,\text{rad},\\
\dot\psi(t_0) &= -0.701\,054\,958\,658\,9918\times10^{-6}\,\text{rad/d},\\
\dot\omega(t_0) &= 0.096\,067\,366\,226\,0632\times10^{-6}\,\text{rad/d},\\
\dot\varphi(t_0) &= 6.300\,388\,130\,413\,1300\,\text{rad/d}.
\end{aligned} \qquad (10)$$

4. Results

Our analytical solution is close to the results of Bretagnon et al. (1997). The differences arise from the use of the complete solution ELP 2000 for the motion of the Moon and of a better accuracy in the computation of the right-hand sides of the equations. So, our analytical solution is more accurate than the former one.

Table 2. Differences between the analytical solution and the numerical solution calculated from ELP2000 and VSOP87A over 50 days and 150 years (1900-2050). Unit is μas (microarcsecond).

interval	$\Delta\psi$	$\Delta\omega$	$\Delta\varphi$
50 days	0.023	0.006	0.022
150 years	1.49	0.52	1.38

Table 3. Differences between the analytical solution and the numerical solution calculated from DE403/LE403 over 50 days, 150 years (1900-2050), and 55 years (1968-2023). Unit is μas (microarcsecond).

interval	$\Delta\psi$	$\Delta\omega$	$\Delta\varphi$
50 days	0.030	0.009	0.029
150 years	11.80	1.70	10.80
55 years	2.20	0.65	2.10

4.1. Comparison to numerical integration using analytical theories

To test the precision of our solution we performed numerical integration using VSOP87A and ELP 2000. Instead of using directly the Fourier and Poisson series of the theories, that would be very time-consuming, we have first at all built a representation with Chebychev polynomials. The variables used are the rectangular coordinates of the bodies in the reference frame defined by the dynamical equinox and ecliptic J2000. The parameters of the representation are given in Table 1. In this table, the accuracy column contains the largest discrepancies between the theories and the Chebychev polynomials over the time span 1800-2100.

We have compared our analytical solution with a numerical integration run over 50 days from J2000 to test the accuracy of the diurnal terms (terms of period 24, 12 and 8 hours) and with a numerical integration run over 150 yr (1900-2050). Over 50 days, the differences $\Delta\psi$, $\Delta\omega$, $\Delta\varphi$ are about 0.02 μas. Over 150 yr they are smaller than 1.5 μas. Those results are given in Table 2 and illustrated by the Fig. 1 and the Fig. 2.

Figs. 1 and 2 illustrate the periodic differences. For this purpose they have been corrected by linear terms. So, the Fig. 2 corresponds to

$$\begin{aligned}
&\Delta\psi + 1.19\,\mu\text{as} - 350.52\,\mu\text{as}\times t\\
&\Delta\omega + 0.05\,\mu\text{as} - 18.83\,\mu\text{as}\times t\\
&\Delta\varphi - 1.08\,\mu\text{as} + 2144.32\,\mu\text{as}\times t
\end{aligned}$$

where t is measured in tjy from J2000.

Those corrections give an estimate of the accuracy of the computation of the secular terms. For instance for $\dot\psi$ given by (1), we have $\dot\psi = (-50\,403''.763\,708\,8052 \pm 0''.000\,350\,52)$/tjy. The relative precision is 7×10^{-9} and the digits of $\dot\psi$ are given

