Rotation curves for spiral galaxies and non-Newtonian gravity: a phenomenological approach

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Abstract. Rotation curves of spiral galaxies are known with reasonable precision for a large number of galaxies with similar morphologies. The data implies that non-Keplerian fall–off is seen. This implies that (i) large amounts of dark matter must exist at galactic scales or (ii) that Newtonian gravity must somehow be corrected. We present a method for inverting the integral relation between an elemental law of gravity (such as Newton's) and the gravitational field generated by a thin disk distribution with exponential density. This allows us to identify, directly from observations, *extensions* of Newtonian gravity with the property of fitting a large class of rotation curves. The modification is inferred from the observed rotation curve and is finally written in terms of Newton's constant or the effective potential of a test mass moving in the field generated by a point-like particle.

Key words: gravitation – galaxies: kinematics and dynamics – methods: analytical – cosmology: dark matter

1. Introduction

When trying to understand the dynamics of large scale astrophysical systems, for example galaxies, we assume that the dominant interaction at that scale is gravity. This implies the use of Einstein's General theory of Relativity (GR), so that in the limit when the speeds involved are much smaller than the speed of light and in the weak field limit, one may legitimately apply the Newtonian limit. For most galaxies these two conditions are met, and therefore Newtonian considerations apply.

Both GR and its Newtonian limit, have been successfully and directly tested at scales not much larger than the Solar System (See, e.g., Will 1993). However, when one tries to apply them to galaxies or even larger systems, the predicted behavior is usually found to be quite different from what is observed. In fact, in order to accommodate the observations, it is customary to assume the presence of a large amount of non-visible matter,

the so-called *dark matter*. The needed amount of dark matter has to be somewhere between 90 and 99 per cent of the total mass of the Universe; furthermore, in order to be consistent with the predictions derived from standard Big Bang nucleosynthesis, it must be non-baryonic. Of course this discrepancy is commonly known as the *Dark Matter Problem*.

In this paper we study the problem of the rotation curves of spiral galaxies, a case in point. When we apply the Newtonian approximation to these systems we find that their rotation curves should fall for large radius as $v^2 \propto r^{-1}$, i.e., in a Keplerian fall-off. Instead, the observed rotation velocity is typically seen to remain constant after attaining a maximum value, as if it were to go to some asymptotic value, different for each galaxy. This is usually explained assuming a halo of *Dark matter* surrounding the visible galaxy, with the adequate shape for accommodating the observed rotation curve.

However, Newton's law of gravity is *just* a phenomenological law that was designed by Newton to explain gravitational dynamics within Solar-system scales. On the other hand, General Relativity (GR) was developed by Einstein with the constraint in mind of recovering the Newtonian potential in the limits of weak fields and small (compared with the speed of light) velocities, assuming that the phenomenological law discovered by Newton for the Solar system could be extrapolated to distances up to *infinity*. These considerations leave open the possibility that, at least while the *Dark Matter* component remains unidentified, the possibility exists that GR, despite its conceptual beauty, would have to be modified in some way, perhaps in the same spirit as it was used to modify Newtonian gravity for strong fields and large velocities or, perhaps, in other ways.

Different alternative theories have already been proposed to solve the *Dark Matter problem* without dark matter, and most of them suggest that Newton's law must be modified either for very small accelerations or for large distances (a good review on some of these alternatives may be found in Sanders 1991). Milgrom's Modified Newtonian Dynamics (MOND) (see Milgrom 1983) suggests that the second law of Newtonian dynamics breaks down for very small accelerations, in such a way that, in that limit, the acceleration is not proportional to the gravitational force but to its square root. This phenomenological theory has

been successfully tested in many galaxies (see Begeman et al 1991) although there is some controversy about its validity in some dwarf galaxies (Lake 1989, Milgrom 1991 and Milgrom 1995). Even a relativistic model that recovers MOND in the adequate limit has been proposed (see Bekenstein 1987 and Sanders 1991). Even though this approach is not accepted by most astrophysicists who seem reticent to assume its validity, it is very interesting on its own right, and should not be discarded without strong evidence against it. However, here we will focus our attention on the possibility that Newtonian gravity could be modified not for small accelerations but for large distances.

