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Ephemerides of the five major Uranian satellites by numerical integration

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Abstract. A catalogue is compiled of all available published and unpublished Earth-based observations of the satellites of Uranus. References are given to each of the published datasets. A brief survey of the observations is made concentrating on the different types of data and their quality. A numerical integration is fitted to observations in the time interval April 1977 to October 1995, which includes some of the best quality photographic data, the Voyager astrographic data, CCD data and meridian circle observations. The mass values found are expressed as the product of the universal gravitational constant times the mass of the body, in units of $km^3 sec^{-2}$. Values determined for the mass of Uranus of 5794054 ± 163, the dynamical flattening J_2 of Uranus 0.003365 ± 0.000013 and for the mass of Miranda of 4.09 ± 0.18 agree well with previously published values. The standard errors are formal errors from the least squares. The values obtained for the masses of the outer four satellites are found, from comparison with further solutions made, to be sensitive to the choice of weights used for the spacecraft data. These solutions are discussed in the light of the dynamics of the satellite system.

Key words: planets and satellites: Uranus – satellites of Uranus – ephemerides – astrometry

1. Introduction

The Voyager 2 fly-by of Uranus was in January 1986. With the approach of this event great impetus was given to both acquiring new series of high quality astrometric observations of the satellites and into developing precise ephemerides for them. A large number of accurate positions were obtained including those by Walker et al. (1978), Veillet (1983a,1985), Harrington and Walker (1984), Pascu et al. (1987) and Walker and Harrington (1988). Each of these series of observations covered several oppositions and were essential in the development of the ephemerides. Analytical and numerical techniques were used in generating the orbits. A general analytical theory for the Uranian satellites was developed by Laskar (1986) (GUST86) and a numerical integration generated by R. A. Jacobson at the Jet Propulsion Laboratory (JPL), specifically for the purpose of aiding the navigation of Voyager during the encounter with Uranus. During the successful encounter, radio and optical data of the satellites were acquired. The optical data were subsequently converted (Jacobson 1992a) into a more convenient, but equivalent, astrographic form. Laskar and Jacobson (1987) fitted GUST86 to Earth-based observations from 1911 to 1986 and the spacecraft data. Jacobson (1986) fitted the JPL integration to spacecraft data and to Earth-based observations spanning the period 1973 to 1985. An important result from each of these analyses of data was the determination of the masses of the five satellites. In fact, prior to the Voyager mission the satellite masses were not known with any great certainty. It is only with the spacecraft data and the precise orbital models which were derived in order to analyse them which changed this.

Since 1986 several new series of observations of the Uranian satellites have been published. The two largest, and which also happen to be the most accurate of these datasets, are the series made using the 1.6m reflector at the Laboratório Nacional de Astrofisica at Itajubá, Brazil (Veiga et al. 1987, Veiga and Vieira Martins 1995) and the observations made using the 1.0m Jacobus Kapteyn Telescope (JKT) at the Roque de los Muchachos Observatory on La Palma, Canary Islands (Jones et al. 1997). The Brazilian series of observations is part of a systematic program of astrometric observations of natural satellites of the outer planets initiated in 1982. The Uranian satellite data cover each opposition from 1982 to 1994 with the exception of the 3 oppositions from 1986 to 1988 and that in 1990. The data from 1989 and after are all made using a CCD. A further paper in this series Veiga and Vieira Martins (1994) was discovered after much work had been done and these observations were not included in the present analysis of observations. They include re-measurements of some of the data given in Viega et al. (1987) but now using photometric methods plus additional photographic observations at the 1987 and 1988 oppositions. The La Palma series of observations, all made using CCD detectors, begin at the 1987 opposition and are also part of an observing programme of the major satellites of the outer planets. The largest number of observations were made at the 1990 and 1991 oppositions.

A semi-analytical theory was developed by Lazzaro (1991). Although compared with recent ephemerides no fit was made of the theory to observations. Other analytical studies by Malhotra et al. (1989) and Christou and Murray (1997) concentrated on theoretical aspects of the dynamics of the Uranian satellite system. Jacobson et al. (1992b) extended backwards in time the numerical integration used in the Voyager encounter with Uranus. The integration was now fitted to data in the period 1960 April to 1985 June. A new determination was made of the satellite masses. The standard errors of the masses quoted were an assessment of the true errors and were based on an examination of previously determined values and on results of a number of sensitivity tests.

In this paper we describe a numerical integration fitted to observations made since 1986 combined with the Voyager observations and some of the best quality pre-Voyager data. We discuss the solutions found, in particular the values for the satellite masses, with previous results from other researchers.

This paper begins by briefly describing the compilation of a catalogue of Earth-based observations of the satellites of Uranus. The catalogue covers the complete time-span of observations from 1787, with the discovery observations of Titania and Oberon, up to the meridian circle observations of Oberon in 1995. A table is given containing the references to observations on which the catalogue is based (note data in Jacobson (1992a) is not entered in the catalogue). A general description of the observations is given, highlighting the gradual improvement in the quality and the orbital models needed to analyse them.

Next we give the equations to be numerically integrated. From the equations of motion the variational equations are derived. The partial derivatives from integration of these equations enable us to fit the integration of the equations of motion to the observations.

In the following section we describe the fit of simple precessing ellipses for each satellite to the catalogue of observations. Although this representation of the orbits is inadequate due to the accuracy of the recent observations they do enable a check to be made of the quality of observations and that they have been entered in the catalogue correctly. The precessing ellipses also provide a means to compute approximate starting conditions to the integration. The numerical integration is then described and the results from the integrations discussed. In determining initial conditions for the fit of the equations of motion to the observations it was found necessary initially to iterate on orbital elements at epoch. The derivation is given of the partial derivatives with respect to orbital elements at epoch. The final section summarises the results obtained and the methods and techniques used in determining ephemerides by numerical integration. Future numerical investigations into the orbits of the Uranian satellites are discussed.

2. The observations

2.1. General remarks

A catalogue has been compiled of Earth-based observations of the Uranian satellites. References to all the data included in the catalogue are given in Table 1. In addition, Table 1 includes references to Veiga and Vieira Martins (1994) which were discovered too late to be included in the catalogue and to Jacobson (1992a) which contains the Voyager observations. The references were gathered from those given in the Ph.D thesis of C. Veillet (1983a), the extensive bibliography and literature search made at JPL (see Jacobson 1985) and the author's own search of the literature. It should be noted some of the observations published were re-reduced several years later and these again were published e.g. this is the case of much of the micrometer observations obtained using the 26-inch refractor at the U. S. Naval Observatory (USNO), Washington. Both published observations have been included in Table 1.

The references marked with an asterisk in Table 1 indicate observations which have not been published and were either received by the author as a personal communication or were used in a Ph.D. dissertation and were left unpublished. The observations in Standish (1996) and Veillet (1985) were received from R. A. Jacobson being originally communications to him.

The Ph.D. thesis of Harris (1949) contains some early unpublished photographic observations. Harris measured and reduced plates taken at Lick Observatory in 1905 and 1914 using the Crossley reflector and at Lowell Observatory in 1914 to 1916 using the 42-inch reflector.

Jacobson (1996) and Standish (1996) are observations made under contract for the Jet Propulsion Laboratory (JPL) in support of the Voyager mission. The data in Jacobson (1996) represents a corrected form of the observations originally sent to JPL by Mulholland (1985). These were made using the 2.1m and 76cm reflectors at McDonald Observatory and cover the oppositions in 1974 to 1982, but excluding that in 1981. Standish (1996) observations are a reduction of raw measures of planet, satellites and stars received from P. A. Ianna which were made with the 66cm refractor at Mt. Stromlo. These observations cover the oppositions 1983-1986.

