Astron. Astrophys. 331, 1103–1107 (1998)

ASTRONOMY AND ASTROPHYSICS

Analytic description of collisionally evolving fast electrons, and solar loop-top hard X-ray sources

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Received 31 July 1997 / Accepted 20 November 1997

Abstract. We present a new approach to the problem of particle transport described by the linearised Fokker-Planck equation. Instead of attempting to solve for the distribution function directly, exact and analytic expressions for the moments of the distribution are derived from the equivalent stochastic differential equation. Although the moments themselves will be of greatest use, we also show how these moments can be used to construct an exact, analytic solution to the Fokker-Planck equation. In addition, we explain how mean scattering theory naturally emerges from the first order moments. The derivation of the second (and higher) order moments means that the spatial spreading of electrons due to the changing pitch angle distribution can be described analytically for any injected pitch angle – previously, such a description was not possible with mean scattering and, in general, numerical simulation was the only method available. The treatment also explicitly reveals a simple scaling relationship between the distribution of particles along the magnetic field and the square of the particle's injection energy. We check our results against numerical simulations and point out how the results here can be extended to more general cases. Uses of these results are illustrated in relation to the spatial distribution of Hard X-Ray (HXR) emission and its relevance to solar HXR "above the loop top" sources.

Key words: Sun: flares – scattering – X-rays: bursts

1. Introduction

Non-thermal particles play a central role in solar flares (eg Brown and Smith 1980) and as such knowledge of how they are distributed along the magnetic fields in flare loops is of great interest. Emission of electro-magnetic radiation, eg Hard X-ray bremsstrahlung, is direct evidence for the presence of a non-thermal particle distribution and heating of the plasma as a result of its presence can cause a variety of observable phenomena, eg chromospheric evaporation. In both cases, to interpret and model the observations, an understanding is needed of how the electrons propagate and consequently how they are distributed spatially. For understanding the directivity and polarisation of emitted radiation, the pitch angle distribution is certainly very important. More fundamentally, understanding of particle transport must be improved before observations can be interpreted meaningfully in relation to the primary energy release and particle acceleration in flares.

As high energy (energy much greater than the thermal energy of particles in the background plasma) particles travel along a magnetic field their pitch angle, ie the angle between their velocity vector and magnetic field, can be affected by: the particles in the "cold" background plasma; waves in the plasma, which they may also drive; and any convergence in the magnetic field. Here we shall concentrate on the first of these processes in the non-relativistic limit. To order Λ (the Coulomb logarithm) in the Fokker-Planck equation, scattering reduces the energy of the particles deterministically but affects the direction of the particle's motion in a stochastic way, due to the effect of very many long distance Coulomb collisions with the particles in the background plasma. In the current work, we consider only charged particles moving in the presence of a constant magnetic field, so that only the pitch angle of the electrons, θ (or $\mu = \cos\theta$) need be considered. This means, for example, that the propagation of the electrons along a field line is a stochastic process determined by both the slowing of the electrons and by the change in their pitch angle distribution.

Without resorting to the Fokker-Planck equation it is possible to determine where, on average, the electrons will be, in terms of column depth measured along the field line, N, and what their average pitch angle, μ, will be, once they have been degraded to energy E. If the injection energy and pitch angle are E_0 and μ_0 resp. then, as was first shown in Brown (1972),

$$\frac{E}{E_0} = \frac{\mu}{\mu_0} = \left(1 - \frac{N}{\frac{2}{3}\mu_0 N_s}\right)^{1/3} \tag{1}$$

where

$$N_s(E_0) = \frac{E_0^2}{4\pi e^4 \Lambda} \tag{2}$$

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e being the charge of an electron in e.s.u. and Λ being the Coulomb logarithm, which we take to be constant. N_s can be interpreted as the column depth measured *along the electron's path* that is required to reduce its energy to zero (in reality the electron will join the thermal distribution of the plasma, the typical thermal energy being much less than E_0). The $\frac{2}{3}$ accounts for the average reduction in pitch angle, effectively reducing the "average" electron's component of velocity along the field. These are the results of "mean scattering", for which further details can be found in Brown (1972), Emslie (1978), Tandberg-Hanssen and Emslie (1984), Craig et al. (1985) and Vilmer et al. (1986). The treatment in this paper shows how these results can be derived directly from the Fokker-Planck equation.