We have posed the following (somewhat longish) question: "*Is it possible to find a phenomenological universal Newton-like law that can explain the observed dynamics in spiral galaxies, without having to assume the presence of an undetected mass component, and whose short distance limit is compatible with the Newtonian law?*" We will offer a positive answer to this question.

This is not the first time such an approach is taken in the literature. Work along these lines has already been done. For example, Kuhn & Kruglyak 1987 and Tohline 1983 suggest to add a correction proportional to $1/r$ to the Newtonian force, and the week field limit of Mannheim's Weyl gravity (Mannheim 1989) implies that a constant correction should be added. In all cases, a specific functional form is assumed (or derived, in the Mannheim's case) for the generalized force. This form is parametrized by some free parameters, and the evaluation of the predicted rotation curves for some known spiral galaxies through the corresponding numerical integrals leads, when compared with observations, to specific values for these parameters. This is a very interesting and direct approach, but its success obviously depends on *a good choice* for the initial form for the "improved" force.

We will present here a procedure that follows the inverse methodology: we will write down an equation such that, once we know the observed rotation velocity of a galaxy we readily obtain which is the force, if any, that is able to generate that rotation curve *without assuming* the presence of any dark matter. In this way we will not have to assume a form for the phenomenological law, we will *infer it* directly from the observational data. The observed data is our starting point, not the final result of some "fit". And, what could be more interesting, the method presented here can be, in principle, equally useful for discarding a non-Newtonian law of gravity as for proving its existence. Once we have the equation that allows us to find the force from the *observed* velocity, we will apply it to a sample of spiral galaxies and check if there exists a common phenomenological law that works for *all* the galaxies in the sample. In Sect. 2 we give the solution to the mathematical problem, in Sect. 3 we apply it to real galaxies and we end offering some conclusions.

2. The analytical method

We write the gravitational field as a generalization of both, the Newtonian potential and the Newtonian force. This we do by introducing two functions $g(r)$ and $g_{\text{eff}}(r)$ defined as:

$$\phi(r) \equiv -\frac{G_0 m_1 m_2}{r} g(r). \tag{1}$$

$$\boldsymbol{F}(r) \equiv -\frac{G_0 m_1 m_2}{r^2} g_{\text{eff}}(r) \frac{\boldsymbol{r}}{r} \tag{2}$$

where $\phi(r)$ and $\boldsymbol{F}(r)$ are the potential and the force experienced by two *point-like* particles of masses m_1 and m_2 separated by a distance r and G_0 is Newton's constant. The two functions $g(r)$ and $g_{\text{eff}}(r)$ are related through:

$$g_{\text{eff}}(r) \equiv g(r) - r g'(r), \tag{3}$$

where the prime denotes a derivative with respect to the argument of the function.

In order to calculate the field due to a given mass distribution Ω described by a density function $\rho(r)$, we must first integrate the microscopic field over the distribution, i.e., we must perform the integral

$$\Phi(\boldsymbol{R}) = -G_0 \int\!\!\int\!\!\int_\Omega d^3\boldsymbol{r} \ \frac{g(|\boldsymbol{R} - \boldsymbol{r}|)}{|\boldsymbol{R} - \boldsymbol{r}|} \ \rho(\boldsymbol{r}), \tag{4}$$

which gives the potential experienced by a point mass at a position $\boldsymbol{R}$ from the center of Ω. For a symmetric distribution (spherical or disk when considering the disk plane) it is convenient to introduce the following notation:

$$\Phi(R) \equiv -\frac{G_0 M_{\text{tot}}}{R} \Psi(R). \tag{5}$$

$$V_{\text{rot}}^2(R) \equiv \frac{G_0 M_{\text{tot}}}{R} \Psi_{\text{eff}}(R) \tag{6}$$

where $V_{\text{rot}}(R)$ is the rotation velocity of a test particle describing circular orbits in the gravitational field generated by the distribution Ω.