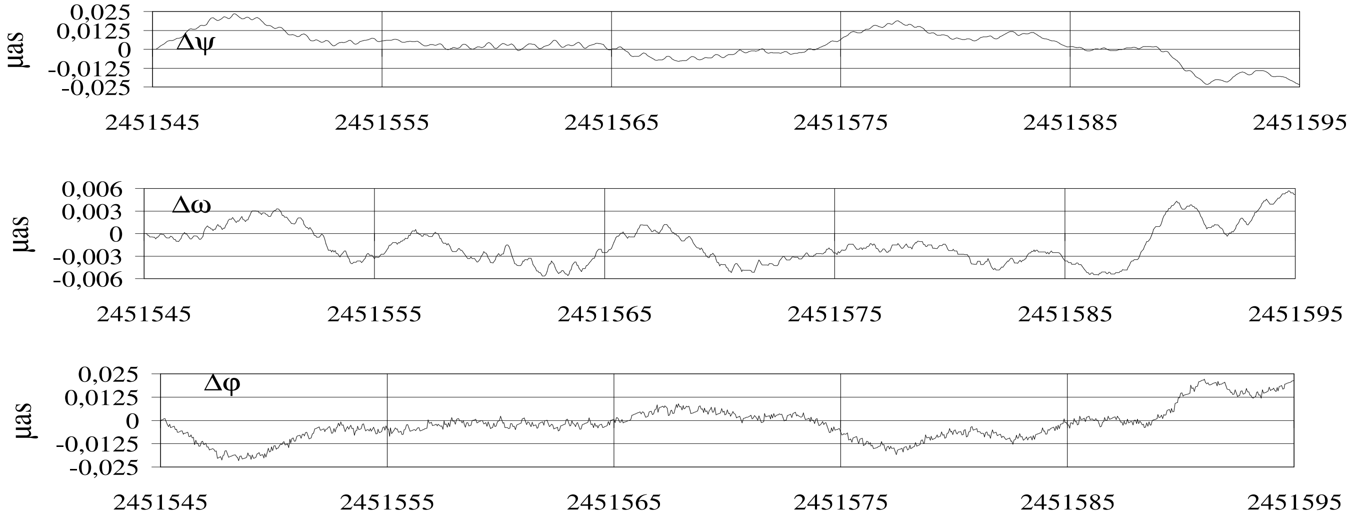

Fig. 1. Theory – numerical integration using ELP2000 and VSOP87A over 50 days

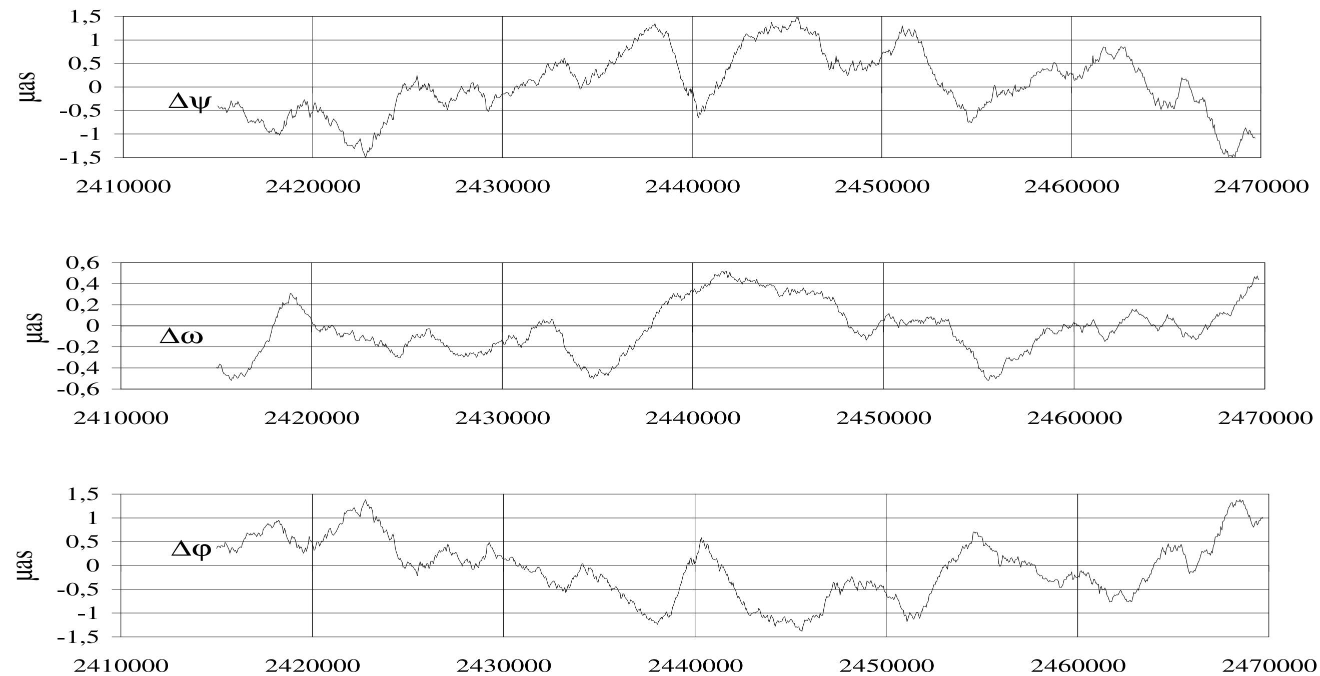

Fig. 2. Theory – numerical integration using ELP2000 and VSOP87A over 1900-2050

only to obtain the value $50''\!.2877/\text{year}$ for the constant of the general precession. The accuracy of H_d given by (3) is $H_d = 0.003\,273\,766\,818 \pm 23 \times 10^{-12}$. The accuracy of the Earth rotation is

$$\begin{aligned}\dot{\varphi} &= (2\,301\,216.753\,651\,536\,317\,\text{rad} \pm 2\,144.32\,\mu\text{as})/\text{tjy} \\ &= (4.746\,600\,278\,245\,062\,081 \times 10^{17} \pm 2\,144.32)\,\mu\text{as}/\text{tjy}\end{aligned}$$

and the relative precision is 4.5×10^{-15}.