An extensive series of unpublished observations are given in the Ph.D. thesis of C. Veillet (Veillet 1983a). These were taken principally by C. Veillet with the remainder by G. Ratier using instruments at Pic du Midi Observatory, Observatory of Haute-Provence, the European Southern Observatory (ESO) and the Canada-France-Hawaii Tel. Corp. on Hawaii. The series covers the oppositions from 1977 to 1982. Further unpublished observations made at ESO in 1984 are given in Veillet (1985).

The observations communicated in Shen (1992) were made in 1990 using the 1m reflector of Yunnan Observatory.

The observations in the published papers in Table 1 are available electronically on request to the author.

2.2. Catalogue of observations

The observations have been entered in the catalogue in a consistent format following closely that used by Strugnell and Taylor (1990) in their compilation of a catalogue of observations of Saturn's satellites. Two additional flags have been added to this format to indicate if the E-terms of aberration are present in the observations and if a refraction correction has been made to the

Table 1. References containing observations of the satellites of Uranus. The papers are ordered in terms of the earliest date of observation given.

Herschel (1815)	Schaeberle (1897)	Struve (1928)
Herschel (1833)	Hussey (1902)	Steavenson (1948)
Lamont (1837)	Aitken (1898)	Whittaker and Greenberg (1973)
von Asten (1872)	Aitken (1899)	van Biesbroeck (1970)
Lassell (1847)	USNO (1911)	Steavenson (1964)
Lassell (1852)	See (1900)	Tomita and Soma (1979)
Lassell (1851)	Aitken (1901)	van Biesbroeck et al. (1976)
Struve (1847)	See (1902)	Soulié (1968)
Lassell (1848)	See (1907)	Soulié (1972)
Lassell (1849)	Dinwiddie (1903)	Soulié (1975)
Lassell (1853)	Aitken (1904)	Soulié (1978)
Lassell (1857)	Frederick and Hammond (1908)	Mulholland et al. (1979)
Lassell (1864)	Aitken (1905)	Jacobson (1996)[1]*
Lassell (1865)	Harris (1949)*	Walker et al. (1978)
Vogel (1872)	Wirtz (1912)	Veillet and Ratier (1980)
Rosse (1875)	Aitken (1909)	Veillet (1983a)*
Newcomb (1875)	Frederickson (1909)	Harrington and Walker (1984)
Davis (1875)	Barnard (1909)	Veillet (1983b)
Holden (1875)	Hall (1911)	Debehogne et al. (1981)
Holden (1876)	USNO (1929)	Pascu et al. (1987)
Hall (1876)	Barnard (1912)	Veiga et al. (1987)
Hall (1877)	Aitken (1912)	Veiga and Vieira Martins (1994)
Holden (1877)	Eppes (1912)	Standish (1996)[2]*
Hall (1878)	Barnard (1915)	Veillet (1985)*
Holden (1878)	Rosanof (1925)	Walker and Harrington (1988)
Holden (1879)	Aitken (1914)	Jacobson (1992a)
USNO (1880-1884)	Nicholson (1915)	Veiga and Vieira Martins (1995)
Hough and Burnham (1880)	Barnard (1916)	Shen (1992)*
Henry and Henry (1884)	Barnard (1919)	Abrahamian et al. (1993)
Perrotin (1887)	Barnard (1927)	Jones et al. (1997)
USNO (1892)	Hall (1921)	Carlsberg Meridian Catalogues(1993-1997)
Schaeberle (1895)	Hall and Bower (1923)	
Barnard (1896)	Sytinskaja (1930)	

* Observations are unpublished
[1] JPL contract observations communicated by J. D. Mulholland
[2] JPL contract observations communicated by P. A. Ianna

observations. The observations have been entered at the full precision of the computer (here 14 figures) for purposes explained later.

Several problems arise in storing diverse observations in a uniform format. Many of these originate from converting the observation time to a UTC time. For observations prior to 1925, astronomical time was in use (and for some data just after e.g. Struve 1928), and care must be taken in deciding the correct day of observation when converting observed sidereal times to universal time. A few observations (Hall 1911; Eppes 1912) give times which have been corrected for the light-time. These observations, although made at Washington, are given in Paris Mean Time and residuals were computed using theories from the Connaissance des Temps. In converting these times to universal time it was assumed the light-times used were those tabulated in the Connaissance des Temps for the appropriate year. Many of the early observations (before 1929) were published in local mean or sidereal times. Most of the longitudes of the observatories for which a conversion from local time is required, whenever possible, have been obtained from the Nautical Almanac (NA) for 1929. For the Lick times both Pacific Standard Time (PST) and Mount Hamilton Mean and Sidereal Times, MHMT and MHST respectively, were used. PST is a zone time and the observation times will be 8 hours slow of Greenwich Mean Time (GMT). The others are local times and the longitude from the Astronomical Almanac for 1984 for the 36-inch refractor is used. MHMT is $8^h\ 6^m\ 34^s$ slow of GMT. Finally, some observatories were relocated like the USNO in 1893 and the Dearborn Observatory in 1887. Thus, for observations made at these observatories in Washington Mean Time (1893 and earlier) and Chicago Mean Time (Hough and Burnham 1880) respectively, the longitudes were obtained from earlier almanacs (NA's of 1891 and 1881 were used).

In order to facilitate the analysis of observations a new catalogue is derived from the first which we will call the differenced catalogue. In this new catalogue all photographic observations consisting of RA and Dec's are differenced to create inter-satellite observations of the form $\Delta\alpha$, $\Delta\delta$ (i.e. type 2 data as designated by Strugnell and Taylor 1990). The photographic data in Abrahamian et al. (1993), Walker and Harrington (1988), Harrington and Walker (1984), Walker et al. (1978), Rosanof (1925) and Nicholson (1915) consisting of observations of a satellite with respect to the planet are also differenced to create inter-satellite ones and it is these which are included in the new catalogue. Thus for these datasets, if for a given date only a single satellite was observed this observation would be omitted. The reason for differencing the photographic observations is to remove any systematic errors resulting from the star positions used. The observations of a satellite relative to the planet are differenced as often accurate measurement of the planet is difficult. Depending on which satellites were observed the choice for reference satellite was in the order Oberon, Titania, Umbriel and Ariel. This order is chosen based on clarity of satellite images and ease of measurement on a photographic plate or CCD frame. As an example, suppose all 5 satellites are observed then 4 inter-satellite observations will be entered, each with Oberon as the reference satellite. The data in the original catalogue were entered at the full precision of the computer to avoid any round-off error in the differenced catalogue. All other observations in the original catalogue are retained unchanged in the new catalogue.

Residuals were computed from fitting simple precessing ellipses (see Sect. 4) to the differenced catalogue. This representation of the orbits is perfectly adequate for checking that the data has been entered correctly in the catalogue and to compute an approximate rms (root-mean-square) for each dataset to assess its quality. Examination of the residuals enabled any errors in the published observations to be found such as wrongly identified satellites or components of an observation given the wrong way round. In computing residuals, quantities such as planet coordinates, precession matrices and light-times are required for each observation time. Computation of these quantities was greatly simplified by processing sequentially dates in the differenced catalogue.

2.3. Overview of the data

A detailed description of the various series of observations is not given here. Instead, an overview of the data is presented concentrating on the type of observations made, their quality and quantity.

The first satellites discovered were Titania and Oberon by William Herschel at Slough, England in 1787 January using his 19-inch reflector. These faint objects ($\approx$ 14mag.) were extremely difficult to observe and the resulting series of visual estimations and micrometer measurements have an accuracy of a few arcsecs. Only a few other observations were made (Herschel 1833; Lamont 1837) until the discovery of the next two satellites.

