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ASTRONOMY AND ASTROPHYSICS

Caustics of the restricted three-body problem

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Abstract. Trajectories of the planar, circular, restricted three-body problem are given in the configuration space through the caustics associated to the invariant tori of quasi-periodic orbits. It is shown that the caustics of trajectories librating in any particular resonance display some features associated to that resonance. This method can be considered complementary to the Poincaré surface of section method, because it provides information not accessible by the other method.

Key words: celestial mechanics, stellar dynamics – chaos

1. Introduction

A commonly adopted model to study the dynamical evolution of small bodies in the solar system is the planar, circular, Restricted Three-Body Problem (RTBP) (see for instance Szebehely, 1967). In the case of asteroidal motion, the Sun and Jupiter are assumed to be in circular orbits around the center of mass of the system, and an asteroid with negligible mass moves under the gravitational influence of these two main bodies.

Several studies on the main asteroid belt have been made using this model. In general, these studies consisted essentially in numerical integrations of this problem aiming at:
– the search of periodic orbits and the families associated to them as well as the study of their stability with the purpose of characterizing the system orbits through periodic orbits (see for instance Szebehely, 1967, chapters 8 and 9);
– computing the variation of the orbital elements, with the purpose of understanding the evolution of such elements (see for instance Winter and Murray, 1997a);
– obtaining the intersection of the orbits with a transversal surface (Poincaré surface of section method), with the purpose of determining the nature of the orbits, regular or chaotic, and their extent on the phase space (see for instance Winter and Murray, 1994a and 1994b).

However, it is rare to find a method being used to describe in a complete and clear way the orbits in the configuration space, which is the physical space where the orbits actually evolve. In principle, this description could be made in a simple and direct way by just plotting the quasi-periodic orbit in the configuration space. This process is naive and does not work, because even for just one trajectory its path becomes very quickly confused, hiding all the details of the trajectory.

In this work we present a description of some RTBP quasi-periodic orbits. This presentation is made in the configuration space using the caustics of the problem. The caustics are envelopes of the quasi-periodic trajectories, or the contours of the torus that contains each one of these orbits. The construction of the caustics is based on a technique recently developed by Stuchi and Vieira Martins (1995, 1996). The method is built from simple properties of the variational equations solutions, which are fulfilled in the case of not degenerated integrable Hamiltonian systems and, in principle, can be used for almost all quasi-periodic orbits of Hamiltonian systems.

As it will be shown, a relatively small number of caustics characterize the various types of orbits of the RTBP and the shape of these caustics are closely related to the existing resonances.

2. Caustics of Hamiltonian systems

In this section we consider the quasi-periodic orbits lying on invariant tori. In particular, we present the essential ideas needed for the computation of the caustics associated with these invariant tori. For further details and more explanation see Stuchi and Vieira Martins (1995).

As it is well known, the limited trajectories of an integrable Hamiltonian system in the phase space are, in general, dense on tori of dimension equal to the number of degrees of freedom of the system (see for instance Ozório de Almeida, 1988). Given a dense orbit on an invariant torus, one can verify that when the solutions of the variational equations associated with the torus are divided by the time, they tend to tangent vectors to the invariant torus when the time interval increases with respect to the initial time. In this way, we can numerically build the tangent space to the torus at each point. Computing the singular points