It can be readily checked that the auxiliary functions Ψ_{eff} and Ψ satisfy the following functional relationship:

$$\Psi_{\text{eff}}(R) = \Psi(R) - R \Psi'(R). \tag{7}$$

The luminous matter in many spiral galaxies can be modelled, as a first approximation, by a thin disk distribution with an exponentially decaying density (this point will be discussed in more detail in Sect. 3, when our results are applied to real galaxies). In cylindrical coordinates this may be written as:

$$\rho(\boldsymbol{r}) \equiv \rho_0 \, e^{-\alpha r} \delta(z) \tag{8}$$

where α is obtained from the luminosity profile for each galaxy.

Our problem can now be paraphrased as follows: "knowing the rotation velocity (i.e., $\Psi_{\text{eff}}(R)$ up to a normalization constant proportional to the mass of the galaxy), what is the elemental

law of gravity, i.e., $g(r)$ or $g_{\text{eff}}(r)$, capable of generating that velocity field in a thin disk galaxy?"

This problem can be solved exactly for a *spherical* galaxy with an exponentially decreasing density. Here the solution can be summarized as:

$$g(r) = \Psi(r) - \frac{2}{\alpha^2}\Psi''(r) + \frac{1}{\alpha^4}\Psi^{(iv)}(r). \qquad (9)$$

This case will be consider in detail in a separate publication (See Rodrigo-Blanco in preparation). The line of reasoning leading to the proof of Eq. (9) is as follows: plug Eq. (8) into (4), and use the Fourier transform of $g(r)$ together with the addition theorem of Bessel functions to decouple the integration variables in the integrals. Once this is done, and after introducing the function $\Psi(R)$, integration by parts yields Eq. (9).

In the thin disk case the problem cannot be solved exactly. For this reason we will use an approximation that we call "*Gaussian approximation*" (we will see that, in the Newtonian case, this approximation is equivalent to using Gauss' law for calculating the gravitational field and hence the name). It should be noted that this approximation improves when one considers a $g_{\text{eff}}(r)$ which is an increasing function of r (which, of course, is a welcome bonus for understanding the rotation curves of galaxies).

In this approximation the appropriate $g_{\text{eff}}(r)$ turns out to be:

$$g_{\text{eff}}(x) = \Psi_{\text{eff}}(x) - \frac{1}{\alpha^2}\Psi''_{\text{eff}}(x) + \frac{2}{\alpha^2 x}\Psi'_{\text{eff}}(x) \qquad (10)$$

where $\Psi_{\text{eff}}(x)$ has the following behavior at the origin:

$$\Psi_{\text{eff}}(0) = \Psi'_{\text{eff}}(0) = 0. \qquad (11)$$

The mathematical formalism applied to obtain Eqs. (10) and (11) is very similar to the one used for a spherical distribution. In both cases, the use of the addition theorem of Bessel functions leads to an infinite series of terms involving Bessel functions of the form $J_{2k+1/2}$ and we truncate the series keeping only the term with $k = 0$. Actually, in the presence of spherical symmetry this is the only term that contributes to the integrals, and therefore the result is exact. In the thin-disk case it can be shown that this term dominates the integrals in the cases of interest. The mathematical details will be given in a separate publication (See Rodrigo-Blanco in preparation). Here we give a qualitative *a posteriori* justification of the goodness of the approximation. First, it can be seen that the solution to Eq. (10), when $g_{\text{eff}}(r) = 1$, (Newtonian limit) is

$$V_{\text{rot}}^2(R) = \frac{G_0 M_{\text{tot}}}{R}[1 - (1 + \alpha R)e^{-\alpha R}] = \frac{G_0 M(R)}{R}, \qquad (12)$$

where $M(R)$ is the disk mass *inside* a sphere[1] of radius R. In order to get an idea of what happens for a growing g_{eff}, let us restrict ourselves to the case when $g_{\text{eff}}(r)$ can be parametrized

