

On the stability of motion of N -body systems: the effect of the variation of particle number, softening and rotation

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Received 6 March 1997 / Accepted 6 October 1997

Abstract. Using the Ricci and scalar curvatures of the configuration manifold of gravitational N -body systems, we study the exponential instability in their trajectories. It is found that the exponentiation time-scale for isotropic Plummer spheres varies very little with particle number if the softening is small. Large softening on the other hand has a marked effect and, if large enough, can cause the curvatures to become positive. This last result confirms the previous observations for self gravitating sheets and suggests that the *qualitative* behaviour of large- N and continuum systems may be different, and that their equivalence is only obtained in the limit of infinite N and finite softening. It is also found that the presence of a large fraction of the kinetic energy in rotational motion increases the exponentiation time-scales significantly – an effect that should be expected given the regular nature of nearly circular motion. In the light of the results of this and of previous studies, it is suggested that the exponential instability may arise from low order resonances between the period of the variation of the gravitational field due to distant encounters and the orbital period of a test particle. For periods long compared to the exponentiation time but short compared to the diffusion time-scales of the action variables, the standard picture of collisionless dynamics may be valid in an averaged sense – nevertheless this time interval need not coincide with that predicted by standard relaxation theory. Instead it is suggested that, at least for systems with well defined final states, the relaxation time should scale as $\sim N^{1/2}$.

Key words: stellar dynamics – galaxies: kinematics and dynamics – galaxies: evolution

1. Introduction

It is well documented in numerical simulations that N -body gravitational systems display an exponential instability with respect to small changes in the initial conditions (e.g., Miller 1964; Goodman et al. 1993; Kandrup et al. 1994). This instability not only appears when the linearised dynamics is studied but also

when the full nonlinear evolution of two originally similar systems is considered. It does not appear therefore to be simply a product of linearisation of the equations of motion – a linearisation instability. It is possible however (El-Zant 1996a) that the exponential instability observed in short time calculations of the divergence of temporal states may result from phase mixing. However there is another way in which the instability manifests itself, namely through the predominantly negative two dimensional curvature of the configuration space of N -body systems (Gurzadyan & Savvidy 1984,1986; Kandrup 1990a, 1990b). This property arises from the qualitative phase space structure of a system and cannot be explained away so easily. It necessarily implies instability *normal* and not just along the phase space¹ trajectory of a system (a property known in dynamical literature as transversality: e.g., Ruelle 1989), which implies *qualitatively* different behaviour from that of regular systems. Since, if real, such an effect could have far reaching implications on the evolution of gravitational systems, it is natural to enquire as to how the predicted instability correlates with various parameters of N -body systems in an attempt to uncover its origin and implications. Since not all two dimensional curvatures are always negative during the evolution of these systems, and in any case, these depend on the full Riemann tensor (difficult to calculate for all but the smallest systems) an averaged chaos indicator still relying on the geometric approach has to be used.

Such a method, involving the Ricci (or mean) curvature, was devised by Gurzadyan & Kocharyan (1987) and was applied to numerically integrated N -body systems by El-Zant (1996a) and El-Zant & Gurzadyan (1997). In these papers it was shown that the stability of motion of N -body gravitational systems as described by the aforementioned method correlated strongly with various parameters of a gravitational system. For although the exponential instability (as evidenced by the negative Ricci curvature) exists for most initial conditions, it was observed for example that this instability was more pronounced when clear macroscopic (e.g., collective plasma type) instabilities were

¹ The term phase space in this paper will refer to the $6N$ dimensional phase space unless otherwise indicated.

present and in situations where one expects a faster evolution rate (e.g., in the presence of central concentration which accelerates the gravothermal evolution of gravitational systems towards still more concentrated states with higher thermodynamic entropy: Saslaw 1985). On the other hand, the predicted instability was much weaker for regular systems (e.g. for ones in uniform rotation before macroscopic instability starts to have a significant effect). Moreover, it was found (El-Zant 1996a) that the negativity of the Ricci curvature is not a result of contributions due to close encounters but in fact in spite of them – the curvature became always negative only when the fluctuations due to the closest encounters were removed. In fact, for a system in statistical equilibrium, the Ricci curvature is constant (up to random fluctuations and neglecting close encounters) and negative and mainly determined by the first derivatives of the potential. This suggests that the instability arising from this property is unlikely to be the result of the linearisation of a system containing fluctuating forces. In other words, it is likely to be the manifestation of chaotic behaviour characteristic of generic N -body gravitational systems. It is often assumed that if this is the case then the effect of the instability mainly concerns the accuracy of numerical simulations of N -body systems (e.g., Goodman et al. 1993; Miller 1994). This problem is also mentioned in many papers dealing with numerical simulation techniques (e.g., Aarseth 1996; Barnes & Hut 1989). Chaos however does not only affect numerical simulations, it usually comes with other deep implications. The most important is that it leads to diffusion in the action variables which determine the physical state of a system. This diffusion time-scale need not be determined by the classical two body theory – even if it is long compared to the exponentiation time – since this theory assumes an integrable system which remains so under perturbations due to discreteness. On the other hand, systems with negative curvature (whether constant or not) usually have strong statistical properties, very different from near integrable systems (e.g., Pesin 1989). It is therefore very plausible that there may be, contrary to what is often assumed, a physical meaning to the exponential instability. This meaning is however not yet completely clear.

