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*Research Note*

# On the runaway instability of relativistic tori*

**Marek A. Abramowicz**[1,3,4], **Vladimír Karas**[2], **and Antonio Lanza**[3]

[1] Department of Astronomy and Astrophysics, Göteborg University and Chalmers University of Technology, S-412 96 Göteborg, Sweden (marek@tfa.fy.chalmers.se)
[2] Astronomical Institute, Charles University, Faculty of Mathematics and Physics, V Holešovičkách 2, CZ-180 00 Praha, Czech Republic (karas@mbox.cesnet.cz)
[3] Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, I-34 014 Trieste, Italy (lanza@sissa.it)
[4] International Centre for Theoretical Physics, Strada Costiera 11, I-34 014 Trieste, Italy

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**Abstract.** We further investigate the runaway instability of relativistic tori around black holes assuming a power law distribution of the specific angular momentum. We neglect the self-gravity of the torus; thus the gravitational field of the system is determined by the central rotating black hole alone (the Kerr geometry). We find that for any positive (i.e. consistent with the local Rayleigh stability condition) power law index of the angular momentum distribution, the tori are always runaway stable for a sufficiently large spin of the black hole. However, before concluding that some (astrophysically realistic) tori could indeed be runaway stable, one should include, in full general relativity, the destabilizing effect of self-gravity.

**Key words:** accretion: accretion-discs – black hole physics – instabilities

## 1. Introduction

Toroidal fluid configurations around black holes, known as thick accretion disks, have been suggested as models of quasars, other active galactic nuclei, and some X-ray binaries. Recently they have also assumed an important role in cosmological models of $\gamma$-ray bursts based on the merging of two neutron stars. A rather simple mathematical class of such configurations has been introduced by Fishbone & Moncrief (1976), Fishbone (1977), and fully described analytically by the Warsaw group: Abramowicz, Jaroszyński & Sikora (1978), Kozlowski, Jaroszyński & Abramowicz (1978), Jaroszyński, Abramowicz & Paczyński (1980), and Paczyński & Wiita (1980). In the last paper, a very practical and accurate Newtonian model for the gravitational field of a non-rotating black hole, known as the Paczyński-Wiita potential, was introduced.

The question of stability of such configurations attracted considerable attention, because it was recognized that some types of instability could have very direct, and quite interesting, astrophysical consequences. Obviously, for the same reason as in the Newton theory, the essentially *local* Rayleigh criterion for dynamical stability with respect to axially symmetric local perturbations demands that the specific angular momentum $l$ should increase with the distance $R$ from the axis of rotation (Seguin 1975). Even the Rayleigh-stable tori with $\mathrm{d}l/\mathrm{d}R > 0$ are dynamically unstable in the presence of a weak magnetic field (Balbus & Hawley 1991). The Balbus-Hawley instability does not destroy the large scale structure of tori, but instead drives a local turbulence which induces viscosity that is needed for accretion to occur. Much more threatening for the global structure was the important discovery by Papaloizou & Pringle (1987) that all *non-accreting* tori are unstable to global non-axisymmetric perturbations. The consequences of this brilliant work have been studied by numerous authors, and the present view is that even a very modest mass loss due to accretion may be sufficient to stabilize the Papaloizou-Pringle modes (Blaes 1988). Also, it has been shown that the self-gravity of the disk has a stabilizing effect (Goodman & Narayan 1988).

## 2. The runaway instability

A general feature of fluid distribution around a black hole, is the existence of a self-crossing equipotential surface, similar to the Roche lobe in binary systems. The circle at which it crosses itself is called the cusp. The material inside the Roche lobe is gravitationally bound and only in such a case is equilibrium

*Send offprint requests to*: V. Karas

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possible. Those configurations that just fill their Roche lobes are marginally bound and thus could be called critical. The location of the cusp and the shape of the relativistic Roche lobe is determined by the combined gravitational potential of the black hole and the torus, and by the centrifugal potential due to the disc rotation.