To develop planetary ephemerides the masses of the planets must be known well. After the discovery of Neptune in 1846 following irregularities in the motion of Uranus, a good value for the mass of Uranus was needed for development of an ephemeris for Neptune. The mass of a planet can be derived from the period and semi-major axis of a satellite using Kepler's third law. New series of observations were made to determine precisely the satellite orbits.

In 1847 at Liverpool William Lassell visually discovered Ariel and Umbriel with his 24-inch reflector and estimated their positions. The discovery observations were followed by a series of observations of the four satellites made both from Liverpool and Malta. The micrometer measurements from Malta were made initially using his 24-inch reflector (Lassell 1853) and then for the later observations a 48-inch reflector (Lassell 1865). The accuracy of the best data from Liverpool (Lassell 1857) is about $1''$and from Malta (Lassell 1865) is about $0\farcs5$.

The major observatories in the U.S.A. now started to regularly observe the satellite system. Long series of observations were made using the 26-inch equatorial refractor at the USNO, the 36-inch refractor at Lick Observatory and the 40-inch refractor at Yerkes Observatory. The measures were always of the satellite referred to the centre of Uranus up until the 1901 USNO observations when in addition measures of one satellite relative to another (inter-satellite observations) were first given. Good quality micrometer measurements were continued to be made well into the twentieth century. The best of the micrometer data has an accuracy of $0\farcs3$ (Newcomb 1875; Struve 1928). In the analysis of the 1874-75 Washington observations in Newcomb (1875), the orbits were represented by precessing ellipses, and it was remarked by Newcomb "...there is but slight evidence of any real eccentricity of the orbits, and no evidence of any mutual inclination".

The earliest photographic observations found were those made by S. Nicholson at Lick Observatory in 1905. These plates were measured and positions determined by Harris (1949). The early series of photographic observations made at Lick Observatory (Nicholson 1915; Harris 1949), Tashkent Observatory (Rosanof 1925; Sytinskaja 1930) and Lowell Observatory (Harris 1949) were few in number. The observations in Rosanof (1925) were re-reduced by Harris (1949) using improved star positions. The accuracy of the best quality of these observations is about $0\farcs2$ (e.g. Lowell data).

The innermost of Uranus's five major moons Miranda was discovered by Gerald Kuiper using the 82-inch reflector of McDonald Observatory in 1948. This prompted several new series of observations to be made initially to determine orbital elements for Miranda but later to investigate the motions of all the satellites. An extensive series of plates were taken at McDonald covering several oppositions in the period 1948 to 1964 with a further series of plates taken in 1966 at the Lunar and Planetary Laboratory Arizona and Kitt Peak National Observatory (van Biesbroeck 1970, van Biesbroeck et al. 1976). Other long series of photographic observations include those made using the 155cm astrometric reflector at the Flagstaff Station of the U. S. Naval Observatory (Walker et al. 1978, Harrington and Walker 1984 and Walker and Harrington 1988) and those made by C. Veillet and G. Ratier (Veillet 1983a, 1985). For the best quality photographic observations and which include measures of all five satellites, an accuracy of about $0\farcs05$ is obtained e.g. observations in Veillet (1983a) made using the 1.5m Danish reflector at ESO.

New orbit determinations by Harris (1949), Dunham (1971) and Veillet (1983a) incorporated photographic observations in the data analysis. The latter study determined values for the eccentricity and inclination of the satellites and included in the precessing ellipse models, Solar perturbations for Titania and Oberon and periodic perturbations in the longitudes of Miranda, Ariel and Umbriel arising from the Laplacian resonance.

In Pascu et al (1987) observations of Miranda are given which were obtained using a CCD camera and cover the 5 oppositions leading up to the Voyager encounter with Uranus. From the Voyager fly-by of Uranus high quality spacecraft based optical measurements (Jacobson 1992a) were obtained as well as the Earth-based radio-tracking data of the spacecraft. Since Voyager, photographic observations continue to be made, but increasingly data have been obtained using CCD detectors (Veiga and Vieira Martins 1995 and Jones et al. 1997). The accuracy of these datasets being $0.''04$ and $0.''05$ respectively. Observations of Oberon at each opposition since 1992 have been made on the Carlsberg Automatic Meridian circle (CAMC), La Palma. These observations are principally used for making corrections to Uranus's position and so enable different ephemerides to be compared.

The increased accuracy of the data makes analytical theories based on simple precessing ellipses no longer adequate. In order to properly analyse the spacecraft, CCD and photographic data a more complete analytical theory is required. Laskar (1986) developed an analytical theory based on a secular perturbation theory with inclusion of some short-period terms. This was fitted to the data by Laskar and Jacobson (1987). An alternative to the development of an analytical theory is to generate the orbits by numerical integration and fit these to the data. Satellite orbits have been generated in this way by Jacobson (1986).

3. The equations of motion and variational equations

The equatorial plane of Uranus is chosen as the reference plane for the numerical integration. This is the natural reference plane for the system, since the orbits have small inclinations to the equatorial plane, the largest being $\approx 4.^\circ 3$ for Miranda. The calculation of the J_2 and J_4 perturbations are also simplified using this reference plane. It is assumed that the theoretical rate of precession of this reference plane is very small, and the Coriolis terms in the equations of motion due to using this non-inertial frame are negligible, certainly over the time-span of a few decades. The equations were formulated in Cartesian coordinates with the x-axis taken to be in the direction of the ascending node of the equator of Uranus on the Earth equator of B1950.0 (JED 2433282.423), the y-axis in the equatorial plane of Uranus, and the origin at the centre of mass of Uranus.

The equations of motion and variational equations are given for 5 mutually perturbing satellites, orbiting an oblate primary and perturbed by the Sun and 2 planets, Jupiter and Saturn. It should be noted these equations can easily be adapted for other satellite systems with different numbers of mutually perturbing satellites and perturbing planets. The equations of motion for the satellites are, for $i = 1, 2, \ldots 5$;

$$\begin{aligned}
\ddot{\mathbf{r}}_i \;=\; & \frac{-GM(1+m_i)\mathbf{r}_i}{r_i^3} + \sum_{\substack{j=1\\ \neq i}}^{5} GMm_j\left(\frac{\mathbf{r}_j-\mathbf{r}_i}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_j^3}\right) \\
& + GM_s\left(\frac{\mathbf{r}_s-\mathbf{r}_i}{r_{is}^3} - \frac{\mathbf{r}_s}{r_s^3}\right) + \sum_{p=1}^{2} GM_p\left(\frac{\mathbf{R}_p-\mathbf{r}_i}{R_{ip}^3} - \frac{\mathbf{R}_p}{R_p^3}\right) \\
& + A_i\mathbf{r}_i + B_i\hat{\mathbf{k}} + \sum_{\substack{l=1\\ \neq i}}^{5}(m_lA_l\mathbf{r}_l + m_lB_l\hat{\mathbf{k}}) \qquad (1)
\end{aligned}$$

where

$$\begin{aligned}
A_i &= \frac{GM}{r_i^3}\sum_{n=2}^{4} J_n \frac{R_u^n}{r_i^n} P'_{n+1}\left(\frac{z_i}{r_i}\right) \\
B_i &= \frac{-GM}{r_i^2}\sum_{n=2}^{4} J_n \frac{R_u^n}{r_i^n} P'_{n}\left(\frac{z_i}{r_i}\right),
\end{aligned} \qquad (2)$$

with

$$P'_{n+1} = (n+1)P_n + P'_n . z_i/r_i \,. \qquad (3)$$

The subscripts $1, 2 \ldots, 5$ have the usual convention of referring to satellites in order of increasing semi-major axis (i.e. 1=Miranda, 2=Ariel, 3=Umbriel, 4=Titania, 5=Oberon). The subscripts $p = 1, 2$ refer to planets Jupiter and Saturn respectively. We have

$\mathbf{r}_i$ position vector of satellite i
$\mathbf{r}_s$ position vector of the Sun relative to the primary
$\mathbf{R}_p$ position vector of the p^{th} planet relative to the primary
$\hat{\mathbf{k}}$ unit vector in the z direction
r_i distance of satellite i from the centre of the primary
z_i the third component of the coordinates for the i^{th} satellite
r_s distance of Sun from the centre of the primary
R_p distance of p^{th} planet from the centre of the primary
r_{ij} distance between satellites i and j
r_{is} distance between satellite i and the Sun
R_{ip} distance between satellite i and the p^{th} planet
G gravitational constant
m_i mass of satellite i divided by the mass of the primary
M mass of Uranus
M_s mass of the Sun
M_p mass of the p^{th} planet
R_u equatorial radius of the primary (26 200 km was used)
J_n oblateness parameters
P'_n derivative of the Legendre polynomial P_n.