Mean scattering deals only with first order statistical properties of the electron distribution function. To find the distribution function itself, the Fokker-Planck equation must be solved. To date, only approximate analytic solutions for some special cases have been found eg Leach and Petrosian (1981), McTiernan and Petrosian (1990a). Also, an exact expression for the spatially integrated distribution function in terms of Legendre polynomials can be derived, see Kel'ner and Skrynnikov (1985) and Lu and Petrosian (1988). Numerical solution of the Fokker-Planck equation is required to yield further information. Several Monte Carlo type methods have been described, eg Bai (1982), Hamilton et al. (1990) and MacKinnon and Craig (1991). The last of these, referred to as MC from here on, shows how the numerical method can be formally and simply related to the Fokker-Planck equation. These numerical methods have now been exploited to model various observations: stereoscopic observations of flares including treatment of directivity, McTiernan and Petrosian (1990b); above the loop top HXR sources, Fletcher (1995); and height distribution of HXR sources, Fletcher (1996).

There is no doubt that numerical methods provide a very powerful tool, even though they can involve lengthy processing times, but an improved analytic approach is still desirable for gaining a clearer insight and intuitive understanding of the problem. This paper describes a method that can provide exact and analytic expressions for the 2nd and higher order moments of the distribution, which in turn can provide an expression for the distribution function itself. We concentrate mainly on the second order properties which are of particular importance since they describe the spread of electrons about the expected mean position. This method takes us beyond the first order properties of the mean scattering approach and allows discussion of situations previously accessible only with numerical simulations.

The problem is stated in Sect. 2 and the method of solving the equations is described in Sect. 3. Sect. 4 compares the first and second order moments with results from a code based on the method of MC. In Sect. 5 we illustrate the usefulness of the results obtained in looking at how a population of large pitch angle electrons injected at the loop top disperse spatially. The results are then summarised and discussed in Sect. 6.

2. Fokker-Planck equation

The Fokker-Planck equation for non-relativistic electrons in a cold hydrogen plasma is

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial z} - C\frac{\partial}{\partial v}\left(\frac{f}{v^2}\right) - \frac{C}{v^3}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f}{\partial \mu}\right) = 0$$

where $C = 4\pi e^4 \Lambda n/m_e^2$, t is time measured from some specified initial condition, z is the physical distance measured *along* the field line, v is speed, m_e is the mass of the electron, μ is the electron's pitch angle and the units of f are (cm s^{-1})$^{-1}$ cm^{-3}. The initial distribution at $t = 0$ is taken to be

$$f(E, \mu, z, t = 0) = \delta(E - E_0)\delta(\mu - \mu_0)\delta(z - z_0)$$

and as described in MC, linear superposition of solutions with different initial conditions can yield the solution at time t for any arbitrary initial distribution function, or with some added source term on the right hand side.

In terms of Îto calculus, the Fokker-Planck equation may be expressed as a set of three stochastic differential equations:

$$\begin{aligned} dz &= \mu v dt \\ dv &= \frac{-4\pi e^4 \Lambda n}{m^2 v^2} dt \\ d\mu &= \frac{-8\pi e^4 \Lambda n \mu}{m^2 v^3} dt + \left[\frac{4\pi e^4 \Lambda n}{m^2}\frac{(1-\mu^2)}{v^3}\right]^{1/2} dW(t) \end{aligned}$$

where $z = 0$,$v = v_0$, $\mu = \mu_0$ at $t = 0$ and dW is a Wiener process (see below). Upon a change of variables to $dx = nvdt/N_s(E_0)$:

$$y(x) = \int_0^x \mu(x_1)dx_1 \quad (3)$$

$$(E/E_0)^2 = 1 - x \quad (4)$$

$$d\mu = -\frac{\mu dx}{2(1-x)} + \frac{1}{2}\left(\frac{1-\mu^2}{1-x}\right)^{1/2} dW(x) \quad (5)$$

where $dW(x)$ represents a Wiener process which has mean zero, variance 2 and auto-covariance given by $2\delta(\Delta x)$, where δ is the Dirac delta function. A Wiener process can be thought of as a continuous white noise (time) series in x that is independently distributed at each x. The variable x is the normalised column depth "seen" by the electron, ie the column depth integrated along the electron's path. Intuitively, x can be thought of as being the electron's "life-time" parameter as it depends solely on the fraction of the electron's energy E/E_0. The variable y is the normalised column depth integrated along the electron's motion projected onto the field line: y can be identified with the column depth N appearing in mean scattering theory (1). The normalisation is by the stopping column depth measured along the electrons path, given by (2).

The aim is to derive the statistical moments E[$y^n(x)$], where n is a positive integer, and where E[] is the expectation operator. These moments, in physical terms, will provide information on the particles' spatial distribution along the field when the particles have been degraded to energy $E = E_0(1-x)^{1/2}$.

At this point, it is clear from the form of (3), (4) and (5) that

$$E[y^n(x)] = g_n(x, \mu_0)$$

so that for a given x the moments only depend on μ_0, and not explicitly on the injection energy E_0. It can easily be seen that the nth moment of the column depth itself is $N_s^n(E_0)E[y^n(x)]$ which scales as E_0^{2n}. From this we can state quite generally that the mean, variance, skewness, kurtosis and so on of the electron's column depth measured along the field, at the point in its life given by x, are all simply proportional to E_0^2, E_0^4, E_0^6, E_0^8 and so on, respectively. For example, if many electrons were injected with energy E_0, then their mean position at the end of their lives would be proportional to E_0^2 (as predicted by mean scattering) and their spread about that position (ie the root variance) would also be proportional to E_0^2. The following section shows how to derive the functions $g_n(x, \mu_0)$.

3. Solving the Îto equations

The general form of the Îto equation for a system of stochastic differential equations in $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_m(t))$, where $X_i(t)$ is a stochastic process in t, is

$$d\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t), t)dt + G(\mathbf{X}(t), t)d\mathbf{W}(t)$$

where $\mathbf{f}$ is a vector of functions, G is a matrix of functions, $d\mathbf{W}(t)$ is a vector of independent Wiener processes and D is the correlation matrix of the Wiener processes.

The moments of the elements of $\mathbf{X}(t)$ can be found from the moment ordinary differential equation (ODE), see Soong (1973), Sect. 7.1.4.1(c):

$$\frac{dE[h]}{dt} = \sum_{j=1}^{m} E\left[f_j \frac{\partial h}{\partial X}\right] + \sum_{i,j=1}^{m} E\left[(GDG^T)_{ij} \frac{\partial^2 h}{\partial X_i X_j}\right] + E\left[\frac{\partial h}{\partial t}\right] \quad (6)$$

where $h = h(X_1, X_2, X_3, \ldots, X_m, t)$.

3.1. Solving for moments in μ and y

The system of equations we wish to solve is (3), (4) and (5). Fortunately, only one of the three equations contains an explicit noise term, and further simplification arises because the energy equation is very simple.

Firstly, we derive the statistical properties of $\mu(x)$. This can be done by using the μ Eq. (5) to construct the moment ODE using (7) with $h(\mu) = \mu^n$

$$\frac{dE[\mu^n]}{dx} = E\left[-\frac{\mu}{2(1-x)}\frac{\partial}{\partial \mu}(\mu^n)\right] + \frac{1}{4}E\left[\frac{1-\mu^2}{1-x}\frac{\partial^2}{\partial \mu^2}(\mu^n)\right] \quad (7)$$

which can be solved to express the nth moment as an integral involving the $(n-2)$th moment:

$$\frac{d}{dx}\frac{E[\mu^n]}{(1-x)^{n(n+1)/4}} = \frac{n(n-1)E[\mu^{n-2}]}{4(1-x)^{1+n(n+1)/4}} \qquad \text{for } n > 1 \quad (8)$$

