

# Spontaneous symmetry breaking of rapidly rotating stars in general relativity: influence of the 3D-shift vector

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**Abstract.** An analytical scheme and a numerical method in order to study the effects of general relativity on the viscosity driven secular bar mode instability of rapidly rotating stars are presented. The approach consists in perturbing an axisymmetric and stationary configuration and studying its evolution by constructing a series of triaxial quasi-equilibrium configurations. These are obtained by solution of an approximate set of field equations where only the dominant non-axisymmetric terms are taken into account. The progress with respect to our former investigation consists in a higher relativistic order of the non-axisymmetric terms included into the computation, namely the fully three-dimensional treatment of the vector part of the space-time metric tensor as opposed to the scalar part, solely, in the former case. The scheme is applied to rotating stars built on a polytropic equation of state and compared to our previous results. The 3D-vector part turns out to inhibit the symmetry breaking efficiently. Nevertheless, the bar mode instability is still possible for an astrophysically relevant mass of  $M_{\text{ns}} = 1.4 M_{\odot}$  when a stiff polytropic equation of state with an adiabatic index of  $\gamma = 2.5$  is employed. Triaxial neutron stars may be efficient emitters of gravitational waves and are thus potentially interesting sources for the forthcoming laser interferometric gravitational wave detectors such as LIGO, VIRGO and GEO600. From a numerical point of view, the solution of the three-dimensional minimal-distortion shift vector equation in spherical coordinates is an important achievement of our code.

**Key words:** gravitation – instabilities – relativity – stars: rotation

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## 1. Introduction

Large scale laser interferometric gravitational detectors such as LIGO, VIRGO and GEO600 are supposed to become operational in a few years, and among the possible astrophysical scenarios that of a rapidly rotating neutron star, deviating progres-

sively from the initially axisymmetric and stationary configuration towards a triaxial one, is of particular interest. In this case, a “bar shaped” neutron star could become an efficient source of continuous wave gravitational radiation. Rapidly rotating stars can break their symmetry, if the ratio  $T/|W|$  of kinetic and potential energy exceeds some critical value (see Schutz 1987 for a review) and the ( $l = 2, m = 2$ ) bar mode gets unstable. This can be the case for a newborn neutron star, having acquired a sufficiently large amount of rotational kinetic energy during its formation. The matter viscosity being low, it may be sensitive to the *Chandrasekhar-Friedman-Schutz* instability which is related to gravitational radiation reaction (Wagoner 1984; Lai & Shapiro 1995). An alternative mechanism is the *viscosity driven secular instability* which is eventually operative in old neutron stars in close binary systems, being spun up by matter accretion from a nearby companion to some critical value of  $T/|W|$  (Ipser & Managan 1984; Bonazzola et al. 1996). As concerns rigidly rotating, homogeneous and self-gravitating fluid bodies, Newtonian theory tells that at moderate angular velocity, the fluid body is shaped like some axisymmetric *Maclaurin* spheroid. At  $T/|W| = 0.2738$ , the Maclaurin spheroids become dynamically unstable. However, at  $T/|W| = 0.1375$  exists a bifurcation point towards two families of triaxial configurations via the above introduced secular instabilities. In the case of the viscosity driven instability, kinetic energy is dissipated whereas angular momentum is conserved. As a consequence, the fluid body evolves along a sequence close to some *Riemann S* ellipsoids towards a *Jacobi* ellipsoid which is the configuration with the lowest rotational kinetic energy for the given angular momentum and mass of the initial Maclaurin spheroid (Press & Teukolsky 1973). The transition towards the Jacobi ellipsoid occurs on the associated *viscous* time-scale which is much longer than the dynamical one. For this reason, it is called a *secular* instability. In the final state, viscous dissipation has ceased, and the fluid body rotates rigidly again about its smallest axis in an inertial frame. It is this viscosity driven secular instability which we are concerned with in the following. In the Newtonian compressible case, it was shown by Jeans (1919, 1928) that for a polytropic equation of state  $\gamma \gtrsim 2.2$  is needed to reach the bifur-

cation point. If the equation of state is softer, the critical angular velocity  $\Omega_{\text{crit}}$  is greater than the mass shedding limit  $\Omega_K$ , and the bifurcation point is inaccessible. James (1964) has given a refined value for the critical adiabatic index of  $\gamma_{\text{crit}} = 2.238$ , a value which has been recently confirmed by Bonazzola et al. (1996) and, independently, by Skinner & Lindblom (1996) who have found a corresponding value of  $\gamma_{\text{crit}} = 2.237$ . The result of Bonazzola et al. (1996) was a by-product of the investigation of the spontaneous symmetry breaking of fully relativistic rapidly rotating stars presented in the same paper. Until then, all studies of the bifurcation point had been carried out for Newtonian configurations (Jeans 1928; James 1964; Ipser & Managan 1981; Hachisu & Eriguchi 1982; Skinner & Lindblom 1996) or at the 1-PN level of a post-Newtonian analysis at most (Chandrasekhar 1967; Tsirulev & Tsvetkov 1982). In view of the highly relativistic character of realistic neutron star models, this approach is rather motivated by technical simplification than by physical reasoning. Our approach consists in perturbing the ( $l=2, m=2$ ) bar mode of an “exact” axisymmetric configuration and studying the growth or the decay of the applied perturbation. We take into account only the dominant non-axisymmetric terms up to a certain order. In Bonazzola et al. (1996), the level of approximation corresponded to an expansion of the Einstein equations of order 0-PN of the three-dimensional terms. In the present paper, we have included the fully three-dimensional treatment of the shift vector  $N^i$ , raising the approximation level to order 1/2-PN. We recall the basic assumption of *rigid rotation* in our approximation (see Bonazzola et al. (1996) for a discussion of the astrophysical context) and the negligibility of gravitational radiation. Approximation schemes which exploit the near equilibrium of the gravitational fields have gained increasing interest in the context of the binary coalescence problem. Wilson & Mathews (1989) were the first ones to propose the solution of dynamical problems in general relativity by integration of a reduced set of field equations, which basically corresponds to the successive solution of initial value problems for the gravitational fields (York 1979) whereas the hydrodynamic part is calculated exactly under the action of the momentary gravitational fields. It has been applied in the following to study the late stages of an inspiralling neutron star binary system (Wilson & Mathews 1995; Wilson et al. 1996). Cook et al. (1996), being interested in the initial value problem of coalescing binaries, have investigated the validity of Wilson’s approach by application to stationary and axisymmetric neutron star models which, by construction, are perfect equilibrium systems. They further modified Wilson’s approach in a way to exploit the near equilibrium of the matter fields as well, which allows to reduce the matter evolution equations to an algebraic first integral of motion. The neutron star models computed by means of the simplified field equations coincide with those based on the full set of Einstein equations within a few percent. One of the essential points of the Wilson approach is to adopt a *conformally flat* three-metric in order to simplify the field equations. This “conformally flat condition” is being justified by the negligible impact of the radiation part of the gravitational fields on the matter motion. Indeed, for problems in spherical symmetry where

no radiative component exists, it is possible to adopt isotropic coordinates without any analytic approximation. However, it has to be pointed out that even for stationary and axisymmetric systems where no radiative part is present either, isotropic coordinates are not sufficient to give a complete description of the problem. A further degree of freedom of the gravitational field is required which is subject to the dynamical Einstein equations. The other gravitational fields are then entirely amenable to determination by means of elliptic equations derived from (1) the Hamiltonian constraint equation for the conformal factor, (2) the momentum constraint equation for the shift vector, and (3) the additional maximal slicing condition which determines the lapse function. Our model improves on Wilson’s scheme in so far as our approximate equations reduce to the *exact* Einstein equations as long as the configuration remains stationary and axisymmetric, so that only the three-dimensional perturbation is treated approximately.