For critical tori, when a small amount of material is accreting into the hole, a natural question arises (Abramowicz, Calvani & Nobili 1983): is the torus stable with respect to the transfer of mass through its cusp-like inner edge? To answer this question, one must determine the new position of the cusp, the shape of the Roche lobe, and the new equilibrium for the torus. If the relativistic Roche lobe shrinks sufficiently enough, matter which was bounded before the mass transfer will become unbounded, falling catastrophically into the hole on a dynamical time scale. This is the runaway instability. A full answer to the question about the conditions for the runaway instability to occur is far more difficult than was originally imagined, and is still unknown today. The claim by Abramowicz et al. (1983) that sufficiently massive $l$ = const tori are all runaway unstable, was based on an approximate model in which the gravitational field of the central black hole was modeled by the Paczyński-Wiita potential. The self-gravity of the torus was included using Newtonian gravity. However, using the Kerr metric for the black hole, but ignoring the self-gravity of the torus, Wilson (1984) demonstrated that the *non self-gravitating* tori do not suffer from the runaway instability. The reason is that accreting matter delivers not only mass but also angular momentum to the black hole in such a way that the ratio $a/M$ (dimensionless specific angular momentum of the black hole) remains approximatively constant thereby leaving unchanged the structure of the space time. By applying another approach (a Newtonian model of a rotating black hole–ring system) Khanna & Chakrabarti (1992) confirmed however that self-gravitating rings are runaway unstable.

It is obvious that the situation is complex because the self-gravity makes tori runaway unstable, while rotation of the central black hole stabilizes them. Thus, both effects are important and neither can be neglected. To clarify the issue, Nishida et al. (1997), using numerically constructed full general-relativistic models of self-gravitating tori with $l$ = const found that those that are sufficiently massive (the ratio $m_{\rm D} = M_{\rm D}/M > 0.1$ of the disc mass $M_{\rm D}$ to the central black-hole mass $M$) are runaway unstable *always*. These authors pointed out that the runaway instability introduces an acute difficulty for the recently considered models for $\gamma$-ray bursts based on the merging of two neutron stars by making the lifetime of the torus (which is a product of the merging) too short to provide enough energy to power the burst.

Most recently, however, Daigne & Mochkovitch (1997) made an important discovery that a non-zero gradient of the angular momentum distribution inside the torus has a strong stabilizing effect. They used models of non self-gravitating tori and the Paczyński-Wiita potential to describe the black hole. In this case, after the mass transfer, the angular momentum content in the innermost part of the resulting configuration is higher and therefore provides a stronger gravitational barrier preventing catastrophic runaway accretion.

To verify that such a stabilizing effect is also present in full general relativistic regime, one should study the stability of equilibrium configurations constructed self-consistently, in a similar way as in Lanza (1992) and Nishida & Eriguchi (1994) but for a non-constant distribution of angular momentum. Constructing general-relativistic equilibrium configurations with an arbitrary Eulerian rotation law is not particularly difficult. Explicit analytic models with general Eulerian radial distribution of angular momentum, $l \equiv l(R)$ (for example with a power-law $l \propto R^q$) have been constructed by Jaroszyński et al. (1978), and later used by Kuwahara (1988), Chakrabarti (1991), and others. If the equation of state is fixed, each particular model in this class is determined by the ratio $m_{\rm D}$ of the disc mass $M_{\rm D}$ to the central black-hole mass $M$ and by the location of its inner radius. However, since we want to study the stability of a torus that is accreting axisymmetrically, it is necessary to mimic the process by means of a quasi-stationary sequence of equilibria along which the Lagrangian distribution of angular momentum per baryon $j(m)$ is conserved. From a numerical point of view this is a difficult task. Already in Newtonian theory finding equilibria with a given Lagrangian angular momentum distribution demands a non trivial effort (Ostriker & Marck 1968, Daigne & Mochkovitch 1997). The self-consistent iteration between the Einstein field equations and the equation describing the equilibrium of matter is complicated by the fact that the assumed distribution of the angular momentum depends on the Lagrangian coordinate $m$, whereas the field equations are expressed in terms of Eulerian coordinates (e.g. Kozlowski et al. 1978). (Of course, this problem would disappear if we had spherical symmetry, since in this case one could easily rewrite Einstein equations in terms of $m$.) Moreover, in general relativity, the surfaces where the specific angular momentum is constant (von Zeipel surfaces) do not have a trivial cylindrical structure $R$ = const, as in Newtonian theory, but must be determined iteratively. Their shape is more complicated, given as an implicit function $\mathscr{R}(R, \theta)$ = const, and it is changed during accretion. This fact is essential for construction of the tori. It is convenient therefore to explore such complicated iterative techniques first in a simpler approximation before solving the full problem.

