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Potentials for the central parts of a barred galaxy

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Abstract. On the grounds of the figure-eight orbits found by Miller and Smith (1979) we obtain a general potential to model the central parts of a barred galaxy. The orbits are given in parametric form $x = x(\lambda, b), y = y(\lambda, b)$ and the potential is obtained as a power series in the small parameter b. It is found that the resulting potential depends on an arbitrary function $V_o(\lambda)$ and its first derivative. As a typical example we present a potential $V(x, y)$ which is a perturbed harmonic oscillator, symmetric to both axes x and y with terms up to the fourth degree in the variables x,y.

Key words: methods: analytical – Galaxy: kinematics and dynamics

1. Introduction

It is usual in Dynamics to try to find the families of orbits in a galaxy model described by a given potential. This procedure is well known as "the direct problem". On the other hand "the inverse problem" in Dynamics, as formulated by Szebehely (1974), is to try to obtain the potentials creating a given family of orbits. Strictly speaking in the above paper Szebehely presented a partial differential equation giving all potentials generating a certain monoparametric family of orbits f(x,y)=c with total energy E=E(f).

After Szebehely's original work a lot of interesting papers were produced on the subject. For relevant references the reader is referred to an interesting review by Bozis (1995).

In spite of all this rich literature very little was done in order to connect the inverse problem to Celestial Mechanics or Galactic Dynamics. In this second interesting field of Dynamical Astronomy, as far as the author knows, only one paper was written. This attempt to find the potential in a prolate barred galaxy, based on a family of orbits, was made by Szebehely et al. (1980). In particular they tried to determine the potential for a monoparametric family of figure-eigth orbits computed by Miller and Smith (1979) in their model representing a barred galaxy. The idea was original and interesting but their results are subject to criticism (see next section and also Bozis 1995).

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In the present work we shall make an attempt to relate the inverse prolem with Galactic Dynamics. This would be much more interesting if orbit families were observable in galaxies but they are not. The best way to overcome this difficulty is to use orbit families found by N- body simulations, where the potential is already calculated. The main difference, that makes our work interesting, is that, using the inverse problem methods, one can find an analytical expression for the potential associated with the given family of orbits, while in N-body simulations the potential is not given in an analytical form.

We shall try to aproach the problem using a recent method and some new techniques (see Bozis and Borghero 1997, Bozis and Caranicolas 1997). Our aim is to find the potential creating the figure-eight orbits computed by Miller and Smith (1979) in a three-dimensional self-consistent model representing a barred galaxy. In the above paper the authors found that about 25% of all orbits show clean or phase shifted figure-eights. It was found that the period of rotation of the bar was about six times larger than the period of the orbit; therefore, in first approximation one can neglect the rotation of the system. It is also true that the orbits in the (x,z) plane show a remarkable likeness to lemniscates (see Fig. 5 of Miller and Smith 1979). On this basis we consider motion only in the (x,z)-plane, neglect the rotation of the system and assume "clean" figure-eight orbits. Note that in our notation we use y instead of z, therefore our motion takes place in the (x,y) plane where the x-axis is along the bar.

2. Inverse problem theory

Let us consider the monoparametric family of orbits

$$f(x, y) = c \quad . \tag{1}$$

Let us also consider a test particle of unit mass moving on any member of the family (1). Then the second order partial differential equation giving all potentials $\Phi(x, y)$ responsible for the above motion (see Bozis 1984) reads

$$-\Phi_{xx} + k\Phi_{xy} + \Phi_{yy} = h\Phi_x + \mu\Phi_y \quad , \tag{2}$$

where

$$k = \frac{1-\gamma^2}{\gamma}$$
$$h = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma} \quad (3)$$
$$\mu = h\gamma + \frac{3\Gamma}{\gamma}$$

and

$$\gamma = \frac{f_y}{f_x} \ , \ \Gamma = \gamma\gamma_x - \gamma_y \quad . \quad (4)$$