The paper is organized as follows: In Sect. 2 we present the basic assumptions of our model and derive the field and matter equations. Sect. 3 introduces the numerical code, and in Sect. 4 we present the results of the improved scheme for a polytropic equation of state, including the fully three-dimensional solution of the shift vector equation, as well as a comparison with the results of our previous study. In Sect. 5 we will draw some concluding remarks.

## 2. Theoretical model

### 2.1. Basic assumptions

The general space-time line element in terms of the quantities of the (3+1)-formalism of general relativity (see e.g. York 1979 for an introduction) is given by

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij} (dx^i - N^i dt)(dx^j - N^j dt), \quad (1)$$

with the *lapse function*  $N$ , the *shift vector*  $N^i$ , and  $h_{ij}$ , the metric tensor induced in the spatial hypersurfaces  $\Sigma_t$ . Before the symmetry breaking, the space-time associated with the rotating star is *stationary* and *axisymmetric*. We briefly recall the main conclusions for the case where the star matter is assumed to be constituted by a *perfect fluid*, the stress energy tensor having the form

$$\mathbf{T} = (e+p)\mathbf{u} \otimes \mathbf{u} + p\mathbf{g}. \quad (2)$$

The involved quantities are the fluid proper energy density  $e$ , the fluid pressure  $p$ , the fluid four-velocity  $\mathbf{u}$ , and the space-time metric tensor  $\mathbf{g}$ . Two Killing vector fields  $\mathbf{k}$  and  $\mathbf{m}$  are linked to the space-time symmetries where  $\mathbf{k}$  is time-like at least far from the star and  $\mathbf{l}$  space-like, its orbits being closed curves. In the case of *rigid rotation*, the space-time is *circular* and the two-surfaces orthogonal to both  $\mathbf{k}$  and  $\mathbf{m}$  are globally integrable (Carter 1973). The coordinates  $t$  and  $\phi$  are associated with the both Killing vector fields  $\mathbf{k} = \partial_t$  and  $\mathbf{m} = \partial_\phi$  whereas the remaining coordinates  $r$  and  $\theta$  can be chosen arbitrarily. The standard coordinates for stationary axisymmetric systems are

*quasi-isotropic* coordinates where a *conformally flat* metric in the  $(r, \theta)$ -coordinate planes is adopted. The general line element (1) specified to these coordinates reads

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + B^2 r^2 \sin^2 \theta (d\phi - N^\phi dt)^2 + A^2 (dr^2 + r^2 d\theta^2) . \quad (3)$$

The shift vector has only one non-vanishing component  $N^\phi$  which represents the dragging of inertial frames by the rotating star. For this coordinate choice, Bonazzola et al. (1993) have exhibited a compact set of elliptic equations for the metric potentials  $N$ ,  $N^\phi$  and  $\tilde{A}$ ,  $\tilde{B}$ , the latter being defined by  $\tilde{A} = AN$ ,  $\tilde{B} = BN$ . Let us also remind that *quasi-isotropic* coordinates satisfy both the *minimal distortion* coordinate condition, introduced by Smarr & York (1978), and the *maximal slicing* condition  $K \equiv 0$  where  $K$  is the scalar of extrinsic curvature. When the star deviates from axisymmetry, it is no more stationary either, as gravitational radiation carries away energy and angular momentum. However, at the very beginning of the symmetry breaking, this deviation is very small, and, consequently, the losses due to gravitational radiation are negligible. Under this assumption and for rigid rotation, there exists a Killing vector field  $\mathbf{l}$  which is proportional to the fluid velocity  $\mathbf{u}$  (Carter 1979),

$$\mathbf{u} = \lambda \mathbf{l} , \quad (4)$$

where  $\lambda$  is a strictly positive scalar function. In the stationary and axisymmetric case, the Killing vector  $\mathbf{l}$  is given by

$$\mathbf{l} = \mathbf{k} + \Omega \mathbf{m} . \quad (5)$$

The constant  $\Omega$  is the angular velocity defined as  $\Omega = u^\phi / u^t$ . In the non-axisymmetric case, we assume (1) the existence of a vector field  $\mathbf{k}$  which is time-like at least far from the star, (2) a vector field  $\mathbf{m}$  which is space-like everywhere, (3) a constant  $\Omega$ , such that  $\mathbf{l}$  defined by (5) is a Killing vector field and (4) the fluid velocity  $\mathbf{u}$  is proportional to  $\mathbf{l}$ . The Killing vector field  $\mathbf{l}$  is associated with the persisting *helical* symmetry of the space-time generated by the non-axisymmetric body which appears still static in the corotating frame.

## 2.2. Matter equations

Based on the assumptions made in the previous section, namely that (1) the star matter is composed of a single constituent perfect fluid, and (2) the star rotates rigidly, it is possible to derive a simple first integral of motion following the procedure outlined in Bonazzola et al. (1996). We first introduce the family of Eulerian observers  $\mathcal{C}_0$  whose four-velocity coincides with the future directed unit vector field  $\mathbf{n}$  orthogonal to the space-like hypersurfaces  $\Sigma_t$ . The Lorentz factor  $\Gamma$  between these local rest observers  $\mathcal{C}_0$  and the fluid comoving observers  $\mathcal{C}_1$  is given by

$$\Gamma = -\mathbf{n} \cdot \mathbf{u} . \quad (6)$$

With the baryon chemical potential  $\mu$  and the mean baryon mass  $m_B$ , the *log-enthalpy*  $H$  is defined as

$$H \equiv \ln \left( \frac{\mu}{m_B c^2} \right) , \quad (7)$$

which is the relativistic generalization of the Newtonian specific enthalpy  $h$ . Introducing  $\nu = \ln N$ , we recover the first integral of motion

$$H + \nu - \ln \Gamma = \text{const} , \quad (8)$$

already familiar from the axisymmetric and stationary case. Note, however, that in the present case all quantities are functions of  $(r, \theta, \psi)$  where  $\psi \equiv \phi - \Omega t$  is the azimuthal angular variable in the corotating frame. At the Newtonian limit, we have  $H \rightarrow h$ ,  $\nu \rightarrow U$ ,  $-\ln \Gamma \rightarrow -1/2 \Omega^2 \rho^2$ , and (8) approaches the classical first integral of motion where  $U$  is the Newtonian potential and  $\rho$  the distance from the rotation axis.

## 2.3. Field equations

As announced in Sect. 1, we apply an approximate set of field equations derived under the assumptions that (1) the helical symmetry of space-time is conserved after deviation from the axisymmetric and stationary configuration, and (2) gravitational radiation is negligible.

In addition, we only retain the dominant non-axisymmetric contributions in the field equations up to order 1/2-PN, their leading relativistic order being less or equal than  $\alpha^{3/2}$  where

$$\alpha = \max \left[ \frac{\nu}{c^2}, \frac{v^2}{c^2}, \frac{p}{m_B n c^2} \right] \quad (9)$$

is the post-Newtonian expansion parameter. At this level of approximation, the lapse function  $N(r, \theta, \psi)$  and the shift vector  $N^i(r, \theta, \psi)$  have to be considered as three-dimensional quantities. The components  $N^r(r, \theta, \psi)$  and  $N^\theta(r, \theta, \psi)$  are genuinely three-dimensional contributions and are absent at the previous approximation level 0-PN. Corrections of higher relativistic order to the metric tensor are included for the diagonal components via the *axisymmetric* potentials  $\tilde{A}(r, \theta)$  and  $\tilde{B}(r, \theta)$ , their sources being essentially dominated by the fluid pressure whereas the extra-diagonal terms are again generically non-axisymmetric quantities and hence are neglected. Accordingly, the spatial metric tensor reads

$$(h_{ij}) = \frac{1}{N^2} \begin{pmatrix} \tilde{A}^2 & 0 & 0 \\ 0 & \tilde{A}^2 r^2 & 0 \\ 0 & 0 & \tilde{B}^2 r^2 \sin^2 \theta \end{pmatrix} . \quad (10)$$

The choice of  $N^{-2}$  as conformal factor of the (nearly flat) conformal three-metric has proven to be particularly advantageous, as has been exposed by Bonazzola et al. (1993). It isolates the lapse function as the predominant part of the gravitational fields, which is underlined by considering the weak field limit of the corresponding space-time line element, given by

$$g_{\mu\nu} dx^\mu dx^\nu = -(1-2\nu) dt^2 + (1+2\nu) \times (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) , \quad (11)$$

which conducts to the Newtonian equation for the gravitational potential  $\nu$ .