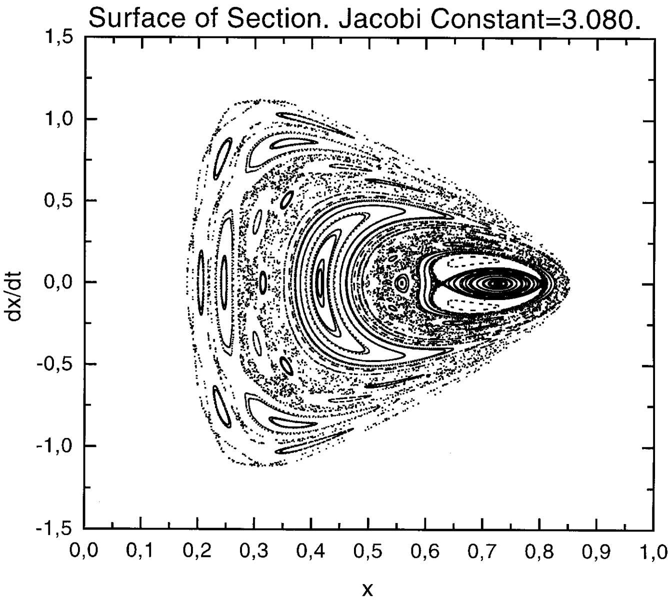

Fig. 1. The RTBP surface of section for $y = 0.0, \dot{y} > 0.0$ and Cj = 3.080 (interior region).

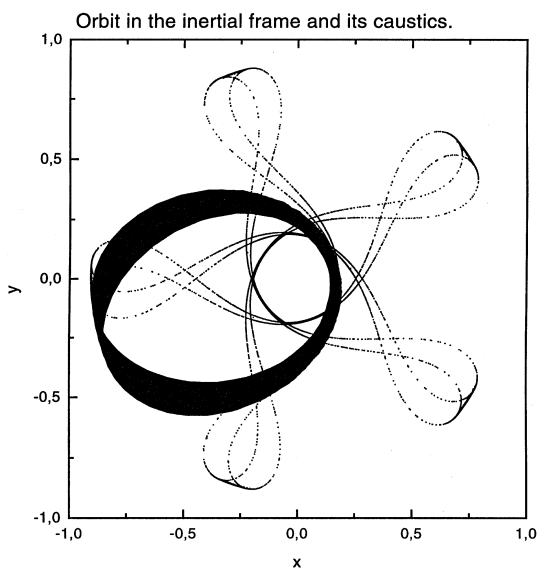

Fig. 2. Orbit in the inertial frame and its caustics in the rotated frame. The initial values in the rotated frame are: $x = 0.18, y = 0.0, \dot{x} = 0.0$ and Cj = 2.940.

of the projection mapping of the torus on the configuration space and taking their projections we have that these points belong to caustics of the torus.

Therefore, to compute the caustics for a particular trajectory of the Hamiltonian system with n degrees of freedom, it is enough to follow the following steps:
– numerically computing a trajectory and n linearly independent solutions of its variational equations (n vectors are sufficient to define, at each point, the tangent space to the n-torus);
– computing the time for which the determinant of the matrix formed by the n first coordinates of the variational solutions divided by t^n goes to zero within a required accuracy;
– determining the points of the trajectory in the configuration space corresponding to these times;
– eliminating the first points, because they correspond to a transient stage.

The remaining points in the configuration space are the points of the caustics associated with the considered trajectory and if this trajectory is dense on a torus, these points correspond to the caustic of the torus.

For a problem of n degrees of freedom, the tori are surfaces limited and differentiable in the phase space of dimension $2n$. Then, their caustics are formed by some closed surfaces of dimension $n - 1$, which are limited, continuous, but not differentiable and with self intersections in the configuration space, that has dimension n. From the point of view of the Hamiltonian systems theory, the invariant tori are Lagrangian manifolds, and the caustics and its singularities are Lagrangian singularities.

In particular, for two degrees of freedom systems the tori are two dimensional surfaces in a four dimensional space and the caustics are formed by one or two closed curves in the configuration space. As the torus is a surface in a four dimensional space, the caustics can be much more complicated than those one gets from their projections on a plane, from a torus of dimension two in a three dimensional space (Ozório de Almeida and Hannay, 1982). The caustic can be composed by one unique curve due to the superposition of the two closed components because of the symmetries of the Hamiltonian. In the algorithm that computes the caustics, both curves correspond to the two senses in which the determinant vanishes.