In this Note we have solved a particular technical problem that is necessary to obtain the full solution, namely we established a method of finding non self-gravitating equilibria with a given Lagrangian distribution of angular momentum in general relativity. Apart from a straightforward replacement of Newtonian formulae by corresponding relativistic expressions, the major difference and difficulty in studying relativistic test-fluid tori which we have faced in this work follows from the fact that the structure of von Zeipel surfaces is not given a-priori and must be determined iteratively.

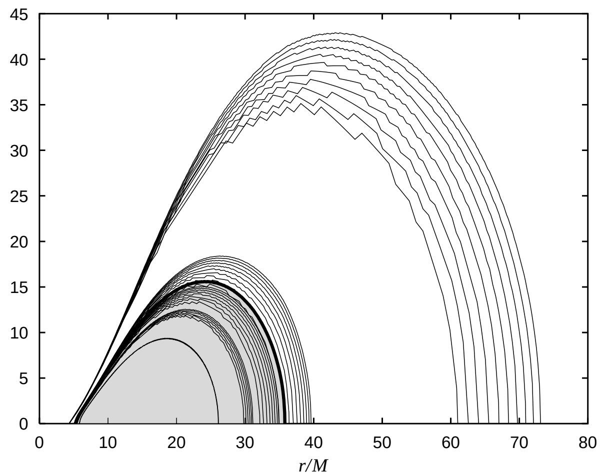


**Fig. 1.** Illustration of the iterative procedure for constructing a model of the torus with a prescribed mass. See text for details.

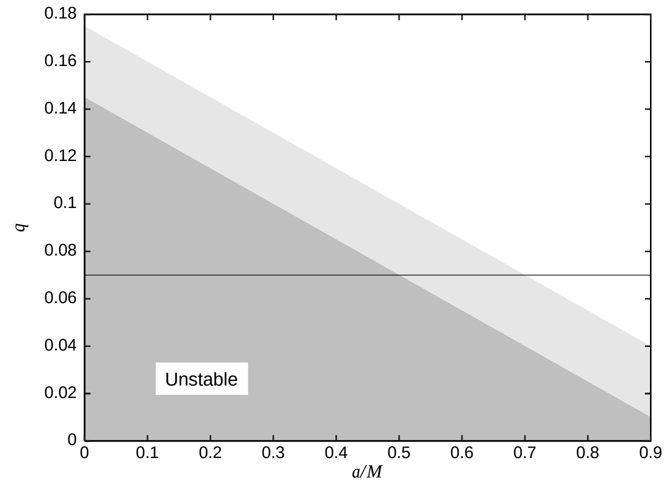


**Fig. 2.** Shading indicates the region in space of parameters $(q, a/M)$ in which the tori filling their Roche lobes are runaway unstable with respect to small overflow. Dark-shaded area corresponds to the initial mass of the torus $M_{\rm D} = 0.36 M_{\odot}$, light-shaded area refers to $M_{\rm D} = 0.18 M_{\odot}$. Black-hole rotation has apparently a stabilizing effect. The horizontal line at $q = 0.07$ indicates where the stability boundary is located in the corresponding pseudo-Newtonian analysis of Daigne & Mochkovitch (1997). See text for details.

## 3. Numerical procedure

We assume that the spacetime is axially symmetric, stationary, and described by the Kerr metric $g_{\mu\nu}$ in Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ (geometric units with $c = G = 1$ in usual notation). The metric is parameterized by the black-hole mass $M$ and its specific angular momentum $a/M$. The torus is described by a relativistic polytrope which relates the pressure $P$ to the density of material $\rho$: $P = \kappa\rho^{\gamma}$ with $\kappa$ and $\gamma$ constant (Tooper 1965). The corresponding energy density $\epsilon = \rho + P/(\gamma - 1)$ is related to the energy per unit mass $\mathscr{E} = (P + \epsilon)\rho^{-1} u_t$ and to Lagrangian angular-momentum density $j = \mathscr{E}\, l$ ($u_t$ denotes the time-component of the four-velocity of the fluid; cf. Kozlowski et al. 1978).