Having specified the space-time line element for the perturbed neutron star models, we can proceed to derive the governing field equations. The temporal evolution of  $h_{ij}$  is determined by

$$\partial_t h_{ij} + N_{i|j} + N_{j|i} + 2NK_{ij} = 0, \quad (12)$$

with the tensor of extrinsic curvature  $K_{ij}$ . For  $h = |h_{ij}| |f_{ij}|^{-1}$ , the determinant of the spatial metric tensor normalized by that of flat space spherical coordinates  $|f_{ij}| = r^4 \sin^2 \theta$ , it follows immediately the relation

$$\partial_t h + 2(N^l{}_{|l} + NK)h = 0, \quad (13)$$

where  $K = K^j{}_j$  denotes the trace of  $K_{ij}$ . Eqs. (12) and (13) enable us to derive the evolution equation of the *conformal* metric tensor  $\tilde{h}_{ij} = h^{-1/3} h_{ij}$

$$h^{1/3} \partial_t [h^{-1/3} h_{ij}] + [N_{i|j} + N_{j|i} - \frac{2}{3}(N^l{}_{|l}) h_{ij}] + 2N [K_{ij} - \frac{1}{3} K h_{ij}] = 0. \quad (14)$$

The lapse function  $N$ , the only three dimensional quantity involved in the product  $h^{-1/3} h_{ij}$ , cancels out in this term. Therefore, the temporal derivative of  $\tilde{h}_{ij}$  vanishes identically. If we further impose the maximal slicing condition  $K \equiv 0$ , we can determine  $K_{ij}$  from the metric potentials according to

$$K_{ij} = - [N_{i|j} + N_{j|i} - \frac{2}{3}(N^l{}_{|l}) h_{ij}] / 2N. \quad (15)$$

Furthermore, the maximal slicing condition yields an elliptic equation for the lapse function  $N$

$$N^l{}_{|l} - N[4\pi(E+S) + K_{kl}K^{lk}] = 0. \quad (16)$$

Here,  $E$  stands for the total energy density and  $S$  for the trace of the stress tensor  $S^i{}_j$ , all of them measured by the Eulerian observer  $\mathcal{C}_0$ .

Inserting (15) into the momentum constraint equation

$$K^{ij}{}_{|j} - 8\pi J^i = 0 \quad (17)$$

leads immediately to the *maximal slicing-minimal distortion* shift vector equation

$$N^{i|j}{}_{|j} + \frac{1}{3}(N^j{}_{|j}){}^{i|} + R^i{}_j N^j + 2K^{ij} N_{|j} + 16\pi N J^i = 0, \quad (18)$$

introduced by Smarr & York (1978) where  $J^i$  denotes the momentum density vector. Indeed, the York minimal distortion gauge condition  $[\partial_t(h^{1/3} h^{ij})]_{|j} \equiv 0$  is trivially fulfilled, since already the interior of the square brackets equals 0. Note that any coordinate system whose conformal metric tensor is time-independent automatically satisfies the minimal distortion gauge condition. This is notably the case for isotropic coordinates, but also for our choice of quasi-isotropic coordinates where the conformal metric is *not* that of flat space like in the Wilson scheme, but time-independent as well. As a consequence, our approximation of keeping the original form of the spatial metric except for the lapse function  $N$ , being treated as a three-dimensional quantity now, ensures the coordinates to

remain maximal slicing-minimal distortion coordinates in the three dimensional case after deviation from the initial stationary and axisymmetric configuration.

The explicit field equations for our particular choice (10) of the spatial metric tensor are then derived after introduction of the auxiliary variables

$$\tilde{\alpha} \equiv \ln \tilde{A}, \quad \tilde{\beta} \equiv \ln \tilde{B}, \quad \text{and} \quad \tilde{G} \equiv \tilde{B} r \sin \theta, \quad (19)$$

following the procedure outlined in Bonazzola et al. (1993). We obtain the following elliptic equation for the logarithm  $\nu$  of the lapse function  $N$

$$\Delta_3 \nu = 4\pi \frac{\tilde{A}^2}{N^2} (E+S) + \frac{\tilde{B}^2}{2N^4} r^2 \sin^2 \theta (\partial N^\phi)^2 - \partial \nu \partial \tilde{\beta}, \quad (20)$$

where  $\Delta_3$  denotes the three-dimensional flat space scalar Laplacian with respect to the coordinates  $(r, \theta, \psi)$  of the corotating frame

$$\Delta_3 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \psi^2}, \quad (21)$$

and where the following abridged notation

$$\partial \nu \partial \tilde{\beta} = \frac{\partial \nu}{\partial r} \frac{\partial \tilde{\beta}}{\partial r} + \frac{1}{r^2} \frac{\partial \nu}{\partial \theta} \frac{\partial \tilde{\beta}}{\partial \theta} \quad (22)$$

is used. We define the *pseudo-physical* components of the shift vector  $N^i$  via the following relations

$$N^{\tilde{r}} \equiv N^r, \quad N^{\tilde{\theta}} \equiv N^\theta r, \quad N^{\tilde{\phi}} \equiv N^\phi r \sin \theta. \quad (23)$$

The shift vector equation (18) associated with this particular frame yields

$$\Delta N^{\tilde{p}} + \frac{1}{3} \nabla^{\tilde{p}} (\nabla_{\tilde{q}} N^{\tilde{q}}) = S^{\tilde{p}}, \quad (24)$$

with  $S^{\tilde{p}}$  specified by (30) below. We further introduce the explicit expressions of the involved derivative operators, associated with the standard orthonormal frame of flat space spherical coordinates. The pseudo-physical components of the three-dimensional flat space vector Laplacian  $\Delta \mathbf{V}$  are specified as

$$(\Delta \mathbf{V})^{\tilde{r}} \equiv \Delta_3 V^{\tilde{r}} - \frac{2V^{\tilde{r}}}{r^2} - \frac{2}{r^2} \left( \frac{\partial}{\partial \theta} + \frac{1}{\tan \theta} \right) V^{\tilde{\theta}} - \frac{2}{r^2 \sin \theta} \frac{\partial V^{\tilde{\phi}}}{\partial \psi}, \quad (25)$$

$$(\Delta \mathbf{V})^{\tilde{\theta}} \equiv \Delta_3 V^{\tilde{\theta}} - \frac{V^{\tilde{\theta}}}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial V^{\tilde{r}}}{\partial \theta} - \frac{2}{r^2 \sin \theta \tan \theta} \frac{\partial V^{\tilde{\phi}}}{\partial \psi}, \quad (26)$$

$$(\Delta V)^{\tilde{\phi}} \equiv \Delta_3 V^{\tilde{\phi}} - \frac{V^{\tilde{\phi}}}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial V_{\tilde{r}}}{\partial \psi} + \frac{2}{r^2 \sin \theta \tan \theta} \frac{\partial V^{\tilde{\theta}}}{\partial \psi}. \quad (27)$$