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[1]   Although this is not the exact result, it is however what we would find if we applied Gauss' law as an approximation for evaluating the gravitational field. That is why we call our approximation *Gaussian*.
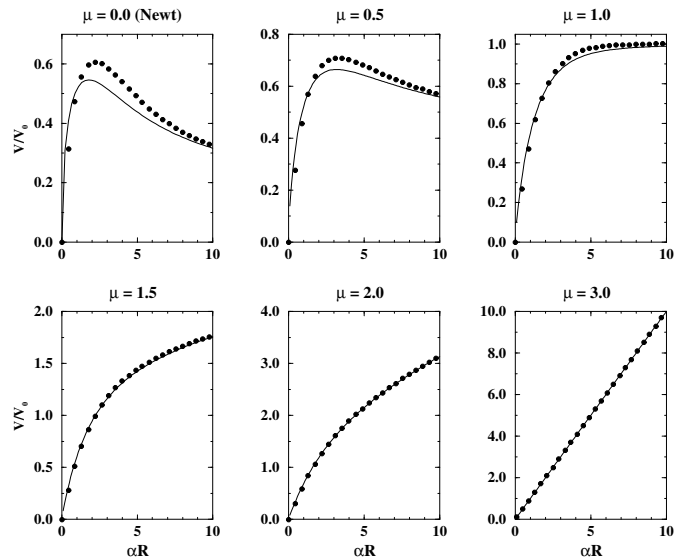

Fig. 1. Exact rotation curves obtained by performing a numerical integration of the forces over a flat disk and obtaining the rotation velocity (full points) and *Gaussian* approximation (solid line) for $g_{\text{eff}}(r) \propto r^\mu$ for some values of μ ($\mu = 0$ is the Newtonian case). In every case, for the sake of clarity, the velocities are normalized by dividing by the appropriate constant: $V_0 \equiv \left(\frac{G_0 M_{\text{tot}}\alpha}{(\alpha a)^\mu}\right)^{1/2}$.

as a power law of the form $g_{\text{eff}}^{(\mu)}(r) \equiv (\frac{r}{a})^\mu$ with μ real and positive. In Fig. 1 we have plotted the rotation curve obtained from Eq. (10) versus the exact solution for six values of μ. It can be seen right away that, the faster $g_{\text{eff}}^{(\mu)}$ grows, the better the approximation.

3. Application to real galaxies

Now we move on to apply our equation (Eq. (10)) to real galaxies. We only can apply the results of the previous section to real galaxies if they can be well described, at least as a first approximation, by Eq. (8). We have chosen a sample of nine galaxies (which we judge to be representative) whose luminosity profiles can be well fitted using a thin disk model with exponential density, with no bulge, and without a very large amount of neutral gas (See refs. Begeman 1987, Carignan 1985, Carignan et al 1988, Persic & Salucci 1995 and Persic et al. 1995). This, assuming a constant value of M/L for the disk, and neglecting both the thickness of the disk and the contribution of neutral hydrogen (so long as it is not distributed as the stellar component), leads to a density function simmilar to that of Eq. (8). However, it must be remembered that these simplifications, necessary for applying Eq. (10) and fairly good for our nine galaxies, are not valid in general. In many spiral galaxies the bulge or the neutral gas components are important and cannot be neglected. It has also been shown that the scale-length $1/\alpha$ in the K-band is, on average, about 15 % shorter than in the B-band (see de Jong 1995 that has reported large colour gradients in a big sample of face-on galaxies), which would mean that assuming a constant

Table 1. Relevant observed properties of selected galaxies. References to the original data: (a) Begeman 1987; (b) Carignan 1985; (c) Carignan et al. 1988; (d) Carignan & Puche 1990; (e) Kent 1987; (f) Mathewson et al. 1992; (g) Metcalfe & Sanks 1991; (h) Persic & Salucci 1995, Persic et al. 1995; (i) Puche et al. 1990

| Galaxy name	Distance (Mpc)	Scale length (kpc)	Luminosity $10^{10}L_{\odot}$	Rot. curve
NGC 2403	3.2 [a]	2.1 [a]	0.8 [a]	a
NGC 3198	9.4 [a]	2.4 [e]	0.9 [a]	a
NGC 0598	0.9 [g]	1.89 [h]	0.36 [h]	h
NGC 6503	5.9 [a]	1.72 [a]	0.48 [a]	a
NGC 0247	2.5 [b]	2.9 [b]	0.24 [b]	d
NGC 0300	1.9 [b]	2.0 [b]	0.24 [b]	i
347-g33	20.9 [f]	1.46 [h]	1.675 [h]	h
UGC 2259	9.8 [c]	1.34 [e]	0.1 [c]	d
NGC 7339	20.6 [h]	1.9 [h]	1.159 [h]	h