Next, using the same procedure, the moment ODE equation for $y(x)$ is

$$\frac{dE[y^n]}{dx} = nE[\mu y^{n-1}] \quad (9)$$

where $E[\mu y^{n-1}]$ can be determined from our final moment equation

$$\frac{d}{dx}\frac{E[\mu y^k]}{\sqrt{1-x}} = k\frac{E[\mu^2 y^{k-1}]}{\sqrt{1-x}} \quad (10)$$

Together, (8), (9) and (10) can be used to find the moments of $y(x)$. Doing so does not present any further difficulties beyond the tedious task of evaluating multiple integrals over x, where, in general, the integrands consist of terms of the form $a(\mu_0)(1-x)^{l/2}$, l being an integer.

To illustrate this, we calculate the first and second order moments. Firstly, the moment equation for μ gives for $n = 1, 2$ respectively:

$$E[\mu] = \mu_0(1-x)^{1/2} \quad (11)$$

$$E[\mu^2] = (\mu_0^2 - \frac{1}{3})(1-x)^{3/2} + \frac{1}{3} \quad (12)$$

After a little reduction, putting $x_1 = 1 - x$ for brevity, the first two moments of $y(x)$ can be expressed as

$$E[y(x)] = \frac{2}{3}\mu_0\left[1 - x_1^{3/2}\right] \quad (13)$$

and

$$E[y^2(x)] = 2\left(\frac{\mu_0^2}{3} - \frac{\mu_0^2 - 1/3}{7}\right) + 2\left(\frac{(\mu_0^2 - 1/3)x_1^{7/2}}{7} + \frac{x_1^2}{3} - \frac{(\mu_0^2 + 1)x_1^{3/2}}{3}\right) \quad (14)$$

Result (13) is the familiar mean scattering result, re-derived from this new rigorous standpoint. Result (14) is completely new, and gives the variance in column depth of those electrons whose energy has been degraded by a factor $(1-x)^{1/2}$: the variance of y is given by $Var[y] = E[y^2(x)] - (E[y(x)])^2$.

3.2. Constructing the distribution function

Now we show how the distribution function itself can be expressed analytically. We note that such an exact, analytic expression is mainly of academic interest, since the complete expression can be very complicated, and may prove rather difficult to use in practise. More often, the moments themselves will prove to be of more immediate use – this is demonstrated in Sect. 5, where the second order moment is used to look at how large pitch angle electrons spread away from their point of injection.

The characteristic function $\phi(u)$ of a random variable X that has distribution function $g(x)$ is defined to be, see Soong (1973),

$$\phi(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} g(x) dx \quad (15)$$

It is clear therefore that ϕ and g form a Fourier transform pair. Now, the MacLaurin series for ϕ is

$$\phi(u) = \phi(0) + \phi'(0)u + \phi''(0)\frac{u^2}{2} + \ldots$$

Differentiating the integral in (15) gives

$$\phi^{(n)} = i^n \mathrm{E}[X^n]$$

ie each term of the MacLaurin expansion of ϕ is simply related to the moments of X.

Applying this to the random variable $y(x)$ we find that its distribution function $Y(y, x)$ is given by

$$Y(y, x) = \frac{1}{2\pi}\int_{-1}^{1} e^{iuy} \sum_{k=0}^{\infty} i^n \mathrm{E}[y^n(x)]\frac{u^n}{n!}du$$

where $Y(y, x)$ is the distribution f integrated over pitch angle. The pitch angle distribution at x can be calculated in a similar fashion and is known to be given by a Legendre polynomial solution, see Kel'ner and Skrynnikov (1985), Lu and Petrosian (1988) and MC.