We further need to compute the covariant divergence  $\nabla_{\tilde{q}} V^{\tilde{q}}$  which reads

$$\nabla_{\tilde{q}} V^{\tilde{q}} \equiv \frac{\partial V^{\tilde{r}}}{\partial r} + \frac{2V^{\tilde{r}}}{r} + \frac{1}{r} \frac{\partial V^{\tilde{\theta}}}{\partial \theta} + \frac{V^{\tilde{\theta}}}{r \tan \theta} + \frac{1}{r \sin \theta} \frac{\partial V^{\tilde{\phi}}}{\partial \psi}. \quad (28)$$

Finally, the gradient of a scalar potential  $U$  is computed according to

$$\nabla^{\tilde{r}} U \equiv \frac{\partial U}{\partial r}, \quad \nabla^{\tilde{\theta}} U \equiv \frac{1}{r} \frac{\partial U}{\partial \theta}, \quad \nabla^{\tilde{\phi}} U \equiv \frac{1}{r \sin \theta} \frac{\partial U}{\partial \psi}. \quad (29)$$

The pseudo-physical components  $S^{\tilde{p}}$  of the actual source, computed by means of (18), read

$$S^{\tilde{r}} = -16\pi N J_r, \quad S^{\tilde{\theta}} = -16\pi N \frac{J_\theta}{r}, \quad \text{and} \quad (30)$$

$$S^{\tilde{\phi}} = -16\pi \frac{N \tilde{A}^2}{\tilde{B}^2} \frac{J_\phi}{r \sin \theta} - r \sin \theta \partial N^\phi \partial (3\tilde{\beta} - 4\nu).$$

Eqs. (20) and (24) together with (30) constitute the 3D-part of our field equations. The remaining gravitational potentials  $\tilde{A}$  and  $\tilde{B}$  are computed by means of the dynamical Einstein and the Hamiltonian constraint equations after integration over  $\psi$ , which conducts to the original equations derived in Bonazzola et al. (1993). They are genuine 2D-equations, intimately related to the axisymmetry and stationarity of the initial configuration (see Gourgoulhon & Bonazzola (1993) for a geometrically motivated derivation of the (3+1)-equations for this case). For the potentials  $\tilde{\alpha}$  and  $\tilde{G}$ , we then have

$$\Delta_2 \tilde{\alpha} = \left\langle 8\pi \frac{\tilde{A}^2}{N^2} S^{\tilde{\phi}}_\phi + \frac{3\tilde{B}^2}{4N^4} r^2 \sin^2 \theta (\partial N^\phi)^2 - (\partial \nu)^2 \right\rangle, \quad (31)$$

$$\Delta_2 \tilde{G} = \left\langle 8\pi \frac{\tilde{A}^2 \tilde{B}}{N^2} r \sin \theta (S^{\tilde{r}}_r + S^{\tilde{\theta}}_\theta) \right\rangle, \quad (32)$$

where  $\Delta_2$  stands for the two-dimensional flat space scalar Laplacian

$$\Delta_2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad (33)$$

and where  $\langle A \rangle$  denotes the average of  $A$  with respect to the angular variable  $\psi$ . This ensures the consistency of the actual sources of (31) and (32) with the axisymmetry of  $\tilde{\alpha}$  and  $\tilde{G}$ .

To complete the analytic description, we add the matter related quantities for a perfect fluid whose stress-energy tensor has been defined by (2), expressed in terms of variables of the (3+1)-formalism. With the Lorentz factor  $\Gamma$ , defined by (6), the “physical” fluid velocity  $U_{\tilde{a}}$  with respect to the Eulerian observer  $\mathcal{O}_0$  along the  $a$ -th coordinate line is given by

$$U_{\tilde{a}} = \frac{1}{\Gamma} e_{\tilde{a}} \cdot \mathbf{u}, \quad (34)$$

where  $e_{\tilde{a}}$  is the corresponding spatial unit vector. For our coordinates, the components  $U_{\tilde{a}}$  then read

$$U_{\tilde{r}} = -\frac{\tilde{A}}{N^2} N^r, \quad U_{\tilde{\theta}} = -\frac{\tilde{A}}{N^2} r N^\theta, \quad \text{and} \quad (35)$$

$$U_{\tilde{\phi}} = \frac{\tilde{B}}{N^2} r \sin \theta (\Omega - N^\phi),$$

and the normalization condition  $\mathbf{u} \cdot \mathbf{u} = -1$  on the fluid four-velocity  $\mathbf{u}$  yields

$$\Gamma = (1 - U^2)^{-1/2}. \quad (36)$$

The matter related variables  $\Gamma$ ,  $E$ ,  $J_i$  and  $S^i_j$ , specified to our coordinate system, take the approximate form

$$\Gamma = (1 - U_\phi^2)^{-1/2}, \quad (37)$$

$$E = \Gamma^2(e+p) - p, \quad (38)$$

$$J_\phi = (E+p) \frac{\tilde{B}^2}{N^3} r^2 \sin^2 \theta (\Omega - N^\phi), \quad (39)$$

$$S^{\tilde{r}}_r = p, \quad S^{\tilde{\theta}}_\theta = p, \quad S^{\tilde{\phi}}_\phi = (E+p) U_\phi^2 + p, \quad (40)$$

whereas the remaining components equal 0. By combining (30) and (39), one obtains the final form of  $S^{\tilde{p}}$ ,

$$S^{\tilde{r}} = 0, \quad S^{\tilde{\theta}} = 0, \quad \text{and} \quad (41)$$

$$S^{\tilde{\phi}} = 16\pi(E+p) \frac{\tilde{A}^2}{N^2} r \sin \theta (\Omega - N^\phi) - r \sin \theta \partial N^\phi \partial (3\tilde{\beta} - 4\nu),$$

where the latter is exactly the same expression as the one presented in Bonazzola et al. (1993) except that the lapse function  $N$ , the shift vector component  $N^\phi$  and the matter term  $(E+p)$  are allowed to depend on  $\psi$  now.

Let us finally mention that our analytic scheme yields the *exact* solution to the general field and matter equations for two limiting cases: (1) at the Newtonian limit for an arbitrary deviation from axisymmetry, and (2) in the axisymmetric case up to arbitrary relativistic order.

#### 2.4. Stability of an axisymmetric configuration

At this point, we can summarize our analytical approach. The elliptic field equations (20), (24), (31), and (32), completed by the first integral equation (8), fully determine an axisymmetric and stationary equilibrium model, having specified e.g. the central value of the log-enthalpy  $H_c$  and the angular velocity  $\Omega$  for some particular equation of state. The above problem can be formulated as a fixed point problem in some appropriate functional Banach space. Under reasonable physical assumptions, the induced mapping  $\mathcal{L}$  is contractive, thus a unique solution exists and the deviation of the sequence members from the fixed point is bounded by some decaying exponential function. We refer to Schaudt & Pfister (1996) for a recent proof of this statement,

though, at present, restricted to weakly relativistic configurations such as white dwarfs. The solution scheme consists in solving the three-dimensional field and matter equations iteratively where as initial guess a spherical, static matter distribution with a parabolic density profile is assumed. The gravitational potentials are initially set to their flat space values. After a few iterations, rotation is switched on, and the solution converges to the stationary and axisymmetric configuration fixed by the model parameters  $H_c$  and  $\Omega$ . A particular approximate solution, obtained from a previous axisymmetric one, remains axisymmetric. The sequence of equilibrium models is therefore restricted to the subspace of axisymmetric and stationary ones which is part of the full configuration space of three-dimensional quasi-equilibrium configurations.