4.2. Comparison to numerical integration using DE403/ LE403

The differences $\Delta\psi$, $\Delta\omega$, $\Delta\varphi$ between our analytical solution and a numerical integration computed using DE403/LE403 are greater than the differences given in Table 2. They come from the differences between the models and from the reference frames used in the solutions. We have compared our analytical solution with numerical integrations run over 50 days, 150 yr and also 55 yr (1968-2023) to test the terms with period less than 50 years. The differences are given in Table 3 and they are illustrated by Figs 3, 4 and 5. For the diurnal terms the differences are

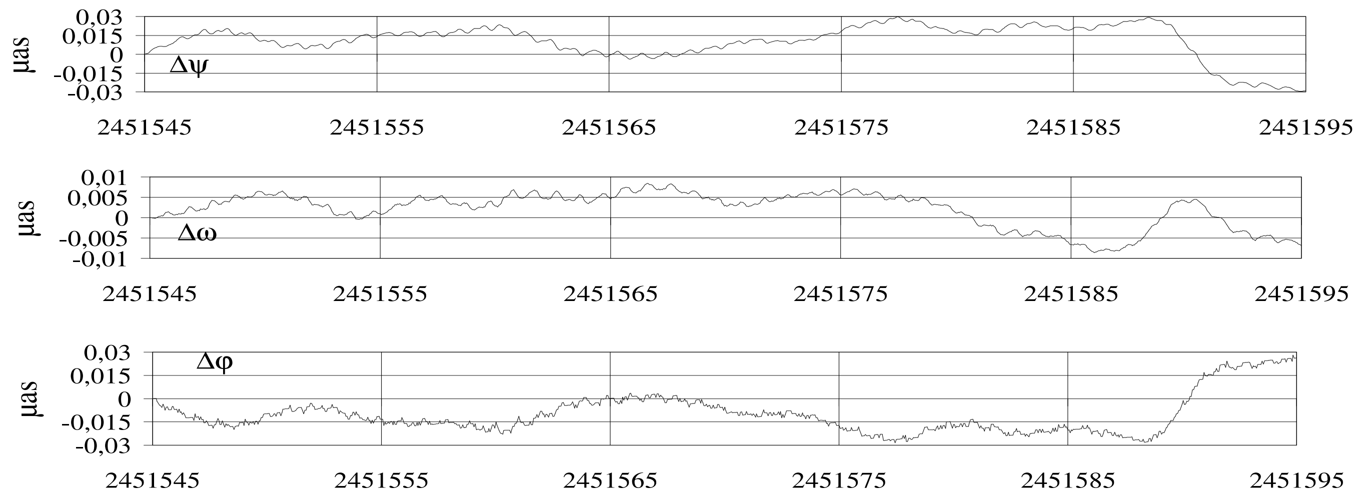

Fig. 3. Theory – numerical integration using DE403/LE403 over 50 days

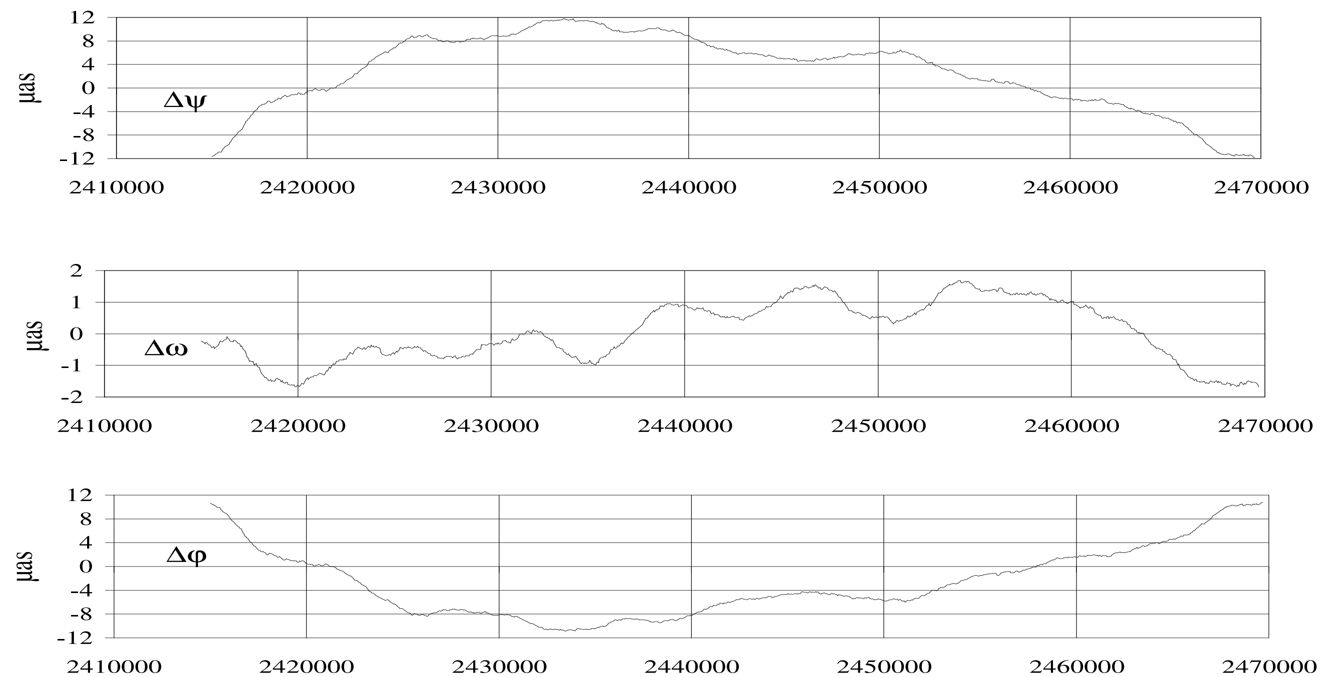

Fig. 4. Theory – numerical integration using DE403/LE403 over 1900-2050

equivalent in Table 3 and in Table 2. Over 150 yr, the differences are about 12 μas for ψ. They correspond to long period and polynomial terms. Over 50 yr the differences are smaller 2.20 μas for ψ, 0.65 μas for ω and 2.10μas for φ.