The masses of the satellites are taken in units of the mass of the primary because it is expected that there will be lower correlation with the mass of the primary when taken in this form than if taken in absolute units.

The Solar and planetary perturbations were computed from the JPL DE200 ephemeris of Uranus,Jupiter and Saturn, referred to the Earth equator and equinox of B1950.0. The Solar and planetary coordinates were each represented in 400 day intervals by a Chebyshev series. Let (X_s, Y_s, Z_s) be the coordinates of the Sun relative to the primary. They are transformed to the integration reference plane by

$$\begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha\sin\delta & -\sin\alpha\sin\delta & \cos\delta \\ \cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta \end{pmatrix} \begin{pmatrix} X_s \\ Y_s \\ Z_s \end{pmatrix} \tag{4}$$

where α, δ are the B1950.0 right ascension and declination of the pole of the primary. The planetary perturbations in the integration reference plane are computed in a similar way; starting with planetary positions relative to the primary and applying the rotation in Eq. (4).

The oblateness accelerations have been modelled using formulae given in Merson and Odell (1975). The perturbation

$$\sum_{\substack{l=1 \\ \neq i}}^{5} (m_l A_l \mathbf{r}_l + m_l B_l \hat{\mathbf{k}})$$

arises from the component of the attraction of each satellite on the primary caused by the oblateness of the primary.

Denote the components of $\mathbf{r}_i$ by

$$\mathbf{r}_i = (c_{3i-2}, c_{3i-1}, c_{3i}) \ (i = 1, 2, \ldots, 5).$$

To fit the numerical integration to the observations we need the partial derivatives of each coordinate c_1, c_2, …, c_{15} with respect to each solved-for parameter. These parameters will be the initial coordinates of the satellites, the initial velocities, J_s $(s = 2, 3, 4)$, the masses of the satellites and the mass of the primary; a total of 39 parameters which we denote by $q_1, q_2, \ldots, q_{39}$. In the fit to observations other solved-for parameters are included such as corrections to the position of the primary and to its pole vector, but these partials are computed at that stage (see sect. 6).

Denoting the differential equation for c_i by

$$\ddot{c}_i = F_i$$

the differential equations for the partial derivatives, the variational equations are

$$\frac{d^2}{dt^2}\left(\frac{\partial c_i}{\partial q_k}\right) = \left(\frac{\partial F_i}{\partial q_k}\right)_{\text{explicit}} + \sum_{j=1}^{15} \frac{\partial F_i}{\partial c_j}\frac{\partial c_j}{\partial q_k}. \tag{5}$$

The explicit partial derivatives of F_i with respect to q_k are zero for $k = 1, 2, \ldots, 30$, since the F_i do not depend on the initial position and velocity of the satellites. A fuller account of the variational equations, including explicit formulae useful for computational purposes, is given in Taylor (1998).

4. Approximate orbits

4.1. Theories

Orbital models based on a precessing ellipse were fitted to the differenced catalogue of observations and Voyager data. As already mentioned, although these orbits are inadequate to analyse the modern data, they are still useful as a quick and easy means to compute residuals and test the quality of a series of observations. They also provide a way to generate approximate starting conditions for the numerical integration.

Ariel, Umbriel, Titania and Oberon have small eccentricities and inclinations to the equatorial plane. Miranda also has a small eccentricity but an inclination of $\approx 4^{\circ}\!.3$ to the equator. For small eccentricities and inclinations the pericentres and nodes are difficult to determine. For Ariel, Umbriel, Titania and Oberon it is better to use in place of e, the eccentricity and ϖ, the longitude of pericentre, $h = e\sin\varpi$, $k = e\cos\varpi$, and in place of i, the inclination and Ω, the longitude of the node $p = \sin i \sin\Omega$ and $q = \sin i\cos\Omega$. For Miranda we use h and k but retain the elements i and Ω.

The general form for the theories of Ariel, Umbriel, Titania and Oberon is

$$\begin{aligned} a &= a_0 \\ \lambda &= \lambda_0 + nt \\ h &= h_0\cos\beta t + k_0\sin\beta t \\ k &= k_0\cos\beta t - h_0\sin\beta t \\ p &= p_0\cos\gamma t + q_0\sin\gamma t \\ q &= q_0\cos\gamma t - p_0\sin\gamma t \end{aligned} \tag{6}$$

where a is the semi-major axis, λ is the mean longitude, n is the mean motion and β, γ are the rates of change of the apse and node respectively.a_0, λ_0,h_0,k_0,p_0 and q_0 are constants determined from the fit to observations. For Miranda the same orbital model is used but p, q are replaced by $i = i_0, \Omega = \Omega_0 + \dot{\Omega}t$, where i_0, Ω_0 are obtained from the fit to data. The longitudes are measured on the Uranus mean equator of 1950.0 from the intersection of this plane with the Earth mean equator of 1950.0.

Let the subscripts i = 1,2,3 refer to satellites Miranda, Ariel and Umbriel respectively. To their mean longitudes we add the periodic terms

$$\Delta L_i = A_i(c_i' \sin(\theta_0 + \dot{\theta}t) + c_i'' \sin 2(\theta_0 + \dot{\theta}t)) \,. \tag{7}$$

These perturbations arise from the Laplacian resonance $n_1 - 3n_2 + 2n_3 = -0.0785^{\circ}/d$ and is the largest perturbation in the satellite system (Laskar and Jacobson 1987). The coefficients c_i',c_i'' are taken from Lazzaro et al. (1984) and the amplitudes A_i and phase and frequency of the resonance obtained from the fit to data.

The satellite vectors are calculated referred to axes O_{XYZ} with the origin the centre of Uranus, O_X the intersection of the mean equators of Uranus and the Earth for 1950.0 and O_Y in the plane of the equator of Uranus. The expressions include the quantities $r\cos f$ and $r\sin f$, where r is the radius vector and

f the true anomaly. $r \cos f$ and $r \sin f$ are expanded in terms of elliptic elements. For Ariel, Umbriel, Titania and Oberon we ignore powers of e and $\sin i$ greater than the second in the satellite vectors. For Miranda we expand the satellite vector in powers of e only, again neglecting powers of e greater than the second. Expressions for the satellite vectors are given in Taylor (1998).