Szebehely's equation

$$E = \Phi(x,y) - \frac{1+\gamma^2}{2\Gamma}(\Phi_x + \gamma\Phi_y) \quad , \quad (5)$$

gives the total energy E=E(c) for the motion on each orbit, member of the family (1). Notice that in order to apply Eqs. (2) and (5) the monoparametric family must be of the (solved for c) form (1). The above two formulae are not, in general, applicable when the monoparametric family is given in an implicit form

$$f(x,y,c) = 0 \quad . \quad (6)$$

If we write Eq. (2) for the family (6), instead of (1), then the coefficients k, h, μ will be functions of x,y,c while the solutions for $\Phi(x,y)$ should not include the parameter c. This is not quaranteed unless the function f(x,y,c) is appropriate. Here we must note that in the paper by Szebehely et al. (1980) equation giving the family of orbits (see Eq. (1) page 880 of the paper) is not solved for the parameter n while the solution for the potential (see Eq. (6) page 881) includes the parameter n, as it should not. Needless to say of course that n cannot be replaced at the final step from Eq. $f(r,\theta) = r^2 - \alpha^2 cosn\theta$ in the expression for the potential, because the potential would be then valid only for the family of figure-eight orbits and not for all families as it should.

In a recent paper Bozis and Borhgero (1997) formulated the inverse problem in case where the family of orbits (1) is given in parametric form

$$x = x(\lambda, b) \ , \ y = y(\lambda, b) \quad . \quad (7)$$

In Eqs. (7) the parameter b characterizes the family while the parameter λ changes along each given orbit $b = b_1$.

Assuming that the Jacobian

$$J = x_\lambda y_b - x_b y_\lambda \quad (8)$$

is not zero, Eqs. (7) describe a transformation establishing locally an one to one correspondence between points in the (x,y) Cartesian plane (where actual motion takes place) and a region in the (λ, b) plane.

In the new variables λ, b the potential is written

$$V(\lambda, b) = \Phi[x(\lambda,b), y(\lambda,b)] \quad (9)$$

while Eqs. (2) and (5) read

$$\delta V_{\lambda\lambda} + \epsilon V_{\lambda b} + (2+\delta_\lambda)V_\lambda + \epsilon_\lambda V_b = 0 \quad (10)$$

and

$$E = V(\lambda,b) + \frac{1}{2}(\delta V_\lambda + \epsilon V_b) \quad , \quad (11)$$

with

$$\delta = \frac{\alpha\beta}{J\Delta} \ , \ \epsilon = \frac{\beta^2}{J\Delta} \quad , \quad (12)$$

where $J \neq 0$ is given by (8) while $\Delta \neq 0$ and α, β are given by

$$\begin{aligned} \Delta &= x_{\lambda\lambda}y_\lambda - y_{\lambda\lambda}x_\lambda \\ \alpha &= -(x_\lambda x_b + y_\lambda y_b) \\ \beta &= x_\lambda^2 + y_\lambda^2 \quad . \end{aligned} \quad (13)$$

Eqs. (10) and (11) have the following meaning: As one goes from the form (1) to the form (7), for the given family of orbits, one can write Eq. (10) for the unknown potential $V(\lambda, b)$.To each solution $V(\lambda, b)$ assosiated with the family (7) there corresponds a total energy of the test particle given by Eq. (11).Because we must have E=E(b) one can derive Eq. (10) from (11) taking into account the condition $E_\lambda = 0$.

Furthermore, for any compatible pair of a family (7) and potential (9), it must be $E \geq V(\lambda, b)$.Taking into account Eqs. (11) and (12) this condition leads to the inequality

$$\frac{1}{J\Delta}(\alpha V_\lambda + \beta V_b) \geq 0 \quad , \quad (14)$$

giving the region on the $\lambda - b$ plane where the family of curves (7) includes real orbits created by the potential $V(\lambda, b)$. Naturally an orbit in the $\lambda - b$ plane is a straight line $b = b_1$ parallel to the λ axis.

3. Potentials associated with figure-eight orbits

In what follows we shall try to find a solution for the potential creating the family of figure-eight orbits given in parametric form by the equations

$$x = \cos\lambda \ , \ y = b\sin 2\lambda \quad . \quad (15)$$

Members of this family for several values of b are shown in Fig. 1. Note that the figure-eight orbits of Miller and Smith (1979) correspond to $b \leq 0.2$.