First, an initial critical configuration is constructed, given the parameters $M_{\rm D}, M, a, \kappa$, and $\gamma$. The starting angular-momentum distribution is specified as a power-law $l \propto \mathscr{R}^q$ with $q \geq 0$ constant. The model is then determined by the above-given parameters, in particular the inner and the outer edges, and the centre of the torus where the pressure is maximum. The matter distribution inside the disc is calculated and expressed in terms of a spline-interpolated function of the von Zeipel coordinate, $m(\mathscr{R})$, where $m(\mathscr{R}(r_{\rm in})) = 0$ and normalisation guarantees $m(\mathscr{R}(r_{\rm out})) = 1$ at the outer edge. Also the Lagrangian angular-momentum density $j$ is expressed as a normalised spline-interpolated function: $\mathscr{J} \equiv \mathscr{J}(\mathscr{R})$, $0 \leq \mathscr{J} \leq 1$. This initial configuration is computed with a pre-determined accuracy of $M_{\rm D}$. Fig. 1 illustrates the above-described iterative procedure. Each bundle of the curves corresponds to a set of surfaces with the inner edge fixed and increasing resolution (the effect of improving resolution is clearly visible). Construction of the models is terminated when the mass of the torus is determined with accuracy better than a pre-determined value, typicaly 0.1%. Location of the inner edge is then changed (another bundle of the curves) and the code iterates again, producing another configuration with a different mass of the torus. The whole procedure is repeated (five times in this example) with the inner edge bounded between the marginally bound and the marginally stable orbits. Iterations are terminated when the torus has a right mass. The final configuration is indicated by shading.

Now we assume that a small amount of material contained near the cusp, between $r_{\rm in}$ and $r_{\rm in} + \delta r_{\rm in}$, is transferred throughout the cusp to the black hole. The mass ratio $m_{\rm D} \equiv M_{\rm D}/M$ is decreased by $\delta m_{\rm D}$, typicaly of the order of one percent, while the angular-momentum parameter of the black hole is changed by a corresponding value, $\delta m_{\rm D} l(r_{\rm in})$. The distribution of $j$ is conserved during the process of accretion: $j'(m') = j(m)$ (primes denote new quantities, after the mass transfer). The next iteration starts with a guess for the new function $l'_0 = j'/\mathscr{E}$. The inner edge of the new critical configuration is iterated until a configuration with the required $m_{\rm D}$ is found; at each iteration $l'_0$ is also updated, so that the final $r'_{\rm in}$ and $l'(\mathscr{R})$ are known only when convergence has been reached. This updating induces a corresponding change of the von Zeipel surfaces, $\mathscr{R}(r, \theta) \rightarrow \mathscr{R}'(r, \theta)$ which provides however only a minor modification and is visible near the inner edge. At convergence then one can verify whether the inner edge of the new critical surface has moved. If $r_{\rm in} + \delta r_{\rm in} > r'_{\rm in}$, then the new configuration is unstable since the crossing equipotential has shrunk.

## 4. Results and conclusions

We have adopted the same values of the starting parameters as Daigne & Mochkovitch (1997): $M = 2.44\, M_{\odot}$, $\gamma = 4/3$, $\kappa = 0.5 \times 10^{15}$ (cgs), corresponding to the models of presumed massive tori which are formed in neutron star mergers. The mass of the torus was chosen to be $M_{\rm D} = 0.36\, M_{\odot}$ and

$M_D = 0.18\,M_\odot$, $\delta m_D = 1\%$. We assumed that the torus corotates with the black hole. Fig. 2 illustrates the results of the calculation, namely the boundary between stable and unstable configurations in the plane of parameters $(q, a/M)$ of the initial configuration. It can be seen that *increasing $q$ and increasing $a/M$ both stabilize the configuration. This is the main result of our present note.* The boundary between stability and instability is positioned at roughly the same value of $q$ as found by Daigne & Mochkovitch (1997) in their work. The difference is factor of 2 which cannot be considered too large given the approximate nature of the Paczyński-Wiita potential and the difference in the iterative procedures. Naturally, Daigne & Mochkovitch could not explore the dependence on $a/M$ which does not appear in the Paczyński-Wiita potential they adopted. The horizontal line in Fig. 2 indicates where the boundary is located in the corresponding analysis of Daigne & Mochkovitch with $M_D = 0.36\,M_\odot$ when pseudo-Newtonian radial coordinate is formally identified with Boyer-Lindquist $r$ in our analysis.

It remains to be discussed how self-gravity of the disc affects this criterion of stability of the accretion process, and what is the interplay between effects of a rotating black hole and a self-gravitating torus within full general relativity (work in progress).

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