Let (ξ, η, ζ) be the components of (X, Y, Z) in a system in which the xy plane lies in the plane of the Earth's equator of 1950.0, and the x-axis points towards the equinox of 1950.0. So we have

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos N & -\sin N \cos J & \sin N \sin J \\ \sin N & \cos N \cos J & -\cos N \sin J \\ 0 & \sin J & \cos J \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{8}$$

where J and N are the inclination and longitude of the node of the Uranus equator of 1950.0 on the Earth equator of 1950.0. The values $J = 74\overset{\circ}{.}96$ and $N = 166\overset{\circ}{.}72$ were used, taken from Dunham (1971). After much work using these values had been done, a more up to date determination of these values in French et al. (1988) was found and this was used in the fit of the numerical integration to observations. For the present purposes it is not important that the more recent values were not used.

4.2. The fit of theories to the observations

The observed quantities are right ascension (RA) α and declination (Dec) δ, differential coordinates $x = \Delta\alpha \cos \delta$, $y = \Delta\delta$ or position angle p and separation s. The RA and Dec observations are either Earth-based (geocentric or topocentric) or Voyager-based. The latter are computed as described in Jacobson (1992a). The differential coordinates are differences of RA's and Dec's of either two satellites (inter-satellite observations) or a satellite and the planet. The $\Delta\alpha$,$\Delta\delta$ observations in the catalogue are analysed as x,y observations, by multiplying $\Delta\alpha$ by $\cos \delta_p$, where δ_p is the declination of Uranus. In fact, due to the precision of the x,y observations, all x components were regarded as factored by $\cos \delta_p$. The position angle and separation measures are either of one satellite referred to another or of a satellite referred to Uranus. The maximum separation between any two satellites is about 80″ (Titania-Oberon). Thus p and s can be calculated from

$$\begin{aligned} \tan p &= x/y \\ s^2 &= x^2 + y^2 . \end{aligned} \tag{9}$$

The theories were fitted to the catalogue and Voyager data by least-squares. For each satellite, corrections were sought to a_0, λ_0, h_0, k_0, p_0 and q_0 (i_0, Ω_0 replacing p_0, q_0 for Miranda), n, β and γ ($\dot\Omega$ replacing γ for Miranda) and in addition corrections to the parameters A_1, A_2, A_3, θ_0 and $\dot\theta$ which model the Laplacian term. Initial values for the parameters were obtained from the solution in Veillet (1983a) with small non-zero values taken for the amplitudes A_2 and A_3.

Residuals were computed in the reference plane in which the observation was made. The ΔT values used to convert a UT time of observation to a TDT time were obtained from the table given in section K of the Astronomical Alamanac for 1997. From the JPL DE200 ephemeris the appropriate coordinates of Uranus were computed – astrometric or apparent, topocentric or geocentric. The satellite coordinates (ξ, η, ζ) were computed at the observation time ante-dated by the light-time from Uranus to the Earth. The planet and satellite coordinates were transformed to the observation reference plane and their RA's and Dec's computed. From these, if necessary,(x, y) or (p, s) were computed and thus residuals were formed.

In the equations of condition the partial derivatives of the observed quantities with respect to (w.r.t.) the solved-for parameters were calculated by differentiating analytically Eqs. (6) to (9) and the expressions for the satellite vectors. The partials were computed by the following sequence of steps:

(i) Derivatives of a, λ, h, k, p, q
(i, Ω instead of p, q for Miranda)
w.r.t. a_0, λ_0, h_0, k_0, p_0, q_0
(i_0, Ω_0 instead of p_0, q_0 for Miranda)
,n, β, γ ($\dot\Omega$ instead of γ for Miranda)
and w.r.t. A_1, A_2, A_3, θ_0, $\dot\theta$
(for Miranda, Ariel and Umbriel).

(ii) Derivatives of ξ, η, ζ w.r.t. a, λ, h, k, p, q
(i, Ω instead of p, q for Miranda)

(iii) Derivatives of α, δ w.r.t. ξ, η, ζ

(iv) Derivatives of x, y or p, s w.r.t. α, δ (if necessary) .

After the derivatives in step (ii) are calculated the partials are transformed if necessary to the reference frame in which the observation is given. Equations of condition for position angle are scaled by multiplying through by s_c, the computed separation.

The Earth-based data vary in accuracy considerably. The different quality of observations is even more apparent when comparing the accuracy of Earth-based data with spacecraft data. Therefore, it is necessary to weight the data. To obtain weights for the Earth-based observations, initially a solution was made where the data was not weighted. From this the rms was computed for each dataset. Then for a dataset with an rms of σ, weights for each observation in this dataset were chosen as σ^{-2}. In our formulation of the least squares the equations of condition were multiplied by σ^{-1}. A new solution was made using these weights and from this a new set of σ's computed. Choosing weights based on these new σ's a further solution was made. After a few iterations there was no significant changes to the σ's and the weights were kept fixed. For the spacecraft data each observation was weighted individually. Weights σ_v^{-2} were chosen, where σ_v is the combined uncertainty of the observation quoted by Jacobson (1992a) and his value for the uncertainty in the spacecraft position. In the least-squares solution for the Earth-based data a rejection level of 3σ for outlying data was used and for the spacecraft positions a rejection level of $3\sigma_v$.

Since the fit of the theories to observations was primarily made to enable approximate starting conditions to be obtained for the numerical integration and as they do not represent accurate orbital models for the satellites the solution is not given. Later in this paper parameters are given so a precise numerical integration of the Uranian satellites can be reproduced.

5. The numerical integration

5.1. Description of the numerical integration

The equations of motion and variational equations were numerically integrated. The unit of mass was taken to be the mass of the Sun, so $M_s = 1$ and $GM_s = 2.95912208286 \times 10^{-4}\mathrm{au}^3\mathrm{day}^{-2}$. The Uranus barycentric coordinates of the spacecraft, needed to process the Voyager observations, are given in kilometres. These are converted to au's using 1au = 149597870.66 km. The equation to give partials w.r.t. J_3 was not integrated, as $J_3 = 0$ was always assumed. From the variational equations we use the partials of the coordinates w.r.t. 38 parameters. These are the initial coordinates of the satellites, the initial velocities, J_2,J_4 and the masses of the five satellites and Uranus. The solved-for parameters $q_1, q_2, \ldots, q_{38}$ are now redefined to represent these. Corrections to the coordinates of the pole of Uranus were also sought. The equations of motion are dependent on them through the computation of the Solar perturbations, but these are relatively weak. The main dependence on the pole vector is in the transformation of satellite coordinates from the integration reference frame to the reference frame of the observations. Partials are calculated at this stage using the rotation matrices obtained from differentiating w.r.t. α and δ the inverse of the matrix in Eq. (4).

Initial conditions for the variational equations, Eqs. (5), at the initial epoch are respectively

$$\frac{\partial c_i}{\partial q_i} = 1, \quad \frac{d}{dt}\left(\frac{\partial c_i}{\partial q_{i+15}}\right) = 1 \quad \text{for} \quad i = 1, 2, \ldots, 15$$

all other $\dfrac{\partial c_i}{\partial q_k}, \dfrac{d}{dt}\left(\dfrac{\partial c_i}{\partial q_k}\right)$ are zero.

A total of 585 second-order differential equations are numerically integrated, 15 for the coordinates and 570 for the partial derivatives. An 8^{th} order Gauss-Jackson method is used, using an iterative starting procedure. A predictor-corrector scheme was used for the integration of the coordinates and a predictor only for the integration of the partial derivatives. Partial derivatives are not needed as precisely as the coordinates, since an iterative least-squares method is used. The indirect oblateness perturbations were omitted from the differential equations for the partial derivatives.

The planetary perturbations were included in the numerical integration of Jacobson (1992b) in a simplified form, but without loss of accuracy, by locating at the Solar System barycentre a fictitious body with a mass equal to the sum of the masses of the Sun and planets interior to Uranus. We found no significant differences between solutions made with the planetary perturbations omitted and those including them, either individually (Jupiter and Saturn perturbations) or as modelled by Jacobson (1992b). The solutions discussed in this paper were derived without including the planetary perturbations.