In this paper we continue our investigation of the stability of motion of N -body systems using the Ricci and scalar curvatures of the configuration manifold of generalised coordinates (the method is briefly described in Sect. 2). We will be interested here in the case of spherical systems. The main questions we would like to ask are as follows. First, it has been observed (both in direct simulations and by using the Ricci curvature method) that the exponentiation time-scales are rather small (usually a fraction of a dynamical time). If this time-scale does not increase with N , then one must conclude that it cannot be directly related to the diffusion of the action variables in a system in virial equilibrium – and therefore any physical meaning will have to be more subtle. In Sect. 3 we examine the variation of that time-scale for N up to 15 000. For flat sheet like systems it was found (El-Zant 1996a) that significant softening destroyed the negative Ricci curvature of the configuration space, meaning that the structure of the phase-space should be different and that

the motion should be more regular. If this is the case, then the exponential instability must arise from discreteness effects and there has to be a discontinuity in the transition between the phase space structure of large (but finite) N -body gravitational systems and their continuum counterparts. In Sect. 4 we study the effect of softening in spherical Plummer models and its possible implications in some detail. Another problem is concerned with the fact that many galaxies are observed to be in fairly regular states where most of the kinetic energy is in the form of ordered rotational motion. Again, a small exponentiation time-scale (if directly interpreted) is incompatible with such motion – especially if the divergence takes place from most initial conditions, in which case there is no room for “stable chaos” or trapping between KAM tori (as for example suggested by Gouda et al. 1994). It is therefore important to see if the presence of large amount of rotational motion does influence the relaxation time-scales significantly. This effect is studied in Sect. 5. In the final section we summarise the results and describe a possible interpretation of the origin and effect of the exponential instability of N -body systems and how this can be tested.

2. Method

The study of the stability properties of gravitational N -body systems using the Ricci curvature was initiated by Gurzadyan & Kocharyan (1987). Details of how the Ricci curvature can be used to study the stability of N -body trajectories are given in El-Zant (1996a). We just mention here a few points that are essential to the interpretation of the results of the following sections.

Through the Lagrangian formulation of dynamics, it is possible to reduce the study of the stability of motion of N -body systems to that of the geometry of the corresponding Lagrangian manifold. When this is done, the Gaussian curvatures on any two dimensional directions normal to the motion are found to be mostly (but not always) negative (e.g., Gurzadyan & Savvidy 1986; Kandrup 1990a). Since for large N the probability any of the two dimensional curvatures being positive becomes exceedingly small, it is possible to replace the full set of two dimensional curvatures by their average over the $3N - 1$ directions normal to the geodesic velocity vector \mathbf{u} . This quantity also happens to coincide with the Ricci curvature of the manifold. In Cartesian coordinates in the enveloping $3N$ configuration space this is given by

$$r_u = 3A \frac{(W_i u^i)^2}{W^2} - 2A \frac{W_{ij} u^i u^j}{W} - \left(A - \frac{1}{2} \right) \frac{\|\nabla W\|^2}{W^3} - \frac{\nabla^2 W}{2W^2} \quad (1)$$

with $A = \frac{3N-2}{4}$. Here W is the kinetic energy calculated as a function of the potential energy while $\nabla W = \sum W_i$ and $\nabla^2 W = \sum W_{ii}$. For a N -body system, the implied summation would be over $i, j = 1, 3N$. If we label by a, b and c the particle numbers (which run from 1 to N) and by k and l the three

Cartesian coordinates of a particle, then for particles with same mass m and with $G = 1$ we have

$$W_i = \frac{\partial W}{\partial q_i} = \frac{1}{\sqrt{m}} \frac{\partial W}{\partial r_a^k} = -m^{3/2} \sum_{c \neq a} \frac{r_{ac}^k}{r_{ac}^3}, \quad (2)$$

$$W_{ij} = \frac{\partial^2 W}{\partial q_i \partial q_j} = \frac{1}{m} \frac{\partial^2 W}{\partial r_b^k \partial r_a^l} = m \left[\frac{\delta_{kl}}{r_{ab}^3} - \frac{3r_{ab}^k r_{ab}^l}{r_{ab}^5} \right], \quad (3)$$

if $a \neq b$ and

$$W_{ij} = \frac{\partial^2 W}{\partial q_i \partial q_j} = \frac{1}{m} \frac{\partial^2 W}{\partial r_b^k \partial r_a^l} = -m \sum_{c \neq a} \left[\frac{\delta_{kl}}{r_{ac}^3} - \frac{3r_{ac}^k r_{ac}^l}{r_{ac}^5} \right], \quad (4)$$

if $a = b$. In these equations

$$r_{ab}^2 = (r_{ab}^1)^2 + (r_{ab}^2)^2 + (r_{ab}^3)^2 + b^2, \quad (5)$$

where b is a softening parameter and $r_{ab}^k = r_a^k - r_b^k$.

A negative Ricci curvature can be interpreted to imply that a system will, in general, display exponential instability to random perturbations (El-Zant 1996b). The associated exponentiation time-scale is then given by

$$\tau_e \sim \left(\frac{3N}{-2\bar{r}_u} \right)^{1/2} \times \frac{1}{\bar{W}}, \quad (6)$$

where a bar denotes a phase space average (can be either static or dynamic).

When the two dimensional curvatures are averaged over all geodesics that pass by a given point as well as over the directions normal to the velocity vector at that point, one obtains the curvature scalar

$$R = \sum_{\mathbf{u}, \mathbf{n}} k_{\mathbf{u}, \mathbf{n}} = \sum_{\mathbf{u}} r_{\mathbf{u}}. \quad (7)$$

This quantity turns out to be always negative for $N > 2$. This means that a N -body system will, in general, display exponential instability when randomly perturbed and for random distribution in velocity space. An associated instability time-scale can also be inferred: One replaces $r_{\mathbf{u}}$ in Eq. (6) by R and $3N$ by $(3N)^2$. Clearly the time-scale derived from the scalar curvature can be equal to that inferred from the Ricci curvature only if no ordered motion is present. We will see that this is indeed the case.

It is the exponentiation time-scales that will be of interest in the coming sections. We will be averaging them either over (pseudo)random realizations of N -body systems or over time-averages for numerically integrated systems. In the latter case, we use the Aarseth (1996) NBODY2 code which is a direct summation code applying the Ahmad & Cohen (1973) neighbour scheme and individual time-steps for each of the particles in the simulation. These refinements speed up the integration considerably, while essentially maintaining the accuracy and simplicity of direct summation codes. Since we are still in the exploratory stages of applying and testing geometric methods to gravitational systems, it is prudent to study the behaviour of the

curvatures when calculated in an accurate and straightforward manner before integrating it into large- N codes. The parameters of the code are fixed at the same values used in El-Zant & Gurzadyan (1997).