At a certain iteration step  $J_0$ , after convergence is considered to be sufficient, a small perturbation

$$\delta\nu \propto \epsilon H_c r^2 \sin^2\theta (\cos^2\psi - \sin^2\psi) \quad (42)$$

is added to  $\nu = \ln N$ , which excites the  $l = 2$ ,  $m = \pm 2$  mode. Here  $H_c$  denotes the central value of the log-enthalpy and  $\epsilon$  is a small constant of order  $10^{-8}$ . The three-dimensional gravitational potentials  $N$  and  $N^i$  respond to this perturbation via the field equations and the matter distribution via the first integral of motion. The non-axisymmetry of a particular configuration is conveniently measured by a parameter  $\beta$ , introduced by means of the Fourier expansion of  $\nu$  in  $\psi$ ,

$$\nu(r_e, \pi/2, \psi) = \nu(r_e, \pi/2, 0) + \beta \cos 2\psi, \quad (43)$$

where  $r_e$  denotes the mean stellar radius in the equatorial plane. As mentioned above,  $\mathcal{L}$  applied to *axisymmetric* configurations is contractive in some neighbourhood of the previously constructed axisymmetric solution. Axisymmetric perturbations will thus decay exponentially. This may be different for the non-axisymmetric perturbation (42) depending on the influence of the three-dimensional terms in the field and matter equations. The stability of the axisymmetric model is decided by inspection of the behaviour of the non-axisymmetry parameter  $\beta$  during the continued iteration. Having in mind that we operate in the *linear* regime, we may introduce  $\kappa$ , the *amplification factor* of the non-axisymmetry parameter  $\beta$  between two successive iterations, and so the following three cases can be distinguished:

1.  $\kappa < 1$  :  $\beta$  decreases exponentially and the perturbed configuration converges to the non-perturbed axisymmetric configuration — the configuration is *secularly stable*,
2.  $\kappa = 1$  :  $\beta$  does not change during the subsequent iterations — the configuration is *secularly meta-stable*,
3.  $\kappa > 1$  :  $\beta$  grows exponentially and the perturbed configuration evolves subsequently away from the unstable axisymmetric configuration towards a new stable triaxial quasi-equilibrium configuration — the configuration is *secularly unstable*.

To infer the actual stability of a certain configuration, one has of course to keep in mind the approximate character of our perturbed equations. In particular, for fully relativistic configurations, relativistic terms beyond the current approximation level

of order 1/2-PN which are not included in the present scheme will possibly alter the stability against the triaxial secular instability in some a priori unknown sense.

### 3. Numerical code

#### 3.1. Pseudo-spectral method

The numerical implementation of the analytic model described in Sect. 2 relies on a *pseudo-spectral* code which has been derived from the three-dimensional code presented in Bonazzola et al. (1996) but enhanced by the fully three-dimensional treatment of the shift vector  $N^i$ . The previous code was itself an extension of an originally axisymmetric code which had been conceived to construct high precision models of rapidly rotating stars in general relativity (Bonazzola et al. 1993) and in the following applied to study neutron star models based on realistic equations of state (Salgado et al. 1994) and to model neutron stars provided with a strong magnetic field (Bocquet et al. 1995). We refer to Bonazzola et al. (1993) for a detailed description of the numerical techniques which apply essentially to the present three-dimensional code as well:

The various quantities are expanded in Fourier series in  $\psi$  and  $\theta$  and in Chebyshev series in  $r$ . The elliptic field equations are solved after expansion of the actual sources in terms of the angular eigenfunction bases of spherical harmonics  $Y_l^m(\theta, \psi)$  for  $\Delta_3$  and  $(\cos l\theta, \sin l\theta)$  for  $\Delta_2$ . The resulting systems of ordinary differential equations in the radial variable are conveniently solved in coefficient space. The elliptic equations are solved exactly in so far as the computational domain covers the whole space. This allows to satisfy the exact boundary condition of asymptotic flatness at spatial infinity and the proper calculation of the source terms which fill the entire space. In this way, the limitations of approximate boundary conditions such as a Robin boundary condition at some finite radius  $R$  (York & Piran 1982) can be completely overcome.

Two grids are used to cover the numerical domain. The inner one embodies the star whereas the surrounding space is compactified thanks to a change of the radial variable  $u = r^{-1}$ , mapping it onto the finite exterior grid.

We further want to point out that the present study nicely demonstrates the ability of a spectral method to exploit the particular nature of the problem where only a small number of angular modes in the  $\psi$  variable is present. As a consequence, such low a number as  $N_\psi = 4$  in the interval  $[0, 2\pi]$  is sufficient to *exactly* represent the various quantities up to linear order in  $\beta$ .

#### 3.2. Vector Poisson equation

As outlined in Sect. 2.3, a major issue for the computation of triaxial quasi-equilibrium configurations at the present level of approximation is the calculation of the shift vector  $N^i$  by solution of a generalized vector Poisson type equation of the following form

$$\Delta V^i + \alpha \nabla^i (\nabla_l V^l) = S^i, \quad (44)$$

where  $\alpha$  is a constant number. For  $\alpha \neq -1$ , (44) yields an elliptic equation for  $V^i$ , conducting to a well posed boundary value problem. The particular case with  $\alpha=1/3$ , hence

$$\Delta V^i + \frac{1}{3} \nabla^i (\nabla_l V^l) = S^i, \quad (45)$$

is generic part of the York procedure for the solution of the initial value problem of general relativity (York 1983). Therefore, numerous attempts of a numerical solution of (45) have been made in the past. Bowen (1979) has suggested a simplified set of equations by solving for the auxiliary variables  $W^i$  and  $U$  defined through

$$V^i = W^i - \frac{1}{4} \nabla^i U. \quad (46)$$

With this definition, (45) reduces to two Poisson equations for  $W^i$  and  $U$ , namely

$$\Delta W^i = S^i, \quad \Delta U = \nabla_l W^l. \quad (47)$$

This approach has been widely used in analytical and numerical work. Note, however, that Evans (1984) was able to construct a suitable Green's function in Cartesian coordinates. It incorporates the boundary condition  $V^i=0$  at spatial infinity and reads explicitly

$$G_{ij}(\mathbf{x}, \mathbf{x}') = -\frac{7}{32\pi |\mathbf{x}-\mathbf{x}'|} \left[ \delta_{ij} + \frac{(x_i-x'_i)(x_j-x'_j)}{7|\mathbf{x}-\mathbf{x}'|^2} \right], \quad (48)$$

where  $\delta_{ij}$  is the flat space metric tensor.  $G_{ij}$  satisfies

$$[\Delta G_{ij} + \frac{1}{3} \nabla_i (\nabla^l G_{lj})] (\mathbf{x}-\mathbf{x}') = \delta_{ij} \delta^{(3)}(\mathbf{x}-\mathbf{x}'). \quad (49)$$

The elliptic equation (45) is hence transformed into the integral equation

$$V^i(\mathbf{x}) = \int_{R^3} G^i_l(\mathbf{x}-\mathbf{x}') S^l(\mathbf{x}') d^3x'. \quad (50)$$

In general, calculations have been carried out in Cartesian type coordinates which considerably simplify the solution of the elliptic equations (47), because the vector Poisson equation reduces to *independent* scalar Poisson equations for the individual components  $V^x$ ,  $V^y$ , and  $V^z$ . Their use allows further to circumvent the difficulties associated with the coordinate singularities of spherical or cylindrical coordinates when using finite difference methods. Nevertheless, in astrophysical applications, the use of spherical or cylindrical coordinates is often much more adapted to the geometry of the problem where, for instance, one encounters a matter distribution with compact support. Note, however, that because the choice of the local vector basis is arbitrary, one might employ Cartesian components that are functions of the spherical coordinates  $(r, \theta, \phi)$  as well. This would allow to adopt Bowen's scheme (47) unaltered. Boundary conditions may, however, take a simpler form for the spherical components  $V^r$ ,  $V^\theta$ , and  $V^\phi$ .