Using the MATHEMATICA computer algebra system, we find that one can can write Eq. (10) in the form

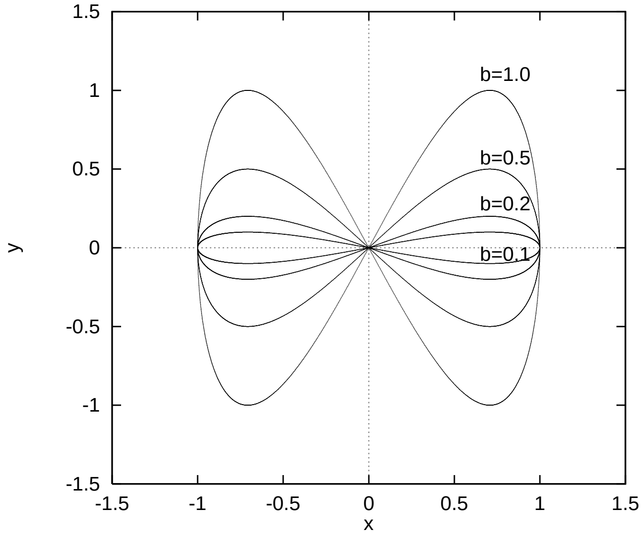

Fig. 1. Figure-eight orbits for various values of b.

$$(\alpha_0 + \alpha_1 b + \cdots + \alpha_3 b^3)V_{\lambda\lambda} +$$
$$(\beta_0 + \beta_1 b + \cdots + \beta_4 b^4)V_{\lambda b} +$$
$$(\gamma_0 + \gamma_1 b + \cdots + \gamma_3 b^3)V_{\lambda} +$$
$$(\delta_0 + \delta_1 b + \cdots + \delta_4 b^4)V_b = 0 \quad . \qquad (16)$$

The calculations show that $\alpha_0 = \alpha_2 = \beta_1 = \beta_3 = \gamma_0 = \gamma_2 = \delta_1 = \delta_3 = 0$ while the remaining are lengthy trigonometric functions of λ. We do not think it was necessary to give these equations here.

As in Bozis and Caranicolas (1997) we shall seek a solution for the potential expressed in the form of a series

$$V(\lambda, b) = V_0(\lambda) + bV_1(\lambda) + b^2 V_2(\lambda) + b^3 V_3(\lambda) + \cdots \qquad (17)$$

In order to determine the functions $V_0(\lambda), V_1(\lambda), V_2(\lambda), \ldots$ etc. we must set equal to zero the coefficients for every power of the small parameter b.

First we take the zero order terms in b from the expansion (16) obtaining

$$(4\sin 2\lambda - \sin 4\lambda)V_{1\lambda} +$$
$$(14 - 8\cos 2\lambda + 2\cos 4\lambda)V_1 = 0 \quad . \qquad (18)$$

The solution of Eq. (18) reads

$$V_1(\lambda) = V_{10}\frac{(2-\cos 2\lambda)(1+cos2\lambda)}{(1-cos2\lambda)} \quad , \qquad (19)$$

where V_{10} is an arbitrary constant.

From the first order terms in b we find

$$A_1(\lambda)V_{2\lambda} + A_2(\lambda)V_2 + A_3(\lambda)V_{0\lambda} + A_4(\lambda)V_{0\lambda\lambda} = 0 \quad , \qquad (20)$$

where $V_{0\lambda}, V_{0\lambda\lambda}$ are the first and second order derivatives with respect to λ of an arbitrary function V_0 while

$$\begin{aligned} A_1(\lambda) &= 4\cos\lambda - 5\cos 3\lambda + \cos 5\lambda \\ A_2(\lambda) &= 36\sin\lambda - 10\sin 3\lambda + 2\sin 5\lambda \\ A_3(\lambda) &= 54\cos\lambda - 26\cos 3\lambda - 14\cos 5\lambda + 2\cos 7\lambda \\ A_4(\lambda) &= \sin\lambda - 3\sin 3\lambda - 3sin5\lambda + \sin 7\lambda \quad . \end{aligned} \qquad (21)$$

Solving Eq. (20) we obtain

$$\begin{aligned} V_2(\lambda) = \; & \frac{\cos\lambda}{\sin^2\lambda}[(V_{20} + V_0)(2\cos 3\lambda - 6\cos\lambda) + \\ & V_{0\lambda}(\sin 3\lambda - \sin\lambda)] \quad , \end{aligned} \qquad (22)$$

where V_{20} is an arbitrary constant. Note that in the solution (22) appears the arbitrary function $V_0(\lambda)$ and its first derivative $V_{0\lambda}$.