Finally, it is important to note here that the negativity of the curvatures is only one mechanism by which chaos can occur according to the geometric formulation of dynamics. Another mechanism is provided by parametric instability as pointed out by Pettini (1993) and Cerruti-Sola & Pettini (1995). However, when N is large and the system considered is near virial equilibrium, the mean potential energy does not undergo significant fluctuations. In that case the second and third terms of Eq. (26) of the latter paper, which depend on the time derivatives of the potential energy, are then very small. This leaves negative curvature as the only effective mechanism for instability.

3. Behaviour of the curvatures as particle numbers change

In this section we look at the effect of varying the particle number on the exponentiation time-scale of isotropic equilibrium Plummer models prepared using the method of Aarseth et al. (1974) and scaled to the units of Heggie & Mathieu (1986) by keeping the total mass and the gravitational constant equal to unity and the total energy fixed at $E = -0.25$. In this case the mean crossing time τ_c is equal to $2\sqrt{2}$ time units.

Direct summation routines are not well suited for integration of large N -body systems (except if used in conjunction with special hardware like GRAPE architecture which was not available to the author) and NBODY2 becomes very slow for particle numbers exceeding a few thousand as to prevent systematic examination of the dynamics for greater particle numbers. Fortunately however, the fact that (give or take random fluctuations) the curvatures are constant for systems in statistical dynamical equilibrium (Eq. (1)) means that, for such systems, it is possible to get an estimate of these quantities by simply calculating them for different (pseudo)random realizations of the same one particle phase space distribution. This in turn is sufficient to give us an idea of the values of the curvatures for fairly large spherical N -body systems in statistical dynamical equilibrium.

For systems consisting of up to $N = 1400$, we integrated the full equations of motion and calculated the Ricci and scalar curvatures along the motion. This is done at intervals of a hundred starting for $N = 100$. The behaviour of the Ricci curvature time series during the evolution of such systems is described in El-Zant & Gurzadyan (1997). As in El-Zant & Gurzadyan (1997) we remove the contributions due to the closest encounters by introducing a short range cutoff in the calculation of the potential energy and its derivatives. We choose a distance (including the softening) of 0.05 for this (i.e., 5% the virial radius). This is typically about a third of the minimum radius of the neighbour spheres as calculated by the NBODY2 algorithm. For systems consisting of up to $N = 15000$ particles we calculate the Ricci and scalar curvatures for ten different realizations of the same equilibrium distribution and take the average. This is done starting at $N = 2000$ at intervals of a

Table 1. Variation with particle numbers of the exponentiation time-scales τ_{er} and τ_{es} calculated from the values of the Ricci and scalar curvatures. In the third and fifth columns the associated errors estimated with the aid of Eq. (8) are given

N	τ_{er}	ϵ_r	τ_{es}	ϵ_s
100	0.28	0.78	0.21	0.06
200	0.34	0.46	0.23	0.04
300	0.32	0.13	0.27	0.01
400	0.28	0.21	0.29	0.01
500	0.31	0.17	0.29	0.01
600	0.28	0.07	0.29	0.01
700	0.31	0.12	0.29	0.01
800	0.31	0.09	0.30	0.01
900	0.29	0.07	0.30	0.01
1000	0.31	0.09	0.30	0.01
1100	0.30	0.07	0.31	0.01
1200	0.31	0.05	0.31	0.01
1300	0.32	0.05	0.31	0.01
1400	0.31	0.05	0.31	0.01
2000	0.29	0.00	0.31	0.00
3000	0.31	0.00	0.31	0.00
4000	0.30	0.00	0.31	0.00
5000	0.31	0.00	0.31	0.00
10000	0.31	0.00	0.31	0.00
15000	0.31	0.00	0.31	0.00

thousand. To compare the results of the two approaches we take the average over the first ten outputs of the low N runs. This corresponds to one crossing time. Over this time interval the structure of the spheres are not expected to have changed much and, if indeed the motion is chaotic over small time-scales, these numbers also correspond to pseudorandom realisations of the same phase space distribution. The softening of the potential (which also follows the Plummer law) was taken as $b = 2/N$ which scales like the ratio of minimum to maximum impact parameters of standard relaxation theory (Binney & Tremaine 1987; Farouki & Salpeter 1994; Gierz & Heggie 1994). This is calculated as to exclude encounters (assumed independent) which lead to deflections beyond a certain maximal value.

Except in the case of $N = 100$ when fluctuations are fairly large, causing it to be occasionally positive, it is found that the Ricci curvature (calculated while excluding the contribution from neighbouring members as described above) is *always* negative. It is therefore easy to extract an average exponential divergence time-scale through the method described in the previous section. The results are shown in Table 1. In the first column are the particle numbers. In the second column and fourth columns are the exponentiation time-scales (in crossing times) averaged over ten different values of the Ricci or Scalar curvatures as described above, while the third and fifth columns contain estimates of the RMS relative dispersion in the ten calculated exponentiation time-scales. This is obtained from

$$\epsilon = \frac{\sqrt{\frac{\sum (\tau_e^i - \bar{\tau}_e)^2}{10}}}{\bar{\tau}_e}, \quad (8)$$

where $\bar{\tau}_e$ denotes the mean of the ten values.

Two things are immediately clear from these results. The first is that the exponentiation time-scale is quite short – being a fraction of the crossing time. The second is that there appears to be no increase in the exponentiation time-scale as N increases. This is true of both the estimates using the Ricci and the scalar curvatures. Although the exponentiation time-scales predicted by the latter quantity slightly increase with N when this number is limited to a few hundred particles, they quickly converge to the same value predicted by the Ricci curvature for larger N .