Also for two degrees of freedom, there is a very simple connection between the curves that appear in the surface of section and the caustics. Note that the closed curves that appear for the quasi-periodic orbits are defined by the intersection of the invariant torus with a plane transversal to the orbits. Therefore, the points of these transversal section curves for which the tangent is perpendicular to the position axis are points of the torus associated to the caustics. Thus one can verify that the surface of section gives an idea of just the local invariant torus. Nevertheless, it is important to point out that the method of the caustics provides information on just one invariant torus, while the Poincaré surface of section is about a set of tori.

3. Caustics for the planar, circular, restricted problem

We considered the normalized planar, circular, restricted three-body problem. The mass ratio of the two primaries is 10^{-3} (which is similar to the Sun-Jupiter mass ratio), the reference frame has its origin at the center of mass of the two primaries and it rotates with angular velocity equal to one such that the two primaries always remain on the x axis. The trajectories of the third body are defined taking the values of the coordinates of position and velocity $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ at time $t = 0$.

In order to choose the starting conditions, we followed the “Atlas of the Planar, Circular, Restricted Three-Body Problem

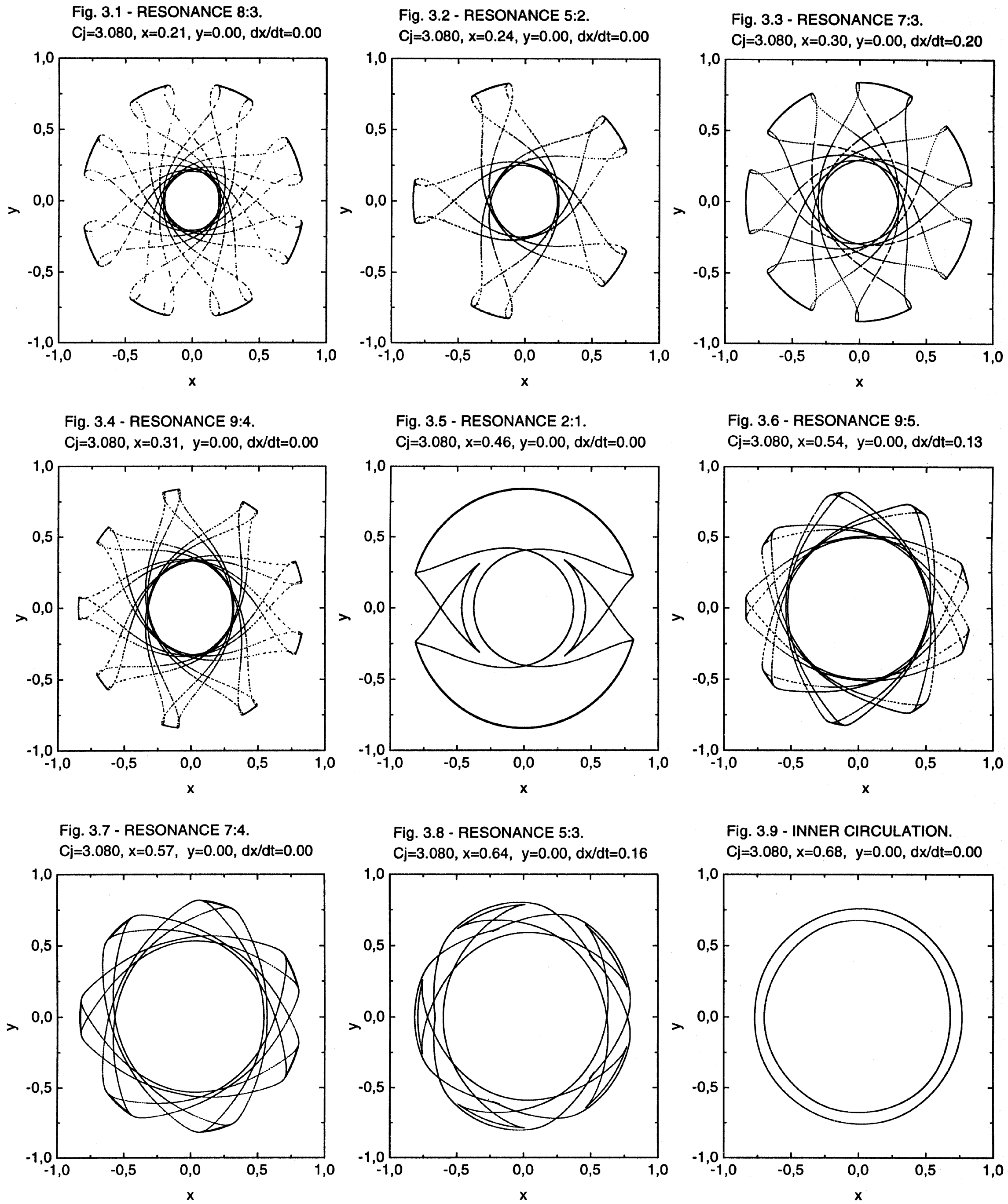

Fig. 3. Caustics for $Cj = 3.080$ (interior region). The values for the resonances and the initial conditions are given for each caustic.

I and II" (Winter and Murray, 1994a, 1994b). As one can conclude from an examination of the surfaces of section for this problem, the trajectories in the phase space are arranged into two large sets: interior orbits and exterior orbits, being, in general, those that move around just one of the primaries and those that move around both primaries. Furthermore, these two sets are subdivided into families that circle a periodic orbit, which in most of the cases are associated to the resonances that correspond to commensurability between the period of the primaries and the third body.

In this work we present a representative sample of trajectories of each set. As it will be seen below, their characteristics are closely related to the type of resonance.

Next, we present some caustics for each set and analyse their main features. The procedure for the computation of the caustics is described as follows. To obtain the caustics, we used the ODE integrator given by Shampine and Gordon (1975) with accuracy 10^{-9}. The zeros of the determinant were computed with accuracy 10^{-8} and were generated around 20,000 points for each caustic. The first thousand points were considered as belonging to the transient stage, and therefore, were not taken into account.