4. Comparison with numerical results

Figs. 1a and 1b show how the theoretical results for $\mu_0 = 0, 0.5, 1$ all agree with the numerical simulation based on the method of MC. 10,000 realizations were used with 10,000 steps in x. For both $\mathrm{E}[y(x)]$ and $\mathrm{E}[y^2(x)]$ the agreement is so good that the theoretical and numerical curves can hardly be distinguished. The very slight differences arise for two reasons. Firstly, because of the finite number of realisations there is a small residual noise component present. Secondly, there is a cumulative error in the numerical integration because of the finite step size in x.

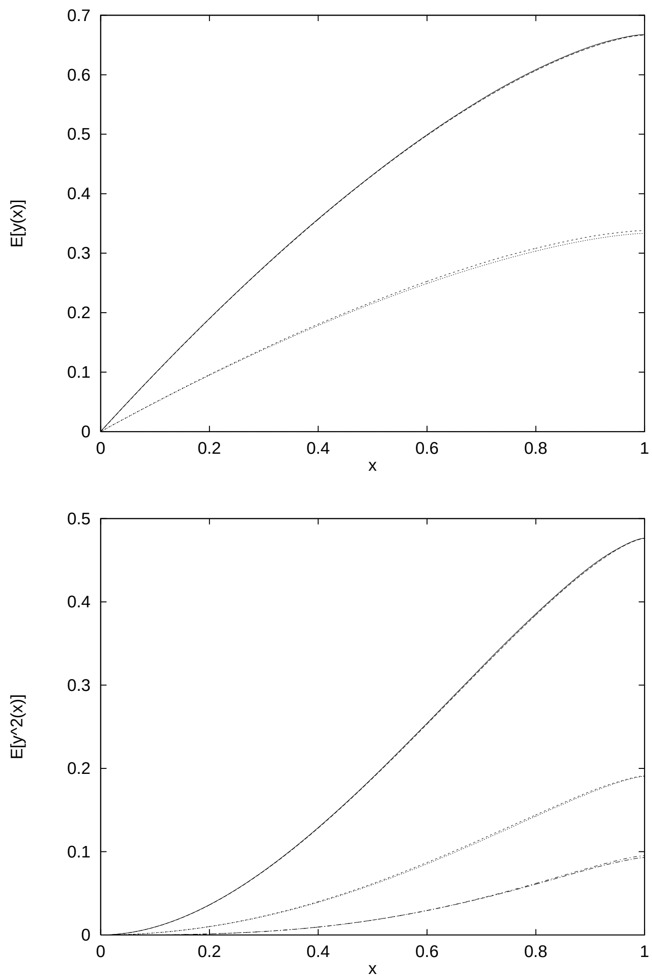

Fig. 1. a $\mathrm{E}[y(x)]$ and **b** $\mathrm{E}[y^2(x)]$ are plotted for $\mu_0 = 0, 0.5, 1$. In both graphs, the upper curve is for $\mu_0 = 1$, the middle curve is for $\mu_0 = 0.5$ and the lower curve is for $\mu_0 = 0$ (for $\mu_0 = 0$, $\mathrm{E}[y(x)]$ is just zero).

5. Application to HXR loop top sources

As mentioned earlier, the distribution function itself is generally not useful because of its complexity. However, the low order moments can be used to investigate problems mathematically, without recourse to numerical simulations. In particular, previously, any problem where the spatial spreading of electrons was of interest could not be treated by mean scattering and therefore required a detailed numerical simulation. Here, we present such a problem and show how it can be addressed using the second order moment of the distribution.