Perplexing as these results may seem, their numerical explanation is actually relatively straightforward. First, since the Ricci curvature is mainly determined by the bulk properties of the system and not by fluctuations due to nearest neighbours, one expects its average over N to remain constant with (large) increasing N , provided that these global properties (described by the one particle distribution function of the system) remain unchanged. This means that the exponentiation time-scale, which depends on that average (cf. Eq. (6)), remains constant for large enough N . The order of magnitude of the exponentiation time-scale can be understood as follows. The last term on the right hand side of Eq. (1) is very small when the softening is – the situation we are interested in here. The first term is also small if the forces on the particles are not aligned with their velocities (a very improbable situation in an equilibrium configuration). We are therefore left with the second and third terms in the expression for $r_{\mathbf{u}}$. For systems near virial equilibrium, and in the absence of ordered motion, the second term is dominated by contributions from close encounters. It is highly fluctuating and averages to zero when the softening is small (this statement however does not appear to hold when ordered motion is present as we shall see in Sect. 5 below). However, although its time averaged contribution is very small, it can have large positive values and, for small N , it occasionally causes the Ricci curvature to be positive if no short range cutoff is introduced. When such a cutoff is introduced however, this term is always small compared to the third. It also becomes small compared to the third term when the particle numbers are large, thus supporting the predictions of Gurzadyan & Savvidy (1984, 1986) and Kandrup (1990a, 1990b) that as N increases the curvature is more likely to become negative. This can be seen from Table 2 where the values of the two terms are shown for single (pseudo)random realizations of Plummer spheres consisting of up to 45 000 unsoftened particles (and no short range cutoff in the calculation of the potential and its derivatives).

The above explains why the exponentiation time-scales described by the Ricci and Scalar curvatures are similar since it is the third term that appears in the formula of the scalar curvature (e.g., Gurzadyan & Savvidy 1986). A simple argument, due to Kandrup (1989), shows why these time-scales should not vary much with particle number. Adopted for use in conjunction with the Ricci curvature, it goes as follows. For large N the Ricci curvature dominated by the last term becomes

$$r_{\mathbf{u}} \sim \frac{3N}{4} N \times \left(\frac{\bar{F}}{W} \right)^2, \quad (9)$$

Table 2. Magnitudes of the second and third term in Eq. (1) and of the resulting Ricci curvature for large N . Values are obtained from a single pseudorandom realizations of the Plummer models

N	$2A \frac{W_{ij} u^i u^j}{W}$	$-(A - \frac{1}{2}) \frac{\ \nabla W\ ^2}{W^3}$	$r_{\mathbf{u}}$
20 000	-1.5×10^5	-3.3×10^5	-4.9×10^5
25 000	-1.5×10^5	-3.9×10^5	-5.6×10^5
30 000	-9.4×10^4	-4.6×10^5	-5.6×10^5
35 000	1.7×10^5	-5.3×10^5	-3.5×10^5
40 000	3.0×10^4	-6.0×10^5	-5.7×10^5
45 000	2.4×10^5	-7.4×10^5	-4.9×10^5

where \bar{F} denotes the average total RMS force per unit mass acting on a test particle. Using Eq. (6), this implies an exponentiation time-scale of $\tau_e \sim \bar{v}/\bar{F}$, where \bar{v} is the RMS speed. This time-scale is of course of the order of a dynamical time.

4. Behaviour of the curvatures as the softening parameter is increased

In the case of $N = 1000$, we have calculated the Ricci and scalar curvatures for a range of softening radii starting from 10^{-3} units (i.e, one thousandth of the virial radius) to 4×10^{-2} units. In Fig 1 we plot the Ricci curvature as a function of the crossing time for some representative values of the softening parameter. These plots show that as this parameter increases so does the Ricci curvature, eventually it becomes positive just as in the case of the flat systems (El-Zant 1996a).

In Table 3 are shown the exponentiation time-scales calculated from the values of the Ricci and scalar curvatures averaged over the first crossing time for the various simulations that were run. Since the exponentiation time-scale has an inverse square root dependence on the Ricci curvature, it is relatively insensitive to this quantity unless it is close to zero. The range of softening parameters in which the transition occurs from negative values of $r_{\mathbf{u}}$ – corresponding to an exponentiation time-scale of one dynamical time – and positive ones is rather small. Therefore, although in principle arbitrarily large exponentiation time-scales can be obtained by fine tuning the softening parameter, in practice a softened system appears to have an exponentiation time-scale of the order of a dynamical time or none at all.

From Table 3 one can also see that, for large softening, the exponentiation time-scales associated with the Ricci curvature are rather large compared to those derived from the average value of the scalar curvature. This is because the second term in Eq. 1, which is not present in the formula for the scalar curvature, is *always* positive when significant softening is present. In turn this is because this term has contributions which contain the second derivatives in the form

$$\sum -\frac{\partial^2 W}{\partial q_i^2} u_i^2. \quad (10)$$

In a spherically symmetric system with positive density decreasing towards the outside, this expression is positive. It can be

Table 3. Variation of the exponentiation time-scales with the softening parameter ϵ_p . Other symbols are as in Table 1. The Ricci and scalar curvatures are positive for the last two values. For this reason no exponentiation time-scale is defined

b	τ_{er}	ϵ_r	τ_{es}	ϵ_s
0.001	0.30	0.09	0.30	0.02
0.002	0.30	0.07	0.30	0.02
0.004	0.31	0.08	0.30	0.02
0.006	0.31	0.09	0.30	0.01
0.008	0.31	0.09	0.30	0.02
0.010	0.32	0.08	0.30	0.02
0.012	0.33	0.09	0.32	0.02
0.014	0.34	0.08	0.33	0.02
0.016	0.35	0.10	0.34	0.02
0.018	0.37	0.11	0.35	0.02
0.020	0.40	0.13	0.37	0.02
0.022	0.42	0.12	0.39	0.02
0.024	0.47	0.17	0.41	0.03
0.026	0.56	0.45	0.44	0.03
0.028	0.67	0.62	0.49	0.06
0.030	0.86	0.43	0.56	0.07
0.032	1.46	0.82	0.66	0.10
0.034	–	–	–	–
0.040	–	–	–	–

fairly large when the softening is large. However it is still highly fluctuating – hence the large errors in the exponentiation time-scales. This is of course in addition to the last term in (1) which is also present in the formula for the scalar curvature and is always positive for systems with positive density.

A completely analogous situation transpires when one increases N with fixed softening and no short range cutoff. Independent of the softening length or its precise functional form, the Ricci curvature is positive and its average over N remains constant as one varies the particle number. This is easy to understand since particles which will contribute significantly to the local density at a given point are those within a distance of the order of magnitude of the softening length b . For high N the number of particles in such spheres increases as Nb^3 while the contribution to the density from each of the particles goes down as $1/b^3$.