5. Form of the analytical solution and comparison with other solutions

The analytical solutions of the precession-nutation and rotation of the Earth are, for each variable, in the form of polynomials in time, Fourier series and Poisson series. The arguments of the Fourier and Poisson series are linear combinations of the angles arising in the analytical theories of the motion of the Moon, of the Sun, and of the planets used in this construction.

5.1. Theory of the Moon

The main part of the perturbations of the motion of the Moon is due to the Sun. These perturbations are represented with the 4 Delaunay angles D, F, l, l' and we have the relations

$$D = W_1 - \lambda_3 + 180°$$

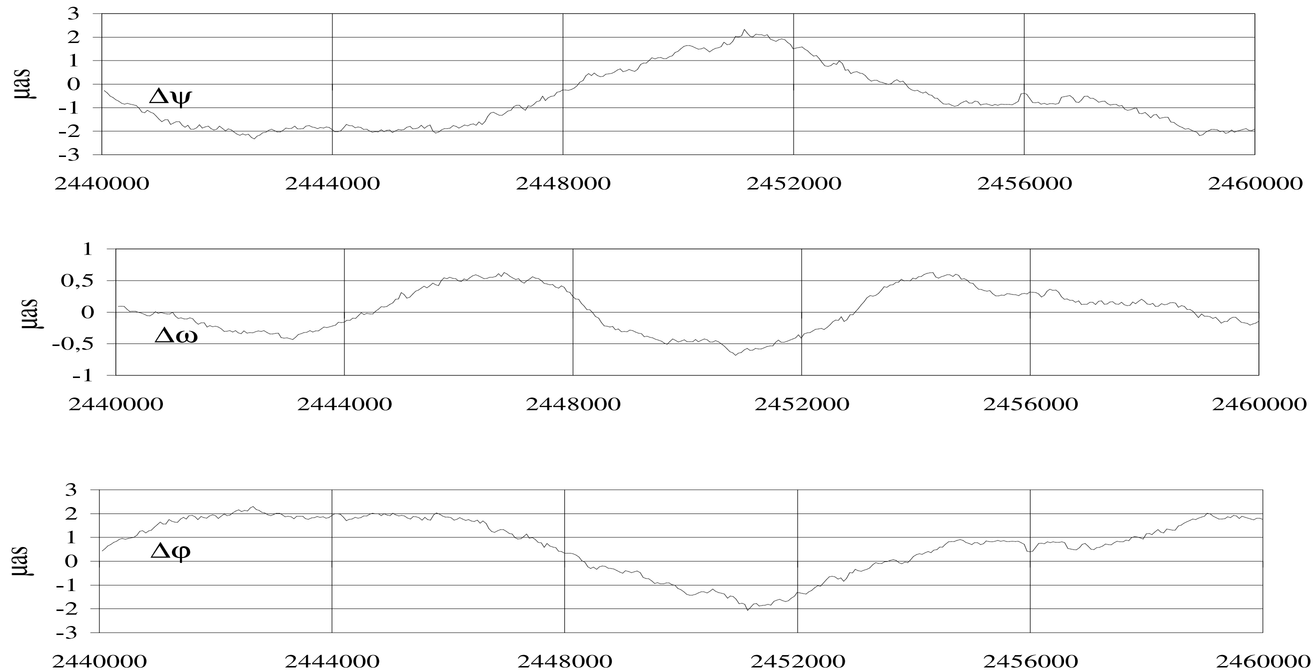

Fig. 5. Theory – numerical integration using DE403/LE403 over 1968-2023

$$F = W_1 - W_3$$
$$l = W_1 - W_2$$
$$l' = \text{mean anomaly of the Sun}$$

in which W_1, W_2, W_3 represent the mean longitudes of the Moon, of the perigee of the Moon, of the node of the Moon, respectively and λ_3 the mean longitude of the Earth.