Our main task is to show that the information provided by caustics complements that provided by the Poincaré surfaces of section. Therefore, we always show our results associated to the surfaces of section. Thus we present the caustics related to the various typical regular orbits that appear in the typical surfaces of section.

For the interior orbits we selected one surface of section corresponding to the value: $y = 0.0, \dot{y} > 0.0$ and for the Jacobi constant Cj = 3.080 (Fig. 1). For this surface of section there are nine main sets of islands immersed in a chaotic region, where the order of resonances varies from first to fifth orders.

For the island located at $x = 0.21, \dot{x} = 0.0$, the caustics are given in Fig. 3.1. Computing the orbital period for the orbit located at the center of the island, it is found that it is in resonance 8:3 with the primaries. Note that the caustics shows 8 lobes, corresponding to the 8 turns that the particle completes while the primaries make 3. The order of the resonance, i.e., the difference between the two numbers of the commensurability, is the number of islands that appear in the surface of section around this orbit. The semi-major axis, a, and the eccentricity, e, of the osculating orbits of the torus represented by the caustics can be evaluated bearing in mind that the smallest distance from the torus to the massive primary corresponds to the pericenter and the largest distance corresponds to the apocenter. This can be seen in Fig. 2, where the caustics in the rotated frame are presented together with the corresponding orbit in the inertial frame. This evaluation improves when the orbit is closer to the periodic orbit. However, a and e can be estimated by using the equations of the two-body problem approximation (Hénon, 1997), i.e. for a $j : i$ resonance, $a = (i/j)^{2/3}$ and $Cj = 2(a(1-e^2))^{1/2} + 1/a$. In the present case, this gives a = 0.51 and e = 0.59.

Note also that due to the resonance the particle's apocenter near the less massive primary occurs always at its greatest distance (see for instance Figs. 3.1 and 3.2). This fact makes the orbit stable. Any other orbit with the same values of Cj, a and e, but without this property will get closer to the less massive primary becoming unstable. This is equivalent to the well known pericentric position on the line of the primaries for the stable internal orbits (see for example Winter and Murray, 1997a).