The starting epoch for the integration was Jan 5.0, 1987 (JED 2446800.5) and integrations were made forwards and backwards in time covering the time-span April 1977 to October 1995. A step length of 0.025 days was used. Tests using a step of half this length showed that the accumulated error by using 0.025 days did not exceed 2×10^{-8}au.

5.2. Determination of starting conditions

To get an initial estimate of the positions and velocities of the integration, it was first fitted to the analytical theories described in Sect. 4. Positions from the analytical theories were output in the reference plane of the integration. Then the initial states were determined from a least-squares fit to these positions over firstly a short time interval ($\pm 10d$ from epoch with positions at 0.5d was used) and then to longer intervals until a set of initial positions and velocities are found from a fit to the time interval covering the period we want to fit to the data. Using these as starting conditions and fitting the integration to the observations we found from successive iterations a large number of observations were rejected, indicating that the initial estimates were not close enough to the solution. To improve these estimates the classical osculating elements at epoch were solved for in place of position and velocity.

The numerical integration of the equations of motion and variational equations was still performed in rectangular coordinates. The initial position and velocity for each satellite at epoch were calculated from the current estimate of the initial osculating elements using two-body formulae. In the equations of condition the partial derivatives w.r.t. initial position and velocity $\frac{\partial c_i}{\partial q_k}, i = 1, 2, \ldots, 15;\ k = 1, 2, \ldots, 30$ are converted to partial derivatives w.r.t. the initial osculating elements using the formulae for $i = 1, 2, \ldots, 5$

$$\begin{aligned}
\frac{\partial c_{3i-2}}{\partial p_k} &= \sum_{j=3i-2}^{3i}\left(\frac{\partial c_{3i-2}}{\partial q_j}\frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i-2}}{\partial q_{15+j}}\frac{\partial q_{15+j}}{\partial p_k}\right)\\
\frac{\partial c_{3i-1}}{\partial p_k} &= \sum_{j=3i-2}^{3i}\left(\frac{\partial c_{3i-1}}{\partial q_j}\frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i-1}}{\partial q_{15+j}}\frac{\partial q_{15+j}}{\partial p_k}\right)\\
\frac{\partial c_{3i}}{\partial p_k} &= \sum_{j=3i-2}^{3i}\left(\frac{\partial c_{3i}}{\partial q_j}\frac{\partial q_j}{\partial p_k} + \frac{\partial c_{3i}}{\partial q_{15+j}}\frac{\partial q_{15+j}}{\partial p_k}\right)\\
&\quad , k = 6i-5, \ldots, 6i
\end{aligned} \tag{10}$$

where $p_{6i-5}, p_{6i-4}, \ldots, p_{6i}, i = 1, 2, \ldots, 5$ are the initial osculating elements for Miranda, Ariel, Umbriel, Titania and Oberon respectively. The partial derivatives $\frac{\partial q_j}{\partial p_k}$ in Eq. (10) are calculated from the formulae for elliptic motion. Referring to the coordinate system for the numerical integration the coordinates of a satellite in terms of elliptic elements are

$$\begin{aligned}
x &= r(\cos\Omega\cos(\omega+f) - \sin\Omega\cos i\sin(\omega+f))\\
y &= r(\sin\Omega\cos(\omega+f) + \cos\Omega\cos i\sin(\omega+f))\\
z &= r\sin i\sin(\omega+f)
\end{aligned} \tag{11}$$

where r is the radius vector, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of pericentre and

f is the true anomaly. From these equations and those from differentiation of them w.r.t. time the partial derivatives of position and velocity w.r.t. the orbital elements(a, e, i, λ, ϖ, Ω) can be derived. These partial derivatives are given in Taylor (1998).

Corrections to the osculating elements p_k,$k = 1, \ldots, 30$ are then found from the fit of the integration to the observations, using least-squares. The corrected elements are then converted to position and velocity by formulae of elliptic motion and these are used as starting values for a new integration of the equations of motion and variational equations. After a few iterations a converged solution was attained. From this solution and reverting to the partial derivatives w.r.t. the initial position and velocity at epoch a solution for the initial state vectors was found. The solutions discussed in the next section were obtained using the rectangular coordinates formulation.

The least-squares fit of the numerical integration to observations is similar to that described earlier for the fit of the approximate theories to observations. Now, the satellite coordinates are from the numerical integration. These are rotated to the reference plane of the Earth equator of 1950.0 using the rotation matrix in Eq. (8). We note that if α_p, δ_p are the RA and Dec of the pole of Uranus $N = 90° + \alpha_p$ and $J = 90° - \delta_p$ and the inverse of matrix in Eq. (8) is the rotation matrix in Eq. (4). The partials in the observation equations are calculated by the following sequence of steps:

(i) Derivatives of $c_{3i-2}, c_{3i-1}, c_{3i}, i = 1, \ldots, 5$ w.r.t. $q_1, q_2, \ldots, q_{38}$

(ii) Derivatives of α, δ w.r.t. $c_{3i-2}, c_{3i-1}, c_{3i}$, $i = 1, \ldots, 5$

(iii) Derivatives of x, y or p, s w.r.t. α, δ (if necessary).

To compute partial derivatives of observational quantities w.r.t. osculating elements the above steps are followed but additionally calculating initially the derivatives of $q_1, q_2, \ldots, q_{30}$ w.r.t. $p_1, p_2, \ldots, p_{30}$. After step (i) the partials are transformed to the reference plane in which the observation is given.

6. Results

In addition to determining the initial state vectors for the satellites, solutions were attempted for a number of other parameters. Good determinations were obtained for J_2, the masses of Miranda, Titania and Oberon, the mass of Uranus and corrections to the position of Uranus. The coordinates of the pole of Uranus were found to be highly correlated with the orbital data. These were subsequently fixed at the values $\alpha_p = 76\overset{\circ}{.}5969$ and $\delta_p = 15\overset{\circ}{.}1117$ from French et al. (1988). The value of J_4 was found to be highly correlated with the J_2 value. J_4 was fixed at the value -0.00002885 from French et al. (1988). A high correlation was found between the masses of Ariel and Umbriel. Both masses were fixed at the values 0.00001558525 and 0.00001349686 respectively, derived from values in Jacobson (1992b) expressed in terms of the mass of Uranus.

The initial state vectors are given in Table 2. The values obtained for the masses of the satellites, Uranus and J_2 are

Mass of Miranda	: $(7.05 \pm 0.31) \times 10^{-7}$
Mass of Titania	: $(3.839 \pm 0.039) \times 10^{-5}$
Mass of Oberon	: $(3.230 \pm 0.067) \times 10^{-5}$
Mass of Uranus	: $(4.36587 \pm 0.00012) \times 10^{-5}$
J_2	: 0.003365 ± 0.000013.

The satellite masses are in terms of the mass of Uranus. In obtaining J_2 a value of 26 200 km was used for the equatorial radius of Uranus. The mass of Miranda agrees well with the values previously published in Jacobson et al. (1992b), Anderson et al. (1987), Tyler et al. (1986), Brown et al. (1991) and Laskar and Jacobson (1987). For Titania and Oberon the values are in each case less than the previously published values, in some cases between 2 to 3 times the standard error less. The value for the mass of Uranus is in good agreement with the value in Jacobson (1992b). The gravity field coefficient J_2 agrees quite well with the value 0.00334343 from French et al. (1988). This latter value was derived in the determination of Uranian ring orbits using Earth-based and Voyager 2 occultation observations.