The planetary perturbations in the solution of the motion of the Moon are represented with the 8 mean planetary longitudes λ_i ($1 \leq i \leq 8$). Lastly, the perturbations of the motion of the Moon by the terrestrial equatorial bulge depend on the angle of precession $\mathscr{P}_A$ the frequency of which is 50″.2877 per year.

Therefore, we have on the whole 13 angles:

$$\lambda_{i,\{i=1,8\}},\ D,\ F,\ l,\ l',\ \mathscr{P}_A.$$

From linear combinations of such angles we find the arguments

$$\lambda_3 - l' + \mathscr{P}_A$$
$$\lambda_3 - l'$$

of which the frequencies are 61″.900 and 11″.612 per year and the periods 20 937 and 111 600 years. There is no meaning to keep neither such arguments nor the angle $\mathscr{P}_A$, in the nutation theory which have to reach a high accuracy for a few ten years. Therefore, in the lunar theory, we have developed as polynomials in time and as Poisson series the arguments $\mathscr{P}_A$, $\lambda_3 - l' + \mathscr{P}_A$ and $\lambda_3 - l'$.

Thus, the solution of the motion of the Moon is expressed as polynomials in time and as Fourier and Poisson series the arguments of which are linear combinations of 11 angles:

$$\lambda_{i,\{i=1,8\}},\ D,\ F,\ l. \tag{11}$$

5.2. Theories of the Sun and the planets

The geocentric solutions of the motion of the Sun and the planets are also expressed as polynomials in time and as Fourier and Poisson series of the 11 angles (11), the angles D, F, l being due to the perturbations of the Earth-Moon barycenter (EMB) and of the planets by the Moon and also to the introduction of the motion of the Moon in the calculation of the vector EMB-Earth.

5.3. Theory of the precession-nutation and rotation of the Earth

The theory of the precession-nutation and rotation of the Earth built with the theories of the motion of the Moon, the Sun, and the planets described in the paragraphs above is also expressed with the 11 angles (11) and the angle of rotation of the Earth φ

$$\varphi = \varphi_0 + \varphi_1 t$$

in which φ_0 and φ_1 are the values given in (5).

With this representation we do not find in the series neither arguments of very long period nor arguments with similar periods as it can be found in the classical nutation tables. For instance, we find in Souchay and Kinoshita (1997) the long period argument

$$-l' + F - D + \Omega \text{ (period 7 646 846.820 d = 20 936 yrs)},$$

and similar period arguments as

argument	period in days
$2\lambda_2 - 3\lambda_3 - 2\mathscr{P}_A$	1455.382
$2\lambda_2 - 3\lambda_3 - \mathscr{P}_A$	1455.160
$2\lambda_2 - 3\lambda_3$	1454.939
$2\lambda_2 - 3\lambda_3 + \mathscr{P}_A$	1454.708
$\lambda_5 - \mathscr{P}_A$	4334.586
λ_5	4332.589
$\lambda_5 + \mathscr{P}_A$	4330.594
$\lambda_5 + 2\mathscr{P}_A$	4328.601

In each group, these terms are in phase every 26 000 years.

In the lunisolar nutation in longitude there also are similar period arguments

argument	period in days	amplitude
Ω	6798.38	$-17''.2806$
$2l' - 2F + 2D - \Omega$	6786.32	$0''.0001$
$2F - 2D + 2\Omega$	182.62	$-1''.2776$
$2l'$	182.63	$0''.0016$
l'	365.26	$0''.1257$
$-l' + 2F - 2D + 2\Omega$	365.22	$0''.0212$

In pairs, these terms are in phase every 20 936 years or 10 468 years.

There is no meaning for keeping terms with periods of 20 936 years or several similar period terms in phase every 26 000 years or every 20 936 years or every 10 468 years because in the analytical solutions of the motion of the Moon, the Sun and the planets, such long period terms are developed as polynomials in time.

Moreover, there is no meaning for representing the solutions with arguments that it is impossible to discriminate over a 20 year time span of the high precision observations.