At first sight, the above results may be taken to mean that the negative Ricci curvature is somehow caused by the singularity in the Newtonian potential or due to the contribution of nearby neighbours – and again the predicted instability may be a mathematical artifact. A closer look however reveals that this is not so. Near the transition from negative to positive curvature, the assumptions justifying the validity of the negativity of the Ricci curvature as an indicator of average instability break down (since in this case many of the two dimensional curvatures will already be positive and the variation in their absolute values may be very large). The positivity of the Ricci curvature in this case only means that the *relative* motion of nearby particles is regular since the (large) force between them is approximately constant with distance. These contributions dominate the value of the Ricci curvature. A similar situation for example occurs if

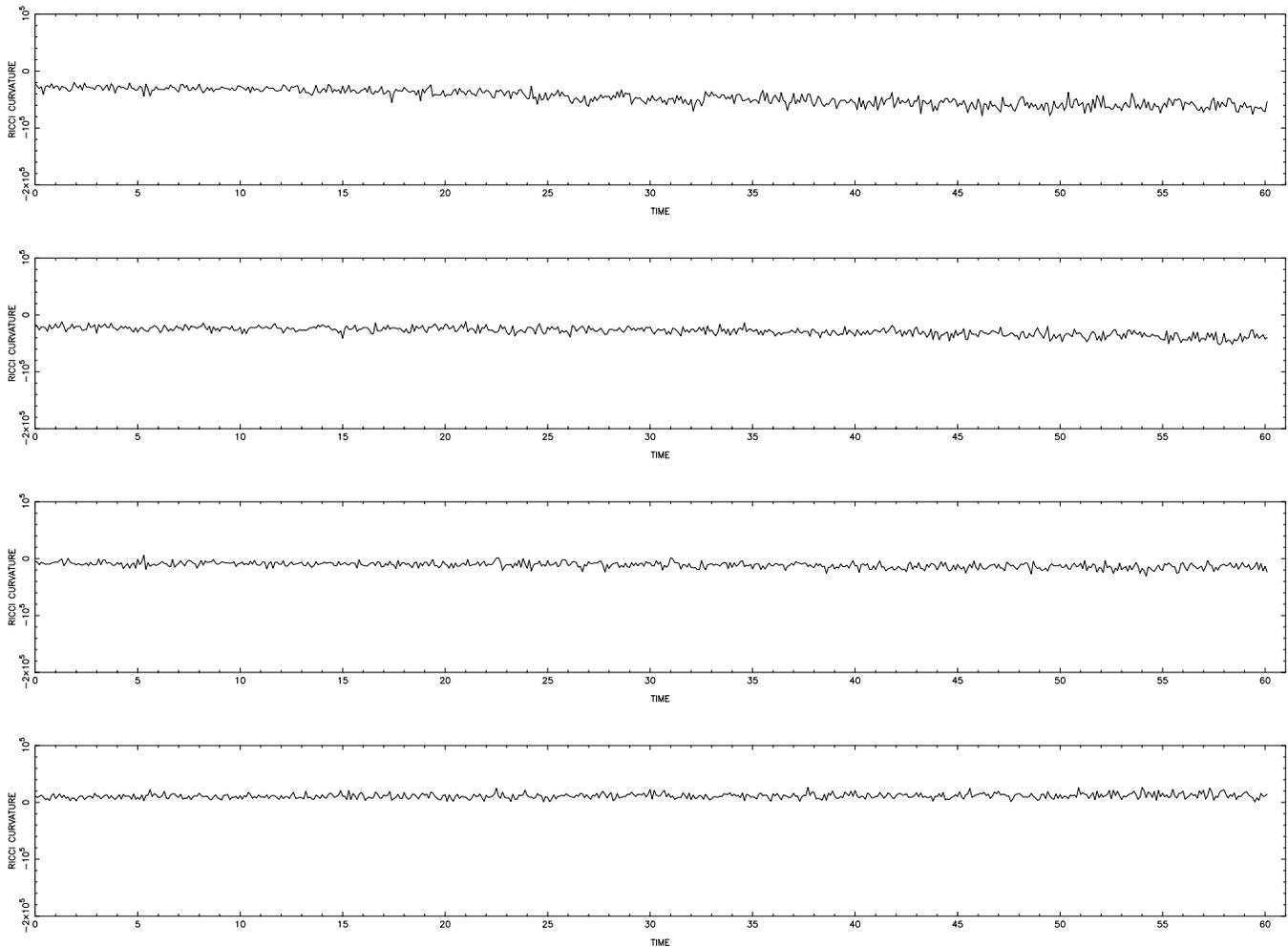


Fig. 1. Evolution of the Ricci curvature for $N = 1000$ systems with different values for the softening parameter b which takes the values of (from top to bottom) 4.0×10^{-3} , 1.6×10^{-2} , 2.8×10^{-2} and 4×10^{-2}

one encloses the system in an “elastic sphere” where particles near the boundary are subjected to a harmonic potential. If the spring constant is large, the curvature is dominated by the resulting positive contributions from these particles (this experiment was actually conducted by the author). Obviously the nature of the gravitational dynamics in this case is not radically modified.

The transition to positive curvature therefore should not be viewed as indicating a sharp switch from a chaotic to an integrable system but from one where the majority of trajectories were highly unstable to one where their instability is somewhat less pronounced (how pronounced can only be determined by calculating the two dimensional curvatures and integrating the full set of linearised equations). Only in the true continuum limit (infinite N and fixed softening) is full integrability recovered (it is interesting to note that the equivalence of the continuum limit to the large- N limit can only be proven for twice differentiable potentials which are bounded everywhere: Braun & Hepp 1977; Spohn 1980). Nevertheless, the fact that the curvature is affected by softening does suggest that the origin in the instability of the trajectories of gravitational systems is related to their discrete

nature – this is in line with the fact that smoothing out the force field eliminates this instability. The effect of softening on the exponentiation time-scale was studied directly by Suto (1991). He found that the exponentiation time-scale was related to the softening by $\tau_e \sim 20b^{1/2}\tau_c$ (where b is in units of average inter-particle distance). Thus, while the softening does increase the relaxation time-scale, it does it in a rather moderate manner. On the other hand, only including the contribution due to nearby neighbours in the calculation of the curvature yields an exponentiation time-scale that increases as $N^{1/3}\tau_c$ (Gurzadyan & Savvidy 1986). We therefore conclude that while the trajectory instability in gravitational systems appears to be related to their discrete nature, what gives rise to this property should be the contribution to discreteness noise from the whole system and not just from neighbouring particles. We discuss this further in the concluding section.