The statistics for the datasets used in the fit are given in Table 3. The weighted rms for each Earth-based dataset was about one. A sizeable number of observations in Standish (1996) were rejected, since some positions are of low accuracy having been reduced with very few reference stars. The differential declination observations in Abrahamian et al. (1993) were down weighted in the analysis and a significant number of the differential right ascension residuals were rejected. The weighted rms for the spacecraft data was 0.373 for the RA observations and 0.358 for the declination observations. The weights used were from Jacobson (1992a).

In order to ascertain the sensitivity of our results to the different types of data analysed and in their weighting a number of further solutions were made. No solution could be found from analysis of the Voyager data alone. The orbital data for each satellite were highly correlated. Attempts were made to obtain a solution from the Earth-based data only but it was found difficult to get convergence. This was probably due to the limited accuracy of the observations and the short time-span covering the data analysed. Further solutions combining the Earth-based data and the Voyager observations were made. Firstly, a solution fixing the value of J_2 at the French et al. (1988) value was made. Solutions were also obtained with the Voyager data weights σ_v^{-2} now chosen so that σ_v is equal to the residual of the observation. The resulting weighted rms for each component of the Voyager data was near to one. For these solutions greater correlations were found for the orbital data, and in particular J_2 was highly correlated with the z component of the velocity for Miranda. Solutions were made fixing J_2 at the value found from the initial solution and at the value from French et al. (1988). These solutions and the initial solution (Solution 1) are given in Table 4 together with the solution of Jacobson et al. (1992b). In order to facilitate comparison with the solution of Jacobson et al. (1992b) the masses are converted to GM values where for a unit solar mass $G = 1.32712438 \times 10^{11}$ km^3 sec^{-2}. The standard errors quoted for the solutions described in this paper are the formal errors from the least-squares analysis. They will almost certainly represent an underestimate of the true error, since there

Table 2. The initial coordinates and velocities of the satellites in units of au and au/day respectively (Solution 1). The epoch is JED 2446800.5. The reference frame is the equator of Uranus and ascending node of equator of Uranus on equator of Earth B1950.0

	Miranda	Ariel	Umbriel	Titania	Oberon
x	−0.0002671269	0.0009780155	0.0009412372	0.0010065934	−0.0034567062
y	0.0008238160	0.0008178558	−0.0014993458	−0.0027361423	−0.0017933166
z	0.0000649225	−0.0000000196	0.0000017092	−0.0000051470	0.0000039824
$\dot{x}$	−0.0036706600	−0.0020393546	0.0022929337	0.0019769101	0.0008391051
$\dot{y}$	−0.0011794554	0.0024466736	0.0014396889	0.0007243742	−0.0016183260
$\dot{z}$	−0.0000717602	0.0000000966	0.0000022138	0.0000021730	−0.0000048763

Table 3. Statistics of residuals for datasets used in the fit of the numerical integration to observations. For each dataset the following quantities are given: μ the mean residual and its standard error, σ the standard deviation about the mean, N_U the number of observations used and N_T the total number available.

Dataset	Type	N_U	N_T	μ ($''$)	σ ($''$)
Veillet and Ratier (1980)	p	57	58	−0.030±0.018	0.138
Veillet (1983a)[1]	$\Delta\alpha\cos\delta$	433	441	0.013±0.004	0.088
	$\Delta\delta$	423	441	0.008±0.004	0.090
Veillet (1983a)[2]	$\Delta\alpha\cos\delta$	61	62	0.059±0.020	0.158
	$\Delta\delta$	62	62	0.048±0.023	0.183
Veillet (1983a)[3]	$\Delta\alpha\cos\delta$	596	617	0.012±0.002	0.050
	$\Delta\delta$	607	617	−0.010±0.002	0.048
Veillet (1983a)[4]	$\Delta\alpha\cos\delta$	145	147	0.022±0.004	0.047
	$\Delta\delta$	147	147	0.023±0.008	0.095
Harrington and Walker (1984)	$\Delta\alpha\cos\delta$	283	291	0.003±0.004	0.064
	$\Delta\delta$	281	291	−0.003±0.004	0.066
Veillet (1983b)[1]	p	21	21	−0.088±0.025	0.114
Veillet (1983b)[5]	p	9	9	−0.003±0.014	0.042
Veillet (1983b)[6]	$\Delta\alpha\cos\delta$	82	82	−0.006±0.013	0.118
	$\Delta\delta$	81	82	−0.034±0.012	0.111
Veillet (1983b)[7]	p	80	82	−0.064±0.009	0.077
Debehogne et al. (1981)	$\Delta\alpha\cos\delta$	13	14	0.077±0.081	0.293
	$\Delta\delta$	14	14	0.046±0.073	0.275
Pascu et al. (1987)	$\Delta\alpha\cos\delta$	76	76	−0.005±0.013	0.115
	$\Delta\delta$	73	76	0.045±0.008	0.070
Veiga et al. (1987)	$\Delta\alpha\cos\delta$	967	1006	0.004±0.003	0.084
	$\Delta\delta$	965	1006	−0.007±0.003	0.080
Standish (1996)	$\Delta\alpha\cos\delta$	439	491	0.012±0.005	0.105
	$\Delta\delta$	422	491	0.002±0.004	0.076
Veillet (1985)	$\Delta\alpha\cos\delta$	401	415	0.007±0.003	0.057
	$\Delta\delta$	401	415	−0.005±0.003	0.056
Walker and Harrington (1988)	$\Delta\alpha\cos\delta$	102	105	0.007±0.003	0.034
	$\Delta\delta$	102	105	−0.001±0.004	0.037
Veiga and Vieira Martins (1995)	$\Delta\alpha\cos\delta$	1417	1472	−0.003±0.001	0.039
	$\Delta\delta$	1410	1472	0.007±0.001	0.036
Shen (1992)	$\Delta\alpha\cos\delta$	14	14	0.003±0.075	0.282
	$\Delta\delta$	14	14	0.059±0.083	0.311
Abrahamian et al. (1993)	$\Delta\alpha\cos\delta$	18	30	0.134±0.052	0.222
	$\Delta\delta$	0	30	-	-
Jones et al. (1997)	$\Delta\alpha\cos\delta$	532	574	−0.006±0.002	0.052
	$\Delta\delta$	533	574	−0.003±0.002	0.055
Carlsberg Mer. Cats. (1993-1997)	α	89	90	−0.000±0.039	0.365
	δ	89	90	−0.000±0.031	0.292
Jacobson (1992a)[8]	α	443	445	−0.011±0.023	0.490
	δ	445	445	−0.078±0.022	0.473

[1] Observations made at Pic du Midi.
[2] Observations made at Observatory of Haute Provence.
[3] Observations made at ESO.
[4] Observations made at Hawaii.
[5] Observations made at ESO in 1980.
[6] Observations made at ESO in 1981.
[7] Position angle observations of the ESO 1981 data.
[8] Observations are Voyager-centred.

Table 4. Comparison of solutions for satellite masses, mass of Uranus and the J_2 gravity harmonic coefficient. The GM values are given for the masses, units km^3 sec^{-2}. For solutions 1 and 2 weights for the Voyager data were taken from Jacobson (1992a) and for solutions 3 and 4 weights σ_v^{-2}, where σ_v approximately equalled the residual of the observation.