5.4. Comparison between SMART97 and other solutions

Roosbeek and Dehant (1997) have carried out comparisons between their solution, the solutions of Souchay and Kinoshita (1997), of Hartmann and Soffel (1994) and our solution. These comparisons display differences included between 500 and 1000 μas or more. This comes from errors that we have identified.

For instance, all the solutions except SMART97 have until now determined a contribution out-of-phase of the 18.6 year term with an amplitude of 135 μas. Their result is wrong by 250 μas (Bretagnon et al, 1997). In Kinoshita and Souchay (1990), this contribution out-of-phase was missing and produced an error of 384 μas (Bretagnon, 1996).

Another error common to the most of the other solutions is found in the calculation of the diurnal terms (24 hour, 12 hour, 8 hour period terms) of the precession-nutation of the rigid Earth:
a) the semidiurnal terms are out of phase by 30 degrees and therefore are wrong by 50%,
b) the 24 hour period terms, more important than the semidiurnal terms, are missing,
c) the 8 hour period terms are also missing.

We give in Table 4 a comparison between the Souchay-Kinoshita (1997) solution and SMART97 for the diurnal terms with an amplitude greater than 1 μas in the nutation in longitude. The 8 hour period terms do not appear in this table, the most important one having an amplitude of 0.14 μas.

We have prefered to express the nutations as functions of the same angles (11) that the ones used in the analytical theories of the motion of the Moon, of the Sun and of the planets and we have not performed the transformation

$$\lambda_3 \longrightarrow F - D + \Omega + 180° \qquad (12)$$

where Ω is the longitude of the node of the Moon. For comparison, we give both forms of the arguments in Table 4.

We verify the absence of the out-of-phase parts in the Souchay-Kinoshita solution (SK97) due to the longitude (α = 14°.9285 West) of the major axis of the terrestrial equatorial ellipse. Moreover, it seems that, in SK97, two terms are wrong: the argument noted $2\varphi + l + \Omega$ is probably the argument $2\varphi + l$ and the amplitude of the argument $2\varphi - 3F - 3\Omega$ is 0.31 μas instead of 5.0 μas. For the others semidiurnal terms, the differences of amplitude are less than 2.1 μas. Recently, the referee, H. Kinoshita, informed us that the last results of Souchay and Kinoshita, not yet published, seem to show a better agreement between SK97 and our solution, at one micro arcsecond level.

Lastly, let us note that it is useless to complete the precession-nutation solutions with the semidiurnal terms if one does not take into account the 24 hour period terms which are more numerous and the amplitude of which is more important.

5.5. Discussion about the choice of the angles

It is indifferent to perform or not to perform the transformation of the arguments (12) nevertheless it seems to us more advisable to keep the only angles introduced by the theories of the motion of the Moon, the Sun and the planets.

On the contrary, it is not correct to keep the longitude ϖ' of the perihelion of the Earth $\varpi' = \lambda_3 - l' = F - D + \Omega - l' + 180°$ in a periodic form. In this way the perihelion of the Earth reckoned from the equinox of date seems to have a period of 20 937 years when it includes many terms the most important of which have the following periods : 23 700, 22 400, 19 000, 19 100, 23 100 years. In the same way, it is not correct to keep the precession angle $\mathscr{P}_A$ in the nutation series. These terms with a period greater than 20 000 have to be developed as polynomials in time.

6. Description of the SMART97 solution

In the construction of the solutions, all the angles are reckoned positively in positive rotation and the obliquity is the one of the equator about the ecliptic J2000.

In the final form of the SMART97 solution, we take up the usual conventions: ψ and $\mathscr{P}$ are reckoned in the negative

Table 4. Diurnal terms of the nutation in longitude. Comparison of the SMART97 solution with the Souchay-Kinoshita solution (SK97). Unit is μas (microarcsecond).