Fig. 2. Fits of the rotation curve of NGC 6503 using ten different functional forms (We have used the class of functions v_1, defined in Eq. (13), with a third-degree polynomial and twelve different values of t going from 1.0 to 2.2 (upper graph) and the $g_{\mathrm{eff}}(r)$ corresponding to each fit (lower graph).

value of M/L for the disk could be dangerous in general. In Table 1 we list the relevant observational data for these galaxies.

Once we have found a sample of galaxies whose mass distributions can be approximated by that of Eq. (8), the next step for applying Eq. (10) is to fit the rotation velocity for each galaxy by some function, so that we can take its derivatives. In order to do this it is important to notice that, in the approach we are describing, there is no specific physical reason or prejudice for choosing one function or another to fit the observed data: we only need that the function fits well the observed velocities within the error bars. Apart from this generic requirement, we can use powers, polynomials or any other function that fits the data so long as the corresponding velocity satisfies the following sufficient conditions: $v(r = 0) = 0$, and $v(r)e^{-\alpha r} \to 0$ as $r \to \infty$ (which are both natural conditions). It is also important to notice that, since second derivatives of the rotation velocity are involved in Eq. (10), small differences in velocities can still lead to large differences in g_{eff}. This can be readily appreciated in Fig. 2, where we have plotted different functional fits to the rotation velocity of NGC 6503 and the corresponding g_{eff} 's (up to a normalization constant).

Although all the fits are *statistically* acceptable, we see qualitative differences between the g_{eff}'s corresponding to each fit. These differences are more significant at short distances. In other words, we have twelve g_{eff}'s that can generate a rotation curve compatible with the observed one, being qualitatively different at small radii. This implies that the exact form of g_{eff} is not constrained by the rotation curve for small distances (where the errors in the observations are larger and small deviations from the Newtonian behaviour are smoothed out by the integration that leads to the macroscopic field). Thus, this region will not be used in our calculations, since any result that could be obtained would, at least, be artificial. We will study the behavior of $g_{\mathrm{eff}}(r)$ for large r with the only restriction of being compatible with $g_{\mathrm{eff}} = 1$ at small distances.

In view of the above we use the following procedure: (*i*) we fit each rotation curve by a wide family of mathematical functions, (*ii*) we calculate the corresponding product $g_{\mathrm{eff}}(r) M_i$ for
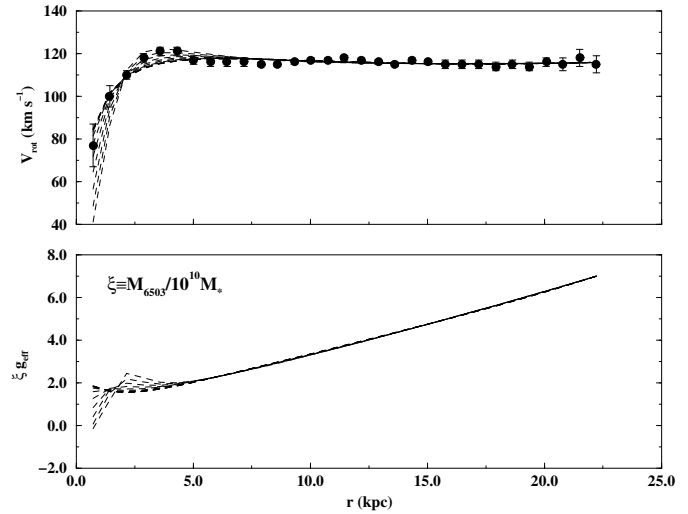
each of those functions (where *i* denotes the particular galaxy). Thus we have a set of $g_{\mathrm{eff}}(r)$'s for each galaxy, each of them capable of generating the observed rotation curve up to some multiplicative constant. Then, (*iii*) we introduce all the $g_{\mathrm{eff}}(r)$'s in a computer program that picks up a $g_{\mathrm{eff}}(r) M_i$ for each galaxy in such a way that, once divided by an appropriate constant, all the $g_{\mathrm{eff}}(r)$'s are as similar as possible. For doing this, we choose a *standard* galaxy among the ones in our sample (in this case we have chosen NGC 6503 because the range of distances for which its rotation curve is observed is the best one to compare with the other galaxies in the sample). Then, for each galaxy, we fit the *mass* proportionality constant for each $g_{\mathrm{eff}}(r)$ minimizing the χ^2 of the comparison with one $g_{\mathrm{eff}}(r)$ for NGC 6503. Finally we pick, for each galaxy, the $g_{\mathrm{eff}}(r)$ for which the final χ^2 is smallest. In this way we obtain a $g_{\mathrm{eff}}(r)$ for each galaxy that is capable of generating its observed rotation curve within the observational accuracy, and we also obtain the total mass of the galaxy relative to the mass of NGC 6503.