	Solution 1	Solution 2	Solution 3	Solution 4	Jacobson (1992b)
Miranda	4.09± 0.18	4.08± 0.18	4.04± 0.18	4.17± 0.17	4.4± 0.5
Ariel	90.3 [fixed]	90.3 [fixed]	86.30± 0.90	87.15± 0.58	90.3± 8.0
Umbriel	78.2 [fixed]	78.2 [fixed]	80.03± 0.80	79.28± 0.47	78.2± 9.0
Titania	222.4± 2.3	222.8± 2.3	222.15± 0.17	234.30± 0.60	235.3± 6.0
Oberon	187.1± 3.9	187.0± 3.9	188.95± 0.58	199.46± 0.13	201.1± 5.0
Uranus	5794054± 163	5794097± 163	5794102± 11	5794011± 34	5793939± 13
J_2	0.003365± 0.000013	0.00334343 [fixed]	0.003365 [fixed]	0.00334343 [fixed]	0.00334343 [fixed]

are significant systematic effects in the Earth-based astrometric observations (Pascu 1977) and probably in the spacecraft data as well (Jacobson et al. 1992b). The standard errors in Jacobson et al. (1992b) solution are their assessment of the true rather than the formal errors and were based on an examination of previously determined values and on the results of a number of sensitivity tests. We see there are no significant differences between solutions 1 and 2. This is to be expected as J_2 is not correlated with the orbital data or the masses. For solutions 3 and 4, the mass of Ariel and mass of Umbriel are not correlated and so values for them can be determined. There is good agreement between solutions 1 to 4 and Jacobson et al. (1992b) for the masses of Miranda, Ariel and Umbriel, mass of Uranus and J_2. For the masses of Titania and Oberon we see the values from solutions 3 and 4 differ significantly depending thus on the J_2 value imposed on the solution. Solution 4 agrees well with that of Jacobson et al. (1992b). It can be seen therefore that when the spacecraft data is given equal weighting to Earth-based data although determinations of the masses of Ariel and Umbriel can be obtained, dependency on the gravity coefficient J_2 results. In fact the orbital data is noticeably more highly correlated in solutions 3 and 4 than in solutions 1 and 2. These results suggest a careful choice of weights for the spacecraft data is needed, based on known uncertainties in these observations, since if they are given equal weight with Earth-based data the solution will become dominated by them.

Ephemerides generated from the initial conditions and physical parameters for Solution 1 are available electronically on request to the author.

The meridian circle observations are referred to an absolute reference frame in right ascension and declination. This reference frame is a smoothed version of the $FK5$, with the zonal errors greatly reduced (Morrison et al. 1990). Residuals for these observations will essentially give information on the accuracy of the planetary ephemeris, as any systematic errors from star positions used in reducing the observations will be small($< 0\farcs05$). These observations cover the period from June 1992 to October 1995. From the graphs of the residuals it is seen a single constant correction is adequate to estimate the deviation of the ephemeris from the observations.

Ephemerides of the planets have been developed at JPL by fitting a numerical integration to a wide variety of data. A series of development ephemerides have been determined. The positions of Uranus in DE200 were used in the present analysis. To compare the accuracy of different ephemerides the numerical integration and fit to data were repeated using the JPL DE245 and DE403 ephemerides. In each fit corrections to the positions of Uranus were determined. These are given in Table 5. The DE200 and DE245 ephemerides are on the FK5 system and the DE403 on the ICRS system. The corrections to the position of Uranus, using the DE403 ephemeris were +0$\farcs$138 in right ascension and $-0\farcs095$ in declination. As the transit circle observations are on the FK5 system a correction to these values is necessary. A correction of $-0\farcs045$ in right ascension and +0$\farcs$025 in declination was made (Morrison 1997). The corrected values have been given in Table 5. It can be seen that DE245 and DE403 give smaller residuals than that from DE200. The declination residuals from using DE403 are markedly smaller than those using DE245 but the right ascension residuals are slightly worse.

7. Conclusions

In order to compute ephemerides for the Uranian satellites it is important to know the accuracy and distribution of the observations. A survey was made of the available published and several of the unpublished observations. References to these datasets are given and this will be a useful source for future developments of ephemerides for the satellites.

Methods are described how to compute a precise numerical integration of the satellites of Uranus. The determination of the partial derivatives for each stage of the orbit fitting is described. These procedures could equally be applied to other satellite systems.

The determination of the mass values for the satellites comes from the secular and periodic variations in their motion and are found from analysing a combination of astrometric and spacecraft data. The spacecraft data covers a short time-span ($<$ 6 mths) and so contributes less than the astrometric data to the determination of the secular perturbations. The spacecraft data does provide a strong fix on the local geometry of the orbits. Combining the two leads to an enhanced determination of the secular perturbations. Since, other than the Laplacian near resonance, there are no true resonances amongst the satellites, the masses cannot be easily determined from periodic longitude variations. There are though, short-period perturbations such as the 145 d term resulting from the near-resonance of orbital pe-

Table 5. Corrections to ephemeris of Uranus

Timespan	Mean	Mean	Corrections to RA				Corrections to Dec			
	RA	Dec	DE200	DE245	DE403	s.e.	DE200	DE245	DE403	s.e
June 92 - Oct 95	$19^h\ 44^m$	$-22°$	$-0''.145$	$+0''.032$	$+0''.093$	$0''.041$	$-0''.177$	$-0''.148$	$-0''.070$	$0''.030$

riods between Oberon and Titania($3n_{Ob} \approx 2n_{Ti}$). These perturbations are seen in the optical data (Jacobson et al. 1992b).

Comparison of the mass determinations from the various solutions made shows the sensitivity of the mass values for the outer four satellites to the weights given for the astrometric data versus the spacecraft data. If weights are chosen such that the weighted rms for each dataset is near to one we find the orbital data for each satellite highly correlated, including a correlation with J_2. The solution in this case is dependent on the J_2 value imposed on it. The mass values from the solution agree well with those in Jacobson et al. (1992b). As the spacecraft data are so much more accurate than the other data analysed the mass values will be dependent overduly on the spacecraft data and consequently the contribution to the mass solutions from the secular perturbations will be downgraded considerably. By lowering the weights for the spacecraft data the mass solutions become independent of the J_2 value, the orbital data are less correlated and greater weight is given to the contribution from the secular perturbations. The weights used were from Jacobson (1992a) and lead to mass values for Titania and Oberon 2 to 3 sigma less than Jacobson et al.(1992b) values. This is using for comparison their more realistic standard errors for the masses. The masses of Ariel and Umbriel are now highly correlated.

The observations used in the fit of the numerical integration cover the time-span April 1977 to October 1995. This time interval was chosen so that the analysis of post-Voyager data could be combined with the Voyager astrographic data and some of the best quality pre-Voyager data. This interval also covers nearly one and a half cycles of the Laplacian near resonance of Miranda, Ariel and Umbriel. However, the choice of interval was also governed by computational considerations. The period covers a time when the satellite orbits are "pole-on" at their greatest but not when they are directly "edge-on". These periods alternate about every 21 yr. Now that good initial conditions have been established, it should be possible to extend the numerical integration backwards to include both geometrical configurations. With an extended baseline of observations a solution should be tried again with the Earth-based data only.

In addition to optical data from Voyager, radio data was obtained from tracking the spacecraft. In particular the Doppler data was used for measuring the gravity field of Uranus and its satellites, and enabled accurate determinations to be made of the mass of the Uranian system and the mass of Miranda (Brown et al. 1991). This data should be included in future developments of ephemerides.

A new spacecraft mission, like Voyager, or preferably one in which an orbiter is put around Uranus would result in the aquisition of new very accurate optical data. These observations, as they would be decades apart from the Voyager data, would greatly improve the determination of the secular perturbations and hence the mass determinations. New accurate ($< 0''.05$) CCD observations would extend the baseline of high-quality astrometric observations and would also contribute to improvement of the long-period motion of the satellites. Further meridian circle observations would enable a simple way for continued monitoring of the deviation of the ephemeris for Uranus from its true position.

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