SMART97				SK97		
argument	$\Delta\psi$(sin)	$\Delta\psi$(cos)	amplitude	argument	$\Delta\psi$(sin)	$\Delta\psi$(cos)
$\lambda_3 + D + \varphi$	38.13	4.69	38.42			
2φ	31.85	−18.28	36.73	2φ	−37.8	0
$\lambda_3 + D - \varphi$	−34.82	4.27	35.08			
$2\lambda_3 + 2D - 2\varphi$	25.36	14.56	29.24	$-(2\varphi - 2F - 2\Omega)$	−27.1	0
$\lambda_3 + D - l + \varphi$	−23.93	−2.99	24.12			
$\lambda_3 + D - l - \varphi$	−19.85	2.49	20.01			
$2\lambda_3 - 2\varphi$	10.63	6.10	12.26	$-(2\varphi - 2F + 2D - 2\Omega)$	−12.5	0
$\lambda_3 - D + l + \varphi$	7.08	0.88	7.13			
$F + \varphi$	−6.01	−0.74	6.06			
$2\lambda_3 + 2D + l - 2\varphi$	5.15	2.96	5.94	$-(2\varphi - l - 2F - 2\Omega)$	−5.2	0
$\lambda_3 + D + F - 2\varphi$	−4.78	−2.75	5.51	$-(2\varphi - 2F - \Omega)$	5.0	0
$F - \varphi$	5.34	−0.66	5.38			
$\lambda_3 + D - F - 2\varphi$	4.32	2.48	4.98	$-(2\varphi - \Omega)$	−4.9	0
$F - l + \varphi$	4.02	0.50	4.05			
$\lambda_3 + D + l - \varphi$	−3.15	0.39	3.17			
$F - l - \varphi$	3.04	−0.38	3.06			
$\lambda_3 + D + l + \varphi$	2.89	0.36	2.91			
$\lambda_3 + \varphi$	−2.18	−0.27	2.20			
$l - 2\varphi$	−1.88	−1.08	2.17	$-(2\varphi - l)$	2.0	0
$2\lambda_3 + 2D - F + \varphi$	1.98	0.24	2.00			
$l + 2\varphi$	1.69	−0.97	1.95	$2\varphi + l$	0	0
$2\lambda_3 + 2D - F - \varphi$	−1.66	0.20	1.68			
φ	−1.12	1.22	1.65			
$\lambda_3 - D + l - \varphi$	−1.60	0.19	1.61			
$2\lambda_3 + D - l + \varphi$	0.13	1.27	1.27			
$2\lambda_3 + 2D - F - l + \varphi$	−1.18	−0.15	1.19			
$\lambda_3 + D + F + l - 2\varphi$	−0.97	−0.56	1.12	$-(2\varphi - l - 2F - \Omega)$	1.0	0
$2\lambda_3 + 4D - l - 2\varphi$	0.97	0.56	1.12	$-(2\varphi + l - 2F - 2D - 2\Omega)$	−0.7	0
$2\lambda_3 + 2D - F - l - \varphi$	−1.06	0.13	1.07			
$2\lambda_3 + 2D + 2\varphi$	0.88	−0.50	1.01	$2\varphi + 2F + 2\Omega$	−1.1	0
$3\lambda_3 + 3D - 2\varphi$	−0.18	0.26	0.31	$-(2\varphi - 3F - 3\Omega)$	−5.0	0
$\lambda_3 + D - F + l + 2\varphi$	0	0	0	$2\varphi + l + \Omega$	2.0	0

Table 5. Number of terms of the series $\mathscr{P}$ and ε.

	$\mathscr{P}$			ε	
truncation	number of terms	accuracy	truncation	number of terms	accuracy
1.00 μas	751	40.0 μas	0.400 μas	653	16.0 μas
0.10 μas	1921	6.4 μas	0.040 μas	1713	2.5 μas
0.01 μas	4929	1.0 μas	0.004 μas	4382	0.4 μas

direction and ω and ε are the obliquities of the ecliptic J2000 and of the ecliptic of date respectively about the equator.

Moreover, we have added the precession to the mean longitudes of the planets in order to have the same frequencies and the same periods as in the other solutions. Thus, the mean longitudes of the planets are reckoned from the equinox of date. We give hereafter the value of the 12 angles of the SMART97 solution:

$$\begin{aligned}
\lambda_1 &= 4.402\,608\,674\,35 + 26\,088.146\,943\,2237\,t \\
\lambda_2 &= 3.176\,146\,528\,84 + 10\,213.529\,347\,8605\,t \\
\lambda_3 &= 1.753\,470\,291\,48 + 6\,283.319\,652\,7950\,t \\
\lambda_4 &= 6.203\,475\,944\,86 + 3\,340.856\,228\,3493\,t \\
\lambda_5 &= 0.599\,546\,329\,34 + 529.934\,766\,7441\,t \\
\lambda_6 &= 0.874\,016\,588\,45 + 213.542\,897\,0875\,t \\
\lambda_7 &= 5.481\,293\,703\,54 + 75.025\,400\,2168\,t \\
\lambda_8 &= 5.311\,886\,118\,71 + 38.376\,837\,2873\,t \\
D &= 5.198\,466\,400\,63 + 77\,713.771\,448\,1804\,t \\
F &= 1.627\,905\,136\,02 + 84\,334.661\,571\,7837\,t \\
l &= 2.355\,555\,638\,75 + 83\,286.914\,247\,7147\,t \\
\varphi &= 4.894\,961\,212\,82 + 2\,301\,216.753\,651\,5365\,t
\end{aligned} \tag{13}$$

The linear parts of the angles are computed from Simon et al. (1994). The differences result from the modification of the tidal model in the lunar theory and of the new inertial ecliptic and dynamical equinox defined by DE403/LE403.