5. The effect of rotation

In El-Zant (1996a) it was found that self gravitating sheets starting from states of solid rotation initially had a larger average

Table 4. Variation of the exponentiation time τ_{er} , averaged over ten (pseudo)random realizations of $N = 1000$, as the fraction of kinetic energy of rotational motion T_{rot}/T is increased. ϵ_r is an error estimate obtained by using Eq. (8)

T_{rot}/T	τ_{er}	ϵ_r
0.00	0.30	0.07
0.02	0.43	0.08
0.15	0.46	0.08
0.33	0.51	0.08
0.55	0.61	0.08
0.70	0.71	0.07
0.81	0.84	0.07
0.90	1.02	0.06
0.95	1.18	0.05
0.97	1.26	0.04
0.98	1.32	0.04
0.99	1.39	0.03
0.995	1.42	0.03

Ricci curvature than ones starting from random initial conditions in velocity space. This property is important since the fact that many flattened disk galaxies consist predominantly of stars moving on nearly circular trajectories near the disk plane suggests that for such systems phase space diffusion time-scales must be relatively long. In this section we study in more detail the change in the Ricci curvature as the energy in rotational motion is increased.

Constructing stable equilibrium spherical self gravitating systems with a wide range of ratios of rotational to random motion and having the same density distribution is not a trivial task (e.g., Palmer 1994), we therefore stick to the simple situation of static averages. The systems chosen here are homogeneous and – in the absence of rotation – have isotropic velocity distribution which does not vary with radius. Solid body rotation is then added to the random motion before all velocities are rescaled so as to have a total energy of -0.25 in accordance with the system of units discussed above (Sect. 3). As we have done before, we calculate the Ricci curvature for ten different pseudorandom realizations of the same distribution, calculate the average, and estimate the RMS error (since the scalar curvature does not explicitly depend on the velocities it remains unchanged when the velocity distribution is changed).

Table 4 shows the variation of the corresponding exponentiation time-scales as the energy in rotational motion is increased. As is clear from these results, the exponentiation time-scales are significantly increased when the rotational motion is increased. Looking at the different terms on the right hand side of Eq. (1), it is easy to see that only the first two are directly dependent on the velocities and may therefore be affected by the reordering of random motion into rotational motion. Of these two terms the first – as discussed above – is small for most equilibrium velocity distributions. It is even smaller for rotating systems since the sum of the scalar product of the particles' velocities and the forces acting on them which this term represents is near zero. The second term consists of two parts. One involves the quanti-

ties in (4) and represents the second derivatives of the potential with respect to coordinates of the same particle multiplied by the velocities of that particle. This term is also relatively small since again the gradient of the force in a corresponding smoothed out system is normal to the velocities (albeit not as small as the first term since the components of the derivatives of the force have larger fluctuations than the components of the force). The other part of the second term (involving the quantities in (3)) consists of the derivatives of the force at one particle's position with respect to another's projected on the velocities of the two particles – in other words it measures correlations in velocities and positions between the trajectories of particles in the systems. This term is small and fluctuating when the velocity field is random but is much larger and has positive sign when the kinetic energy is in the form of ordered rotational motion.

The fact that the exponentiation time-scale is significantly larger when ordered motion is present confirms that, except when effects arising from the non-compactness of the phase space are important (e.g., escape of particles), higher thermodynamic entropies appear to be related to higher values for the dynamical entropy. Thus confirming that N -body systems will, in general, evolve towards higher dynamical entropy states as was found to be the case in El-Zant & Gurzadyan (1997).

If directly interpreted to mean mixing on an exponential rate as is expected in the standard case of gravitational systems with negative two dimensional curvatures, the derived time-scales predict that a rotating system will evolve on a time-scale of about 45 crossing times (El-Zant 1996a). This is still compatible with the age of average disk galaxies at about 10 kpc, and it is possible that for realistic density and velocity distributions the predicted evolution time-scale may be still larger. In addition, enough two dimensional curvatures may be positive so as to restrict the motion. Therefore, as in the case of softened systems, the variation of the Ricci curvature exponentiation times with rotation should only be interpreted as representing a trend. However, as we shall see in the next section, it appears that the relation between the exponentiation time-scales and the macroscopic evolution of gravitational systems may not be so direct.

6. Conclusions and possible interpretation

In this paper we continued our investigation of the behaviour of the Ricci and scalar curvatures of the configuration manifolds of N -body gravitational systems. These relate the geometry of the phase space to the stability properties of trajectories on it. It was found that, for spherical systems with isotropic velocities, the inferred exponentiation time-scale is rather short (less than a crossing time) and did not depend on the particle number when the softening length decreased as $1/N$ and a short range cutoff in the potential was introduced. The exponentiation time-scale however was found to be affected by the presence of ordered rotational motion or when the softening was increased (while keeping the particle number fixed). In the first case it was found to increase significantly while in the second case it could even become undefined because the curvature became positive. A similar process takes place if the softening radius is fixed and

the number of particles is increased (if no short range cutoff is introduced).

The exponentiation time-scale being so short and not varying with particle number means that it is difficult to uncover its significance. The main problem is that these properties apparently contradict the intuitive idea that particles in large systems should move essentially unperturbed in the mean field potential. In spherical potentials this happens to mean that they all lie on regular trajectories. It then should follow that the divergence between nearby trajectories is, on average, linear and not exponential.