We propose a new solution method that has been designed to fit specifically to problems in a spherical coordinate system

when the use of spherical vector components is desirable. Because our spectral method takes into account the intrinsic regularity properties of analytic functions, the first step consists in decoupling the (irrotational) scalar from the (divergence-free) pure vector part of the involved vector quantities in order to obtain a system of ordinary Poisson equations analogous to (47). We introduce vector fields  $\tilde{V}^i$  and  $\tilde{S}^i$ , representing the divergence-free part of  $V^i$  and  $S^i$  respectively, as well as two scalar potentials  $\Psi$  and  $\Phi$ . Appropriate boundary conditions supplied, one has a unique decomposition

$$V^i = \tilde{V}^i + \nabla^i \Psi, \quad S^i = \tilde{S}^i + \nabla^i \Phi. \quad (51)$$

In a first step, we obtain a Poisson equation for  $\Phi$  by computing the divergence of (51)

$$\Delta \Phi = \nabla_l \tilde{S}^l. \quad (52)$$

Eq. (44), expressed in terms of the new variables, then reads

$$\Delta \tilde{V}^i + \nabla^i ((1+\alpha) \Delta \Psi - \Phi) = \tilde{S}^i. \quad (53)$$

Taking the divergence of (44),  $H = ((1+\alpha) \Delta \Psi - \Phi)$  turns out to be a harmonic function, thus satisfying  $\Delta H = 0$ . We seek regular and bounded solutions to the initial equation in  $R^3$  and therefore choose  $H=0$ . Consequently,  $\Psi$  is determined by the Poisson equation

$$(1+\alpha) \Delta \Psi = \Phi. \quad (54)$$

Combining (54) with (53), we obtain a vector Poisson equation for  $\tilde{V}^i$  which is just (44) applied to divergence-free vector fields  $\tilde{V}^i$  and  $\tilde{S}^i$ ,

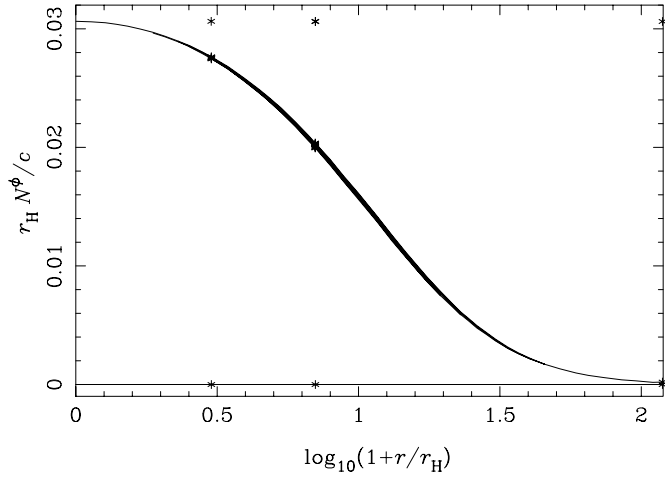
$$\Delta \tilde{V}^i = \tilde{S}^i. \quad (55)$$

Eqs. (54) and (55) resemble (47), but the additional constraint  $\nabla_l \tilde{V}^l=0$  considerably simplifies the solution of (55) in spherical coordinates when the standard orthonormal frame is used. No distinction between contra- and covariant vector components is made in this case. We further drop the tildes on top of the vector indices which had been introduced in Sect. 2.3 to distinguish the pseudo-physical components. The explicit form of  $\Delta V^i = S^i$  then reads

$$\Delta_3 V^r - \frac{2V^r}{r^2} - \frac{2}{r^2} \left( \frac{\partial}{\partial \theta} + \frac{1}{\tan \theta} \right) V^\theta - \frac{2}{r^2 \sin \theta} \frac{\partial V^\phi}{\partial \phi} = S^r, \quad (56)$$

$$\Delta_3 V^\theta - \frac{V^\theta}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial V^r}{\partial \theta} - \frac{2}{r^2 \sin \theta \tan \theta} \frac{\partial V^\phi}{\partial \phi} = S^\theta, \quad (57)$$

$$\Delta_3 V^\phi - \frac{V^\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial V_r}{\partial \phi} + \frac{2}{r^2 \sin \theta \tan \theta} \frac{\partial V^\theta}{\partial \phi} = S^\phi. \quad (58)$$



**Fig. 1.**  $N^\phi$  for a rapidly rotating Kerr black hole at  $a/M=0.99$  where  $r_H$  denotes the radius of the horizon. The different curves correspond to various values of  $\theta$  ranging from  $0^\circ$  to  $90^\circ$ . The asterisks indicate the sub-domain boundaries.

The solution of (56) to (58) is obtained in the following way: combining the divergence-free condition on  $V^i$ ,

$$\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r} \frac{\partial V^\theta}{\partial \theta} + \frac{V^\theta}{r \tan \theta} + \frac{1}{r \sin \theta} \frac{\partial V^\phi}{\partial \phi} = 0, \quad (59)$$

with (56) eliminates all components but  $V^r$  and leads to an ordinary scalar Poisson equation for the auxiliary variable  $\hat{V}^r = r V^r$ ,

$$\Delta_3 \hat{V}^r = r S^r. \quad (60)$$

In order to simplify (57) and (58), we express  $V^\theta$  and  $V^\phi$  according to

$$V^\theta = \frac{1}{r} \frac{\partial U}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi}, \quad V^\phi = \frac{1}{r} \frac{\partial W}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi}, \quad (61)$$

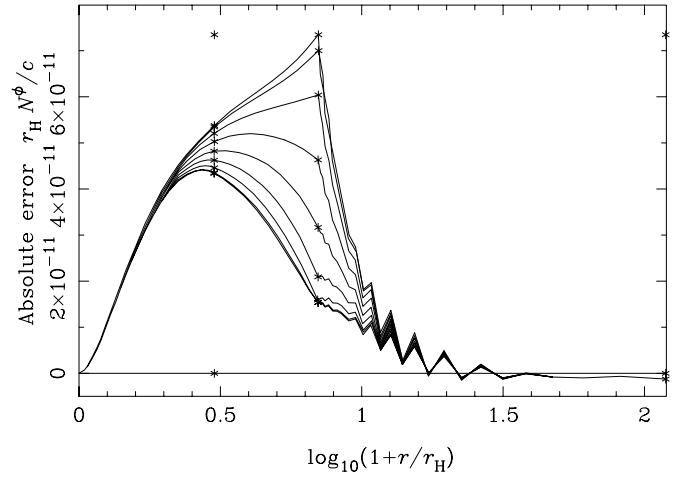
where  $U$  and  $W$  are two auxiliary scalar potentials. Making once more use of (59), the original equations (57) and (58) are replaced by

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{\partial^2 U}{\partial r^2} - \frac{\partial V^r}{\partial r} \right] - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left[ \Delta_3 W - \frac{2}{r} \frac{\partial W}{\partial r} \right] = S^\theta, \quad (62)$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left[ \frac{\partial^2 U}{\partial r^2} - \frac{\partial V^r}{\partial r} \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \Delta_3 W - \frac{2}{r} \frac{\partial W}{\partial r} \right] = S^\phi. \quad (63)$$

An equation that only involves  $(\partial_r U - V^r)$  is obtained after differentiation of (62), multiplied by  $r \sin \theta$ , with respect to  $\theta$  and of (63), multiplied by  $r$ , with respect to  $\phi$ . After integration over  $r$  where the integration constant is set to zero in order to make the source term vanish at spatial infinity, one is left with an equation that only involves the angular part of the scalar Laplacian, namely

$$\frac{1}{r^2} \left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \left[ \frac{\partial U}{\partial r} - V^r \right] = -S^r. \quad (64)$$



**Fig. 2.** Absolute error in  $N^\phi$  corresponding to Fig. 1.

Since  $V^r$  has already been determined by means of (60),  $U$  can be computed immediately after solution of (64). In order to fix the lacking potential  $W$ , an ordinary integration of (63) over  $\phi$  is carried out, and the integration constant is set to zero to comply with the required vanishing behaviour of the source terms of the resulting equation. The final equation for  $\hat{W}^r$ , defined by  $W = r \hat{W}^r$ , then becomes

$$\Delta_3 \hat{W}^r = \int_0^\theta \left[ S^\phi - \frac{1}{r \sin \theta'} \frac{\partial}{\partial \phi} \left( \frac{\partial^2 U}{\partial r^2} - \frac{\partial V^r}{\partial r} \right) \right] d\theta'. \quad (65)$$

The system of equations constituted by (60), (64), and (65) is equivalent to (55) and significantly simplifies the numerical solution. Note the absence of  $S^\theta$  in the source terms of the final equations. This reflects the dependency of the components of  $S^i$  due to the constraint  $\nabla_i S^i = 0$ .