The following set of islands is around the resonance 5:2 and the associated caustics (Fig. 3.2) show similar characteristics to the one in Fig. 3.1. Next, we have Figs. 3.3 - 3.8, corresponding to resonances 7:3, 9:4, 2:1, 9:5, 7:4 and 5:3.

For the resonances 8:3, 5:2 and 7:3, the pericenter distance is less than 0.3, what, in the solar system, would correspond to the location of Mars orbit if we consider the less massive primary as Jupiter.

Note that the caustics associated to the resonances 7:4 and 5:3 have smaller eccentricities and therefore, the lobes are less noticeable, while for the 8:3 and 5:2 resonances the orbits have bigger eccentricities and the lobes are very noticeable. We also have inner circulation (Fig. 3.9), where the particle's orbit has a small eccentricity.

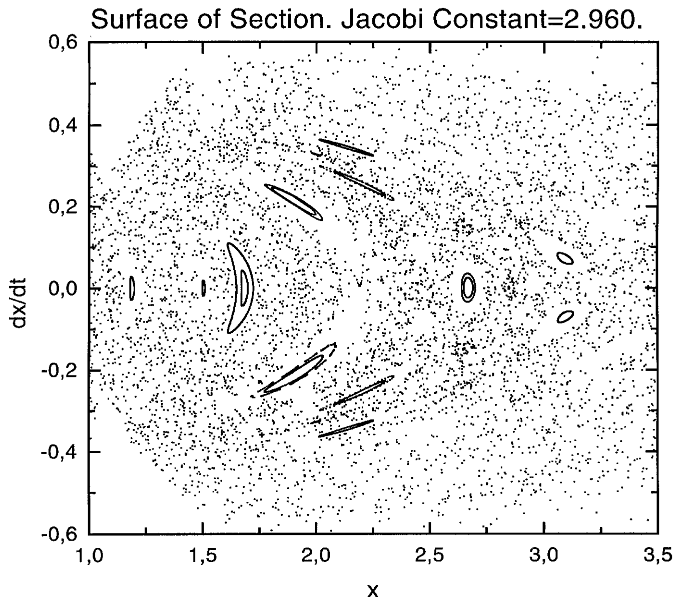

Fig. 4. The RTBP surface of section for $y = 0.0, \dot{y} < 0.0$ and Cj = 2.960 (exterior region).

For the exterior orbits, we also selected one surface of section corresponding to the value: $y = 0.0, \dot{y} < 0.0$ and for the Jacobi constant Cj = 2.960 (Fig. 4)

The caustics of the exterior orbits show similar characteristics to the interior orbits, but the lobes are internal instead of external. For the island located at $x = 1.20, \dot{x} = 0.0$ (Fig. 5.1), the orbit on the invariant torus moves just around the less massive primary. This case looks like a simple circulation, but in fact it corresponds to the resonance 1:1 (non-coorbital).

A particularity of this region is the existence of asymmetric orbits associated with $1 : n$ resonances. This can be seen in Figs. 5.4 and 5.6 corresponding to 1:2 and 1:3 asymmetric resonances. A particularity of this kind of resonance is that the symmetric island that appears in the surface of section corresponds to two families of trajectories (Winter and Murray, 1997b). Otherwise, symmetric islands have symmetric lobes.

Notice that to exterior non asymmetric orbits the particle's pericenter is more distant from the less massive primary (see for instance Figs. 5.2 and 5.3). We can also note that the lobes are not in opposition to the less massive primary and they are symmetric for the symmetric islands.

4. Conclusion

We showed that caustics complement the information provided by the Poincaré surface of section. If we consider both methods, we can determine the values of p and q for the $p : q$ resonance for every orbit. So we conclude that the method can help us to understand a bit more about the complexity of non-linear Hamiltonian systems.