One may like to relate the exponential instability to the process of achieving *dynamical* equilibrium, the time-scale of which does not vary with particle number. (e.g., Kandrup 1989; El-Zant 1996c). While this could be the case, one expects that even completely smooth spherical systems achieve such an equilibrium. The exponential instability on the other hand appears to be inherently related to the discreteness of N -body gravitational systems. At the same time this does not imply that it is mainly a result of close encounters as was explained near the end of Sect. 4. In the light of that discussion, and looking at the relative importance of the terms of the formula for the Ricci curvature (see Sect. 3), one comes to the conclusion that, for spherical systems with isotropic velocity distribution and in virial equilibrium, the negative curvature is related to the fact that N -body systems have a large mean field force (because the interaction is long range) and at the same time have locally peaked density distribution (because of their being composed of particles). Thus we may expect that the cause of the exponential instability is the discreteness effects due to the long range full N -body interaction.

To see how long range gravitational interaction can trigger chaotic behaviour, we follow Chirikov (1979) and divide the Hamiltonian of the system under consideration into an unperturbed part H_0 and a non-integrable time dependent perturbation V . In terms of action angle variables this reads

$$H(\mathbf{J}, \Theta, t) = H_0(\mathbf{J}) + V(\mathbf{J}, \Theta, t). \quad (11)$$

For our purposes H_0 will be the smooth spherically symmetric potential and V would be the perturbation arising from discreteness effects: $V = V_{nbody} - V_0$. If we assume that V is periodic in time with phase $\tau = \Omega t + \tau_0$, there will be a whole set of resonance conditions given by

$$m_i \omega_i(\mathbf{J}_i) + n\Omega = 0, \quad (12)$$

where ω_i ($i = 1, 3$) are the natural frequencies of oscillation of the action variables of the motion in the background potential V_0 . Around the neighbourhood of these resonances a stochastic layer on which chaotic motion can take place forms (e.g., Lichtenberg & Lieberman 1983 (LL)). If $\Omega \gg \omega_i$ however, the resonance conditions are satisfied only for very large m/n and no lower order resonances occur. In this case the effect of these resonances is limited because the stochastic layer around them is small (LL; Meiss 1987). The vast majority of non-resonant trajectories will therefore remain stable. In the six dimensional

phase space this will mean the perturbation causing trajectories to move between KAM tori. In that case the original idea of Chandrasekhar (1942) is justified: he considered discreteness effects due to short lived encounters with nearest neighbours where resonance effects are negligible and where there is a clear separation of time-scales justifying the assumption of independence. However it is now thought that weak distant encounters dominate two body interactions in N -body systems and that distant encounters are more important for larger N systems (BT). Thus, although the strength of perturbations due to discreteness decreases as $\sim 1/\sqrt{N}$, the density of resonances (per action per particle) increases as $\sim N$ and is increasingly dominated by more effective terms. This might explain the observed persistence of chaotic behaviour for large N . For, according to LL, this issue reduces to “the question of whether the density of important resonances, as projected at a single action, increases faster than the width of the resonances decreases. If this happens then we would expect resonance overlap and strongly chaotic motion to occur for N degrees of freedom as $N \rightarrow \infty$ ”.

It is important to note here that although the main cause of the chaotic behaviour may be distant interactions between particles in a N -body system, this does not mean that the exponential instability will not be affected by short range encounters. For, even if one considers these as additional high frequency noise added to the system, according to Pfenniger (1986) such perturbations completely change a chaotic system’s trajectory on relatively short time-scales. One therefore expects high frequency discreteness noise to be an additional source of instability which will affect the exponentiation time-scale. This would explain why, even if the instability is mainly caused by distant encounters, the exponentiation time-scale increases when the force softening is increased this suppressing the high frequency noise.

It is clear that as N increases, quantities such as the energy and angular momenta of individual particles will be better conserved. This however does not imply that the decorrelation time-scale defined by the exponential divergence should become smaller. For example a perturbed pendulum can display highly chaotic oscillations which decorrelate very fast while changes in the amplitudes of these oscillations are much slower. In this type of situation one can average over the fast phase variable which may (because of the short decorrelation time) be considered as random. The evolution of the actions can then be regarded as a diffusion process. In fact this approximation is only valid when the dynamics is *strongly chaotic* so that chaos occurs for the vast majority of initial conditions (Chirikov 1979; LL; Shlesinger et al. 1993). This is of course what appears to be the case for gravitational N -body systems.

In the case of gravitational systems, it is actually not that surprising that the exponentiation time-scales are very different from the diffusion time-scales of the action variables. For in the standard case of a system with negative two dimensional curvatures, the exponential instability will guaranty that the system visit all regions of the available phase space (Anosov 1967). In the case of a N -body system this will be the whole subspace defined by the conservation of total energy and momentum. Since

in a Hamiltonian system the phase density is conserved along the motion, this will imply that the phase space (N particle) distribution function will become constant when averaged over progressively smaller volumes in that space – that is the system becomes more and more likely to be found in any of its microscopic states. For an open gravitational system this clearly cannot be the case since, instead of evolving towards a definite thermodynamics equilibrium, this type of system continually evolves towards more and more concentrated configurations when it divides into a contracting core section surrounded by an expanding halo. In this type of evolution, the Poincaré recurrence theorem is not valid and the system need not, even in principle, return to less inhomogeneous states. Instead it continually moves into new areas of the phase space characterised by larger entropy. In this case, if the evolution time was directly related to the exponentiation time-scale, then there would be no chance for the distribution function to be constant on any region of the phase space. Therefore, no type of equilibrium would be possible and gravitational systems would disintegrate in a few dynamical times! What prevents this from happening of course is that some states are long lived because they are stable dynamical equilibria.

Indeed, it was found that, for closed systems which did have a definite *thermodynamic* equilibrium state, this state was reached on a time-scale comparable to the exponentiation time if no dynamical equilibrium existed between the initial state and the final equilibrium (this happened even when the virial ratio remained nearly constant during the evolution thus ruling out violent relaxation). On the other hand, if there existed intermediate *dynamical* equilibrium states between the final equilibrium and the initial configuration, the relaxation to the final equilibrium state took much longer (El-Zant 1997). It may be therefore that the exponential instability acts as to smooth out the distribution function on the subspaces where these equilibria are defined while at the same time driving the evolution towards higher entropy states. This means that although the instability is locally normal to the phase space motion, the global stretch and fold of that space is such that the decorrelation is faster along the motion – justifying the diffusion description for the evolution of the action variables. For larger N , the dimension of the phase space grows and there are more states (corresponding to a given dynamical equilibrium) to be covered while the exponentiation time-scale is constant. Thus, one expects the diffusion rate of the action variables to be larger as N increases.