Based on the above approach, we have realized a numerical scheme that solves (44) in a multi domain configuration. The computational domain may be extended to spatial infinity thanks to a suitable transformation of the radial variable in the exterior zone (see Bonazzola et al. (1993) for a previous application of this method).

Numerical tests based on simple analytic functions revealed the well known *evanescent* error characteristic for spectral methods where the numerical errors quickly reach the round-off limit of  $\simeq 10^{-14}$  of the employed machine. It has further been applied to solve the general relativistic field and matter equations for numerical models of self-gravitating matter surrounding a rotating black hole which will be presented elsewhere (Bonazzola & Gourgoulhon, in preparation). For vanishing matter density, the Kerr vacuum solution is recovered and allows a direct comparison with the analytic solution expressed in quasi-isotropic coordinates. The *coordinate base* shift vector component  $N^\phi$  near the horizon of a rapidly rotating black hole, characterized by  $a/M=0.99$ , is shown in Fig. 1 whereas the involved numerical error is illustrated in Fig. 2. The outside of the black hole is covered by three grids where the exterior compactified one extends to spatial infinity. The actual grid res-



olution is  $N_r = 33$  in each zone and  $N_\theta = 7$ , covering the interval  $0 \leq \theta \leq \pi/2$ . The numerical error committed on  $N^\phi$  nowhere exceeds  $10^{-10}$  with respect to the analytic solution and decreases even further to  $\simeq 10^{-13}$  for higher values of  $N_r$ . Note the very high precision of the numerical model in spite of the low number of collocation points, which shows the power of spectral methods for problems where the involved quantities can be represented by analytic functions.

### 3.3. Validation of the code

The numerical results in the axisymmetric case have been verified repeatedly by different tests, based on comparison with analytic solutions or numerical models provided by other authors (Eriguchi et al., in preparation), as well as by exploiting intrinsic error indicators like the general relativistic virial identities (Bonazzola 1973; Bonazzola & Gourgoulhon 1994; Gourgoulhon & Bonazzola 1994) or analytic properties of involved quantities. Typical global errors for rapidly rotating neutron star models based on a  $\gamma = 2$  polytropic equation of state are about  $10^{-6}$ . This value has to be confronted with a global error of  $10^{-14}$  for non-rotating, spherical configurations. This difference stems from a certain deficiency of the spectral approximation of *non-analytic* functions. In the spherical static case, the boundary of the inner spherical grid can be chosen to coincide with the star surface. As a consequence, all quantities are analytic in the adjacent sub-domains, and the numerical error is dominated by the roundoff error of the employed machine for a moderate number of grid points thanks to the exponential decrease of the residual numerical errors as a function of the number of grid points. For *non-analytic* functions, the convergence slows down considerably and is worst for discontinuous functions — this behaviour corresponds to the well known Gibbs phenomenon of Fourier series. In the rotating case, the star surface is located *inside* the central grid zone. Therefore, matter variables like the energy density, whose derivatives are discontinuous across the star boundary and even become singular for  $\gamma > 2$ , are no more analytic functions in this domain and suffer hence from the above deterioration. For the current study which is only concerned with a polytropic equation of state  $p = \kappa \rho^\gamma$ , this signifies that  $\gamma$  has to be limited to values for which the spectral approximation still works reasonably well. In practice,  $\gamma$  is bounded by a maximal value  $\gamma_{\max} = 3$ . In particular, incompressible fluid configurations such as the Maclaurin or Jacobi spheroids are beyond the scope of the present code. Since the range of astrophysical interest is still well within in the current limits, these restrictions are not prohibitive.

Because of the lack of concurrent studies in the relativistic regime, the validation of the non-axisymmetric part of the code is limited to the Newtonian limit where some results for compressible and incompressible fluids are available, either obtained by analytic or numerical studies. The different tests (Bonazzola et al. 1996) include in particular a confirmation of James' value of the critical adiabatic index, yielding  $\gamma_{\text{crit}} = 2.238 \pm 0.002$ , which is in perfect agreement with the predicted value. A classical result from the theory of Newtonian incompressible fluid

bodies is the particular value  $T/|W|_{\text{crit}} = 0.1375$  at the bifurcation point of the secular bar mode instability. Since we cannot verify this result directly because of the above restrictions imposed on the equation of state (the incompressible case corresponds to  $\gamma \rightarrow \infty$ ), we studied the dependence of  $T/|W|$  for increasing  $\gamma$  and found the successive values to converge to the asymptotic one. A comparison with results of Ipser & Managan (1981) and Hachisu & Eriguchi (1982) in the compressible polytropic case for a fixed  $\gamma = 3$  showed an agreement of better than 0.5% for the critical angular velocity  $\Omega_K$  and of about 2% for the values of  $T/|W|_{\text{crit}}$  (cf. Table 1 of Bonazzola et al. 1996).

The solution of the 3D-shift vector equation is computed following the method presented in Sect. 3.2 and its implementation is completely independent from the former scheme where only one axisymmetric equation for the only non-vanishing shift vector component  $N^\phi$  was solved. The deviation of  $N^\phi$  for identical models is of order  $10^{-6}$  and corresponds to the global numerical error of a particular configuration labeled by the model parameters  $H_c$  and  $\Omega$ .

## 4. Results for polytropes

As mentioned in Sect. 1, earlier investigations in the Newtonian or at most post-Newtonian regime in general made use of polytropic equations of state which represent simplified but within certain limits reasonable and *consistent* templates of realistic equations of state. The *critical* adiabatic index  $\gamma_{\text{crit}}$  is defined as the value of  $\gamma$  for which the critical angular velocity  $\Omega_{\text{crit}}$  coincides with the Keplerian angular velocity  $\Omega_K$ , the maximum angular velocity supported by a rapidly rotating star without mass shedding from the equator. For  $\gamma < \gamma_{\text{crit}}$ , the star thus cannot rotate rapidly enough for the secular instability to develop. If  $\gamma > \gamma_{\text{crit}}$ , there always exists a certain  $\Omega_{\text{crit}} < \Omega_K$ . Provided some mechanism that fuels the star with rotational kinetic energy, it may be affected by the secular instability.

Newtonian polytropic stars obey a scaling law and as a consequence,  $\gamma_{\text{crit}}$  is a global constant that does not depend on other model parameters. For relativistic configurations, effects of general relativity are likely to influence the bar mode instability. Therefore,  $\gamma_{\text{crit}}$  will be a function of the strength of the relativistic fields which we measure by the value  $N_c$  of the lapse function at the centre of the star. Fig. 3 shows this dependence for the case of the 2D-shift vector equation of our previous investigation as well as the modified graph for the present approach which includes the solution of the 3D-shift vector equation.  $\gamma_{\text{crit}}$  increases only slightly in the first case. One even notes a tiny decrease in the weak field regime. For the present approach, the situation changes quite dramatically. In the Newtonian regime, both graphs converge to James' classical value, which, of course, is required by consistency: for decreasing field strength the shift vector contribution dies out more rapidly than that of the lapse function, so that in both cases the asymptotic space-time line element is given by (11). In the relativistic domain, however, the shift vector shows to inhibit the secular instability more and more efficiently. The calculations are not continued beyond  $\gamma = 3$

where the steepening gradients of the matter variables severely affect the spectral approximation.