Masuda et al. (1994) reported and analysed observations of an "above the loop top" HXR source during the limb flare that occurred on 13 January 1992, which have been re-analysed recently by Alexander and Metcalf (1997). In this event, three distinct patches of HXRs were observed, two corresponding to the footpoints, the third apparently being situated above the soft X-Ray loop. One of several models for this was suggested by Fletcher (1995) who pointed out that the loop top source could be explained using electron transport effects, without any need for assuming a magnetic trap or plasma density enhancement at the loop top (eg Wheatland and Melrose 1995). In this model, any electrons injected into the loop top with large pitch angles ($\mu_0 \sim 0$) remain there until they are scattered. The resultant concentration of electrons at the loop top causes an increase in the thin target HXR bremsstrahlung emission there. We shall refer to this as "μ-trapping" for obvious reasons. Intuitively, it might be thought that this cannot yield a bright enough source, and that the situation is not improved by increasing n, since both the rate of electrons being scattered out of the loop top and the thin target HXR flux are proportional to the density n – hence the emission is independent of density. We show here that this assertion is *not* correct and that increasing the density can increase the source brightness. We shall consider how $\mu_0 = 0$ electrons spread away from the loop top, where they are injected. We shall also assume that $x \ll 1$, a reasonable assumption for electrons with > 20keV and if the density is less than $10^{12}\mathrm{cm}^{-3}$, since the loop top's dimensions are $\sim 10^8$cm. We will also assume that the density n is constant across the loop top.

Starting with (15) for $\mu_0 = 0$:

$$Var[y(x)] = \mathrm{E}[y^2(x)] = \frac{2}{21}\left[1 + 7(1-x)^2 - 7(1-x)^{3/2} - (1-x)^{7/2}\right]$$

With x assumed to be small, expanding to the the lowest non-zero power of x gives $Var[y(x)] \sim x^3/6$. Since the density is uniform, we can return from our de-dimensionalised column depth variables x and y to t (time after injection) and z (distance from loop apex). This gives

$$Var[z] \sim \frac{nv_0^3}{6N_s}t^3$$

This means that we would expect an electron to leave the loop top region in time τ_L given by:

$$\tau_L = \left[\frac{6N_s L^2}{nv_0^3}\right]^{1/3}$$

where we take the loop top to be a region that extends from $z = -L$ to $z = L$. In a steady state situation, ie where the number of electrons in the loop top region M_L is not changing, and there is a constant rate of injection R, we have

$$M_L = R\,\tau_L$$

This means that the thin target HXR emission from these electrons will be proportional to nM_L, which is proportional to $n^{2/3}$. That is a denser background plasma *does* cause the source to be brighter.

The treatment presented here is only supposed to be illustrative and a more complete development of these ideas will be presented in a future paper.

6. Conclusions and discussion

We have shown how to find an exact analytic solution to the Fokker-Planck equation using moments of the distribution function. However, the moments themselves, which are relatively simple in form, are almost certainly of much greater use. In particular, as illustrated in the previous section, the 2nd order moment allows a mathematical description of the spreading of electrons, reducing the need for lengthy numerical simulations.

Also, we note that (4), (11) and (13) can be combined to give the standard mean scattering results (1). The current treatment therefore provides a formal link between the Fokker-Planck equation and standard mean scattering theory. Note that, like mean scattering theory, since we deal with column depth, the density distribution is arbitrary. The method developed here may be extended in various ways. Non-relativistic forms of drift and diffusion coefficients were employed here for simplicity, but the relativistic form is well known (e.g. Hamiltron and Petrosian 1990) and their employment presents no extra difficulty of principle. Inclusion of spatially varying magnetic field strength would be more involved and we note that even in the mean-scattering case (Chandrasekhar and Emslie 1987), no analytical solution can be found.

Here we have shown how the first and second order statistical properties of the electron distribution in pitch angle cosine μ and field line normalised column depth y can be obtained. The method can, in principle, be extended to the higher moments, which should only involve a greater degree of algebraic complexity. We note that since the distribution in y is not symmetric about the mean (see numerical results in MC), the skewness of the distribution will be non-zero.

Acknowledgements. AJC would like to thank T. Sakao and T. Kosugi at the National Observatory of Japan, where most of this work was carried out. The authors also acknowledge PPARC for funding this work, via Rolling Grant no.GR/K59071. AJC also thanks Chris Clarkson of the Astronomy & Astrophysics group for helping with the algebraic reduction. The referee, N. Vilmer, is thanked for her improving and constructive criticism of the paper.

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