More precisely, it can be shown using symbolic dynamics (Alexseev & Yakobson 1981; McCauley 1993; Chirikov 1994) that for hyperbolic systems (a class which includes systems with negative two dimensional curvatures), with well defined final states, the time-scale for a system to achieve such a state with coarse grained volume resolution μ is

$$\tau_r = \frac{\ln(1/\mu)}{h}, \quad (13)$$

where $h \sim \langle \sqrt{-2r_{\mathbf{u}}} \rangle$ is the Kolmogorov entropy in the case when $r_{\mathbf{u}}$ is dominated by the largest two dimensional curvatures. (The above relation is exact and follows from Eq. (14.3)

of Alexseev & Yakobson when h is constant along the N -body trajectory, otherwise it is valid in an averaged asymptotic sense). Fixing the linear resolution d and using Eq. 6 one finds

$$\tau_r = 2\sqrt{3} \ln(1/d) W \sqrt{N} \tau_e \sim \sqrt{N}. \quad (14)$$

Therefore, in systems for which definite statistical equilibria exist, the time-scale to achieve such equilibria does indeed in general increase with N and appears to scale as the inverse of the strength of the perturbation due to discreteness. This is the case even though the exponentiation (or Liapunov) time-scale as derived from the value of the Ricci curvature (or as directly inferred from the divergence of N -body trajectories) does not vary much with N . This is essentially because, unlike the case of low dimensional systems, the Kolmogorov entropy is dominated by the maximal exponent and volume resolution becomes very different from linear resolution. Thus, for large- N systems, the Liapunov time-scale does not necessarily coincide with the mixing time-scale. A more detailed discussion of the mathematical origin of this important effect, which in retrospect has been a major cause of the confusion regarding the meaning of the Liapunov time-scale for N -body systems, will be given elsewhere.

Because of the exponential instability, the (6-d) phase space trajectories of particles quickly decorrelate so that the probability of finding a particle in a given state is independent of the state of the other particles. This being so even if the motion is affected by the discrete nature of the system. For times longer than the decorrelation time τ_e the N particle distribution function can then be given in terms of the individual one particle distribution function by

$$f^N = f(\mathbf{x}_1, \dot{\mathbf{x}}_1, t) f(\mathbf{x}_2, \dot{\mathbf{x}}_2, t) \dots f(\mathbf{x}_N, \dot{\mathbf{x}}_N, t). \quad (15)$$

This is of course the requirement that a system be completely described by the collisionless Boltzmann equation (CBE). Moreover, since steady state solutions of the CBE will only depend on the action variables (Jean's theorem), these solutions will be equivalent to those produced by the mean field dynamics – for times long compared to the exponentiation time-scale but short compared to the diffusion rate of the action variables. In this context the continuum (collisionless) approximation may then be valid in an *averaged* sense: the exact trajectories in smoothed background potentials would not be valid but the time averaged *orbits* would be correct as long as the time-scales considered are small compared to the diffusion times of the action variables.

Although for many purposes the situation described above is similar to that of standard galaxy dynamics, it differs in one important respect: *the underlying motion is intrinsically chaotic and cannot be expressed as a linear superposition of regular motion and independent binary encounters*. For this reason the diffusion time of the action variables need not be accurately represented by the two body relaxation time. Moreover, it is now well known that chaotic trajectories can respond to external perturbations in a manner that is different from that of regular trajectories (Pfenniger 1986; Kandrup 1994; Merritt & Valluri 1996; El-Zant 1996c). Therefore, the response of the trajectories

to additional perturbations (e.g., high frequency discreteness noise or global asymmetries in the background potential) is not necessarily identical to that of the trajectories in the smoothed out potential – which for spherical systems happen to be all regular.

To sum up. In this section it has been suggested that resonances between the orbital frequencies of particles and forcing caused by the discreet nature of the global potential give rise to chaotic trajectories which decorrelate over a time-scale of the order of a dynamical time. Nevertheless some quantities may decorrelate on the much larger time-scale. In the case of systems of having a well defined final equilibrium state the maximum such time-scale is that needed to reach a statistical equilibrium when such a state exists which scales as $N^{1/2}$.

The above ideas can be tested in more than one way. As mentioned in the introduction, it is well documented that N -body trajectories decorrelate over a dynamical time or less. It may be useful however to also examine the stability of *individual* particle trajectories in N -body simulations – preferably by methods that do not require linearisation of the equations of motion (e.g., Laskar's frequency analysis). To find out the dominant range of encounters causing the exponential divergence in N -body systems, it may be possible to calculate the frequency spectrum of perturbations a N -body particle is subjected to and examine the effect of its different regions on individual trajectories in smoothed out potentials. Alternatively, one may want to bin the contributions to the discreteness noise $\mathbf{F} = \mathbf{F}_{smooth} - \mathbf{F}_{nbody}$ acting along a particle's trajectory into impact parameter ranges and again examine their effect on the stability of test particles' trajectories in the smoothed out density distribution. The time-scales over which the physical characteristics of a given system changes can be studied directly by examining the relaxation of the particles' integrals of motion in the smoothed out potential. This can be done either by computing the diffusion coefficients of these variables for particles in N -body simulations or by studying the macroscopic relaxation of numerically simulated systems. The latter approach is most effective for closed systems where definite thermal equilibria are well defined (El-Zant 1996b). One would then look at the time-scale for attaining isothermal equilibrium and the time-scale of relaxation of initial anisotropic velocity distributions (El-Zant 1997) and how these time-scales vary with N (El-Zant & Goodwin 1997). In fact, the existence of a well defined final state that ceases to evolve also means that all the aforementioned tests are easier to conduct for closed systems.

Acknowledgements. I would like to thank Vahe Gurzadyan for many interesting discussions on the subject of this paper and for commenting on the manuscript. It is also a pleasure to thank John Papaloizou and Peter Thomas for helpful discussions. Thanks must also go to Sverre Aarseth for providing a copy of his NBODY2 code and Simon Goodwin for helping with its use.

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