The question which arises immediately addresses the origin of the very different behaviour for the both approximations. Let us recall the classical limit of the first integral of motion

$$h + U - \frac{1}{2} \Omega^2 \rho^2 = \text{const} . \quad (66)$$

The evolution of a perturbed configuration is driven by the mutual influence of the gravitational potential  $U$  and the enthalpy  $h$  which are the only three-dimensional quantities in this case. The centrifugal potential is axisymmetric and remains unchanged. Since relativistic effects tend to increase the overall gravitational forces, one might expect that relativistic corrections to  $U$  inhibit the symmetry breaking. However, the lower graph in Fig. 3, which illustrates the results of our previous investigation, shows an only slight growth of  $\gamma_{\text{crit}}$  and, moreover, even a *destabilizing* effect in the weakly relativistic regime. In the general relativistic case, (66) is replaced by

$$H + \nu - \ln \Gamma = \text{const} . \quad (67)$$

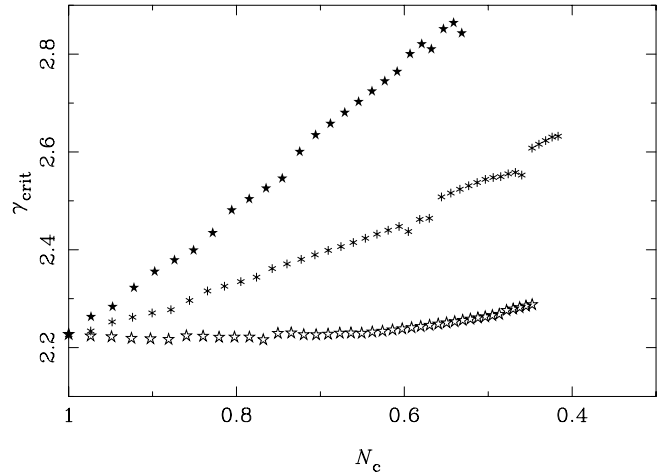
The crucial point is that now the “centrifugal” potential  $U_\Omega = -\ln \Gamma$  depends on the azimuthal angle as well. This means, the angular modulation of  $U_G = \nu$  is superimposed by that of  $U_\Omega$ , the contribution from the latter being a purely relativistic effect without Newtonian counterpart. The variation of  $U_\Omega$  with respect to  $N$  may then enhance or diminish the triaxial perturbation of the effective relativistic potential, depending on its sign and, via the first integral of motion, prevent or favour the triaxial deformation of the fluid body. The relativistic centrifugal potential  $U_\Omega$  according to (37) reads

$$U_\Omega = -\ln \Gamma = \frac{1}{2} \ln(1 - U_\phi^2) , \quad (68)$$

where  $U_\phi$ , the physical fluid velocity in  $\phi$  direction as measured by the local Eulerian observer  $\mathcal{C}_0$ , specified in (35), has the form

$$U_\phi = \frac{\tilde{B}}{N^2} r \sin \theta (\Omega - N^\phi) . \quad (69)$$

In the 2D-shift vector case, the only three-dimensional quantity in (69) is the lapse function  $N$ . The variation  $\delta U_\Omega$  of  $U_\Omega$  with respect to  $N$  adds to  $\delta U_G$  with a *positive* sign: the triaxial perturbation of the relativistic potential is enhanced. This explains the destabilizing effect in the weak field regime before the overall growth of the relativistic forces related to  $U_G$  dominates in the strong field region. The computation of the shift vector in a three-dimensional fashion leads to an additional variation of the term  $(\Omega - N^\phi)$  which contributes to  $\delta U_\Omega$  with a *negative* sign. It competes with the contribution of the lapse function  $N$  to  $\delta U_\Omega$ , acting in the opposite sense. It turns out that the combined effect of both contributions is dominated by the stabilizing effect of the shift vector, which corresponds to the upper curve in Fig. 3. From the Newtonian limit on, the symmetry breaking is increasingly suppressed. The exclusive influence of the relativistic gravitational potential  $U_G$  can be studied by averaging  $U_\Omega$  over  $\psi$ . This corresponds to the Newtonian situation, and the



**Fig. 3.** Critical adiabatic index  $\gamma_{\text{crit}}$  as function of  $N_c$ , the lapse function at the centre of the star. The upper curve is obtained with solution of the full 3D-shift vector equation whereas the lower one illustrates the dependence for the axisymmetric computation of the shift vector. The intermediate curve shows the analogous curve for the first case with the axisymmetrized centrifugal potential.

resulting curve for the critical adiabatic index in this case has been added to Fig. 3. It is located between the two other curves, obtained with a three-dimensional  $U_\Omega$ , and validates our above reasoning. We have confirmed numerically that the fluid velocity  $U_\phi$ , appearing in the relativistic centrifugal potential  $U_\Omega$ , is the *only* quantity where the three-dimensional character of the shift vector actually affects the bar mode instability. We have further checked the robustness of our results by adding further three-dimensional terms to the field equations due to an additional extra-diagonal element in the metric tensor or to treating  $\tilde{A}$  and  $\tilde{B}$  as “pseudo” 3D-variables. No noticeable modification of the results has been observed. These numerical tests give us confidence in the analytic approximation and indicate that an additional 3D-treatment of the tensor part of the space-time metric tensor is unlikely to modify the present results significantly.

As concerns Wilson’s approach, we have carried out the above calculations by imposing  $\tilde{A} \equiv \tilde{B}$  which mimics the “conformally flat condition” except that we do not solve for the fully three-dimensional conformal factor but rather adopt  $\tilde{A}^2 N^{-2}$  as conformal factor where  $\tilde{A}$  remains axisymmetric. The global numerical error of the initial axisymmetric configurations as measured by means of the general relativistic virial theorems is of order  $10^{-4}$  which has to be compared with values of  $10^{-6}$  for solution of the exact equations. The results for the critical adiabatic index  $\gamma_{\text{crit}}$  appear to be rather insensitive to this simplification. The actual values coincide with those by solution of the original equations within a relative error of about  $10^{-3}$  and confirm the validity of Wilson’s simplified scheme for quasi-equilibrium configurations at least for this particular application.

## 5. Conclusion

The present work has studied the viscosity driven secular instability of rapidly rotating stars in general relativity by solution of an approximate set of field equations related to Wilson's scheme for quasi-equilibrium configurations. It has been revised to admit a non-conformal metric tensor which allowed us to recover the exact equations for stationary and axisymmetric configurations in contrast with the simplifying "conformally flat condition". The main improvement with respect to our previous study consists in the fully three-dimensional treatment of the shift vector. The numerical scheme has been applied to configurations built on a polytropic equation of state. The shift vector shows to strongly enhance the stability of relativistic configurations compared to the contribution of the lapse function only. The weak variation of the critical adiabatic index  $\gamma_{\text{crit}}$  in the previous investigation can be explained by a partial cancellation of certain non-axisymmetric terms in the first integral of motion. Comparative calculations adopting the simplifying "conformally flat condition" essentially yield the same results. This is intelligible, having in mind that even for maximum rotation neutron star models the deviation of the geometry of curved space from conformal flatness is of order  $10^{-3}$ .

Nevertheless, the symmetry breaking is still possible for a mass of  $M_{\text{ns}} = 1.4M_{\odot}$  and an adiabatic index of  $\gamma = 2.5$ . A future investigation has to address the question, if realistic models of neutron star matter still admit the symmetry breaking for astrophysically relevant masses, as it was the case in our previous investigation with an axisymmetric shift vector.

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