Although, as mentioned before, it is irrelevant what class of mathematical functions are used for the fit, it is nevertheless interesting to mention what functions we have used here. We have used two kinds of combinations between powers and polynomials, labelled as v_1 and v_2, and defined by:

$$v_1^2(r) = \frac{r^t}{P_m(r)} \tag{13}$$

$$v_2^2(r) = r^t P_m(r) \tag{14}$$

where, in each case, $P_m(r)$ is a polynomial of degree m in r and t is an integer greater than or equal to one.

In Fig. 3 we show the fit to the rotation curve for each galaxy in our sample, and in Fig. 4 we plot the corresponding $g_{\mathrm{eff}}(r)$
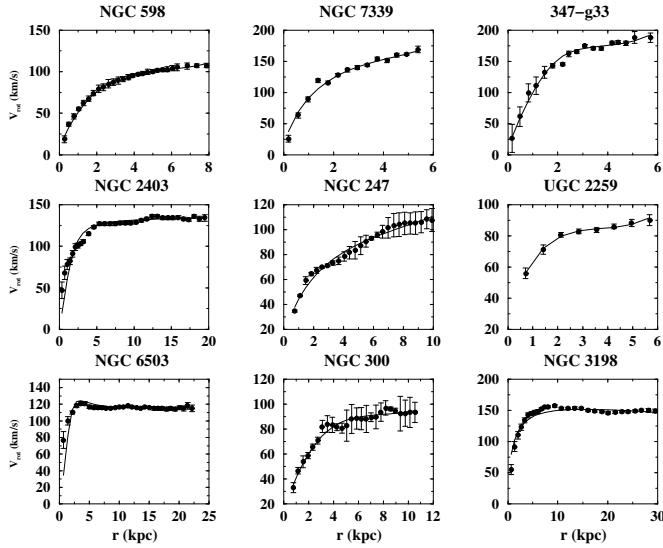
Fig. 3. Observed rotation curve *vs.* the rotation curve *generated* by the $g_{\mathrm{eff}}(r)$ selected for each galaxy.
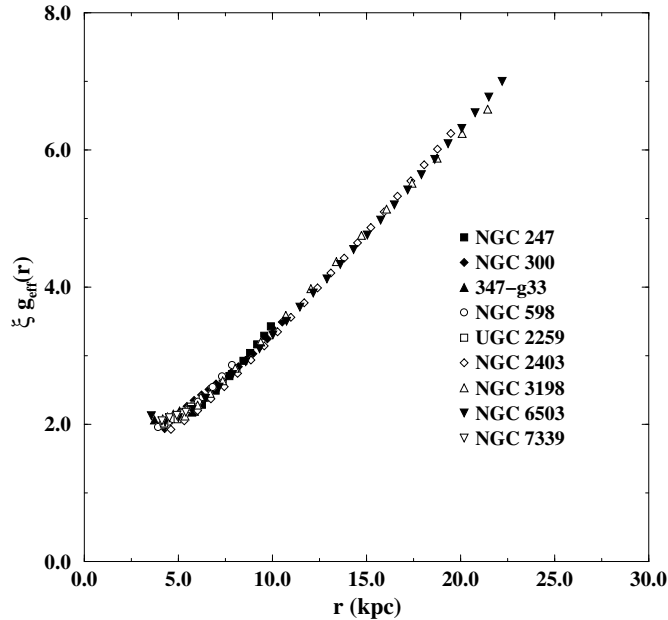