6.1. List of the variables of the solution

The solution of the Euler equations gives the analytical expressions of the angles ψ, ω, and φ fixing the Earth with respect to the ecliptic and equinox J2000 in the dynamical reference frame. As in Bretagnon et al (1997), from these 3 angles we have determined the analytical solutions of $\mathscr{P}$, ε, and χ, and consequently of the sidereal time ST ($\varphi = ST + \chi$), fixing the Earth with respect to the ecliptic and equinox of date.

6.2. Dynamical and kinematical reference frames

From the 7 quantities ψ, ω, φ, $\mathscr{P}$, ε, χ, and ST defined in the dynamical system, we have computed the corresponding 7 quantities in the kinematical system with the Brumberg's (1996) formulas. These formulas use the 3 rotations of geodesic precession-nutation.

6.3. Axis of figure, axis of rotation, axis of angular momentum

The solution of the equations (6) gives the variations of the axis of figure. In order to compare SMART97 with other solutions we have determined, for the quantities ψ and ω in the dynamical system, the differences between the axis of figure and the axis of rotation and between the axis of figure and the axis of angular momentum.

6.4. Presentation of the files of the SMART97 solution

The SMART97 solution is displayed in the form of 18 files. 14 files correspond to the variables linked up with the axis of figure in the dynamical system and in the kinematical system. There are also 2 files giving the difference between the axis of figure and the axis of rotation and 2 files giving the difference between the axis of figure and the axis of angular momentum for the variables ψ and ω expressed in the dynamical system.

In each file, the solution for a variable x is represented in two forms:

$$x = \sum_{k=0} t^k \sum_{i=1} \left(S_{k,i} \sin\beta_i + C_{k,i} \cos\beta_i\right) \tag{14}$$

and

$$x = \sum_{k=0} t^k \sum_{i=1} a_{k,i} \cos\left(b_{k,i} + c_{k,i}\, t\right) \tag{15}$$

with

$$\beta_i = \sum_{j=1}^{8} r_j\, \lambda_j + r_9\, D + r_{10}\, F + r_{11}\, l + r_{12}\, \varphi \tag{16}$$

The coefficients $S_{k,i}$, $C_{k,i}$, $a_{k,i}$ are in μas/tjy^k; t is counted in tjy from J2000. The angles λ_j, D, F, l, φ are given in (13).

Each record contains:

$$ia, ib, ic, k, i, r_{j,\{j=1,12\}}, S_{k,i}, C_{k,i}, a_{k,i}, b_{k,i}, c_{k,i}$$

with

ia = 1 for the axis of figure
= 2 for (axis of figure − axis of rotation)
= 3 for (axis of figure − axis of angular momentum)

ib = 1 for the dynamical system
= 2 for the kinematical system

ic = 1 for ψ
= 2 for ω
= 3 for φ
= 4 for $\mathscr{P}$
= 5 for ε
= 6 for χ
= 7 for ST (sidereal time).

k is the power of time of the formulas (14) and (15).

i is the number of the term.

The quantities r_j, $S_{k,i}$, $C_{k,i}$, $a_{k,i}$, $b_{k,i}$, $c_{k,i}$ are defined by the formulas (14), (15), (16).

In the files of SMART97, we have only kept the terms with an amplitude greater than 0.01 μas (longitude) and 0.004 μas (obliquity) over the time span 1900-2100. This truncation involves an uncertainty of about 1 μas that we have to add to the residuals of Tables 2 and 3 obtained with the complete series. We give in Table 5 the number of terms and the accuracy for different levels of truncation.

In the form (15) of the SMART97 solutions, the substitution of time in the 4929 terms of $\mathscr{P}$, for example, uses 0.0055 second of computer.

The 18 files of SMART97 solutions are available in electronic form given in:

```
>ftp cdsarc.u-strasbg.fr (or) ftp 130.79.128.5
username : anonymous
password : (type your e-mail address here)
ftp>cd pub/A+A/[ this volume number ]/
    [ this page number ]
or
>ftp bdl.fr (or) ftp 193.48.190.1
username : anonymous
password : (type your e-mail address here)
ftp>cd pub/ephem/ref_frame/smart97
```

7. Conclusion

The objective that we had fixed is reached, that is to say the analytical SMART97 solution describes the rotation of the rigid Earth with an accuracy of about 2 μas. The good agreement between this analytical solution and the numerical integration of the complete equations entirely warrants the used method and the obtained results.

From SMART97, we are going to take into account the geophysical contributions by convolution with the Earth's transfer functions of Wahr (1981) and of Dehant and Defraigne (1997). It will be also interesting to build an analytical solution of the non-rigid Earth rotation by using the geophysical models in form of analytical expressions.

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