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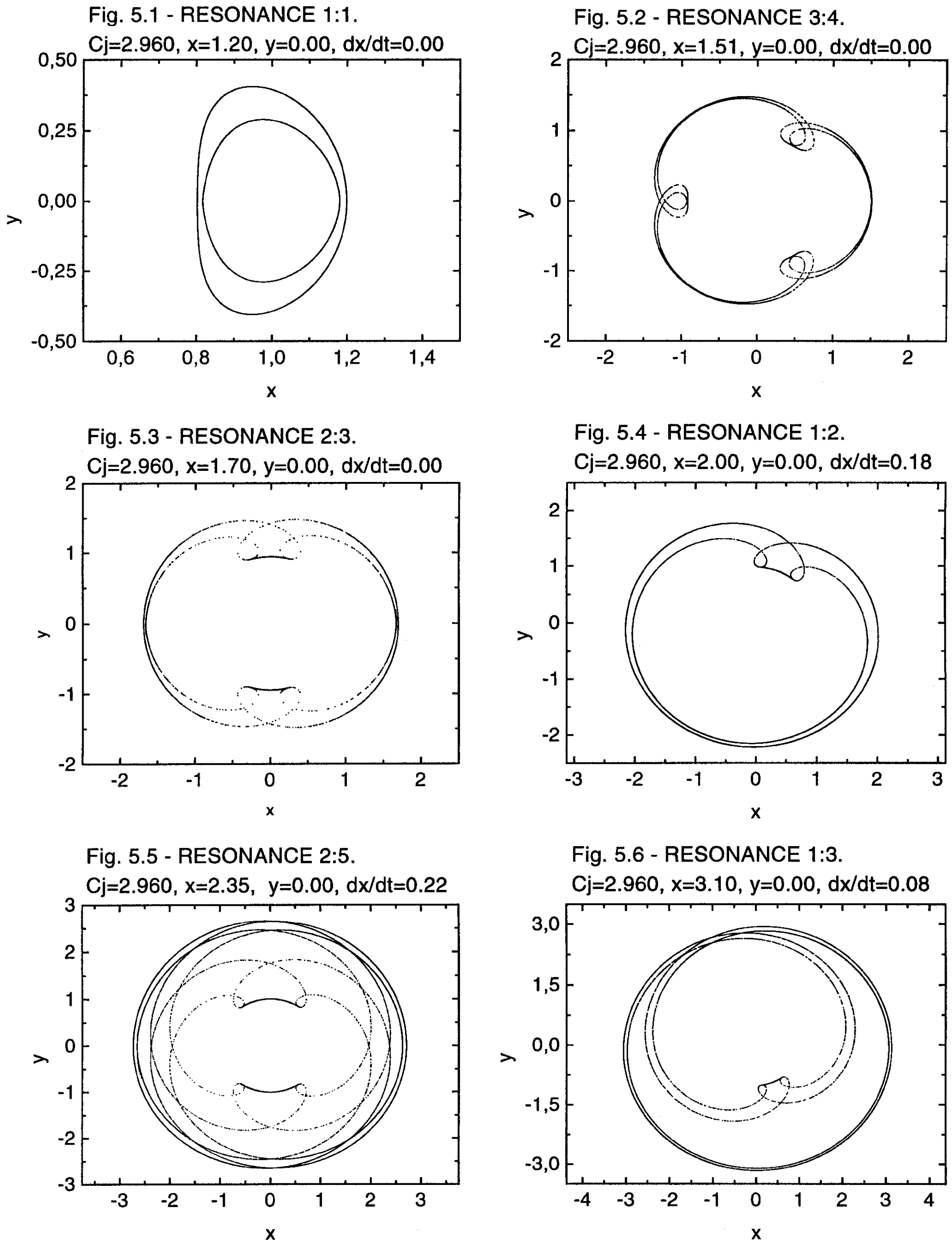

Fig. 5. Caustics for $Cj = 2.960$ (exterior region). The values for the resonances and the initial conditions are given for each caustic.

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