Fig. 4. Normalized $g_{\mathrm{eff}}(r)$ for all the galaxies in the sample. All of them are multiplied by a common constant factor $\xi \equiv M_{6503}/10^{10} M_{\odot}$.

for the galaxies (multiplied by the mass of NGC 6503 in units of 10^{10} solar masses). In Table 2 we list the mass of each galaxy in terms of the mass of NGC 6503 as well as the corresponding mass to light ratio for each galaxy. This mass-to-light ratio is in units of $M_{\odot}/L_{\mathrm{B}\odot}$ and $M_{6503}/10^{10} M_{\odot}$. In this table we also indicate which type of function v_1 or v_2 (See Eqs. (13) and (14)) was finally chosen to fit the observed rotation curve of each galaxy.

Table 2. The first column indicates which kind of function v_1 v_2 (See text) was chosen to fit the observed rotation curve for each galaxy. The second and third columns show the parameters t and m that better fit the data. In the fourth column the mass corresponding to the $g_{\mathrm{eff}}(r)$ chosen for each galaxy is given. The last column shows the corresponding mass-to-light ratio calculated using the observed value of L_{B} this mass-to-light ratio is in units of $M_{\odot}/L_{\mathrm{B}\odot}$ and $M_{6503}/10^{10} M_{\odot}$.

Galaxy		t	m	M/M_{6503}	(M/L_{B})
NGC 2403	v_1	2.1	2	1.30	1.62
NGC 3198	v_1	1.1	2	1.75	1.94
NGC 0598	v_1	1.2	3	0.73	2.02
NGC 6503	v_1	2.2	3	1.00	2.08
NGC 0247	v_2	1.1	2	0.85	3.54
NGC 0300	v_1	1.0	3	0.62	2.58
347-g33	v_1	1.2	3	1.66	0.99
UGC 2259	v_2	1.0	2	0.39	3.9
NGC 7339	v_2	1.2	2	1.62	1.39

4. Conclusions

We have found the solution to the problem of inverting the integral relation between an elemental (two-body) law of gravity and the gravitational field generated by a thin disk distribution with an exponentially decaying density. This problem has been solved in an approximation that leads to good results at large distances, although it fails to be useful at short distances where Many Body effects may be relevant and overshadow the physics of few bodies. This, together with the facts that rotation curves are, in some cases, poorly determined for small radii (and thus they do not give enough information to constrain the exact form of g_{eff} in that range of distances) and that the law of gravity must be assumed to be Newton's at short scales, suggests that we discard this range in our phenomenological study. However it is still interesting to develop new methods that allow to study the short distance behavior in a more accurate way, and we intend to do so in the near future.

We have selected a sample of nine galaxies such that the luminous matter inside them can be well described, as a first approximation, by a thin disk with exponential density. We have applied our equation to the rotation curve of *each* of these galaxies and have found a law of gravity that can generate the observed curves without the need for dark matter (or at least, with a moderate quantity of dark matter distributed with the same exponential law as the luminous matter). These "nine laws" are statistically compatible among themselves, and point in the direction that a single, non-Newtonian, universal (i.e. the same for all the galaxies) law may be at work in the realm of the galaxies.

In order to check if this phenomenological law of gravity is a real alternative to dark matter, a more detailed study is still necessary: (i) much smaller systems must be studied for finding at what scale Newtonian gravity breaks down (a precise determination of the luminous mass of NGC 6503 would also help in this direction, eliminating the scaling factor ξ in Fig. 4), (ii) the law must be checked, through numerical integration, in galaxies with different morphologies (not thin disks), where our *inverse* method cannot be applied, and (iii) the consequences

of a gravitational law like the one plotted in Fig. 4 for all the dynamical problems where gravitation at large scales is involved (spiral structure formation, disk stability, groups and clusters of galaxies, etc.) must be considered. All these problems are out of the scope of this paper and are left for future work.

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