Setting equal to zero the coefficients of b^2 we find

$$\begin{aligned} & B_1(\lambda)V_{3\lambda} + B_2(\lambda)V_3 + \\ & B_3(\lambda)V_{1\lambda\lambda} + B_4(\lambda)V_{1\lambda} + B_5(\lambda)V_1 = 0 \quad , \end{aligned} \qquad (23)$$

where

$$\begin{aligned} B_1(\lambda) &= 12\cos\lambda - 15\cos 3\lambda + 3\cos 5\lambda \\ B_2(\lambda) &= 108\sin\lambda - 30\sin 3\lambda + 6\sin 5\lambda \\ B_3(\lambda) &= 2\sin\lambda - 6\sin 3\lambda - 6\sin 5\lambda + 2\sin 7\lambda \\ B_4(\lambda) &= 148\cos\lambda - 44\cos 3\lambda - 4\cos 5\lambda - 4\cos 7\lambda \\ B_5(\lambda) &= 192\sin\lambda - 224\sin 3\lambda - 32\sin 5\lambda \quad . \end{aligned} \qquad (24)$$

Taking into account (19), we solve (23) obtaining

$$V_3(\lambda) = \frac{\cos^2\lambda}{sin^4\lambda}[V_{30}(6\cos 2\lambda - \cos 4\lambda - 5) - 8V_{10}] \quad , \qquad (25)$$

where V_{30} is again an arbitrary constant.

Taking the first four terms of the series (17), we obtain the following general expression for the potential

$$\begin{aligned} & V(\lambda, b) = V_0(\lambda) + bV_{10}\frac{(2-\cos 2\lambda)(1+\cos 2\lambda)}{(1-\cos 2\lambda)} + \\ & b^2\frac{\cos\lambda}{\sin^2\lambda}[(V_{20} + V_0)(2\cos 3\lambda - 6\cos\lambda) + \\ & V_{0\lambda}(\sin 3\lambda - \sin\lambda)] + \\ & b^3\frac{\cos^2\lambda}{\sin^4\lambda}[V_{30}(6\cos 2\lambda - \cos 4\lambda - 5) - 8V_{10}] \quad . \end{aligned} \qquad (26)$$

4. A special galactic model

Now we proceed to a special case of the general potential (26). First we set $V_{10} = V_{30} = 0$ and choose

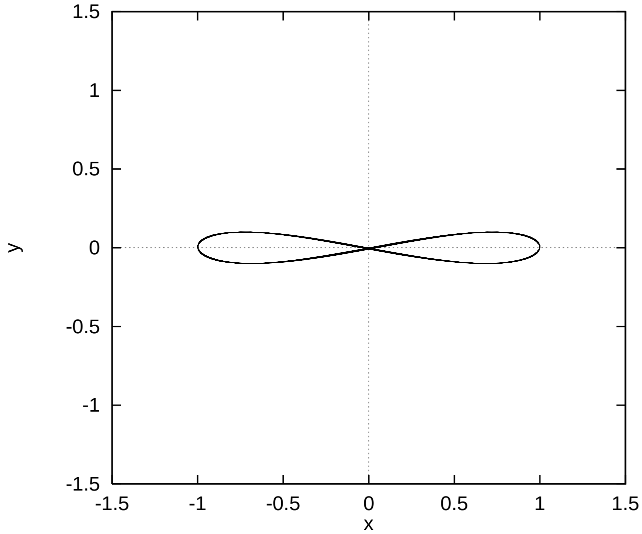

Fig. 2. A figure-eight orbit found by numerical integration. The values of the parameter and initial conditions are given in the text.

$$V_0 = V_{00}\cos^2\lambda - \epsilon_1 \sin^4\lambda \tag{27}$$

as the arbitrary function, where V_{00} is a positive constant while $\epsilon_1 > 0$ is considered as a small parameter.Taking $V_{20} = -V_{00}$ we obtain the potential

$$\begin{aligned} V_s(\lambda, b) &= V_{00}(\cos^2\lambda + 4b^2\sin^2 2\lambda) - \epsilon_1[(1-\cos^2\lambda)^2 + \\ &\quad b^2\sin^2 2\lambda(6\cos^2\lambda - 5)] \quad , \end{aligned} \tag{28}$$

which, translated in the x,y variables and with $V_{00} = 1/2$, becomes

$$\begin{aligned} V_s(x, y) &= \frac{1}{2}(x^2 + 4y^2) - \\ &\quad \epsilon_1[(1-x^2)^2 + (6x^2 - 5)y^2] \quad . \end{aligned} \tag{29}$$

We observe that we have a two dimensional perturbed harmonic oscillator potential, where the ratio of the unperturbed frequencies is 1:2 while ϵ_1 can be considered as the perturbation strength. Potentials of the form (29) have been frequently used to describe local motion in galaxies (see Caranicolas 1990, 1994).

Using (11) and (28) we find that the energy of the figure-eight orbits is given by

$$E = \frac{1}{2} + 2b^2 - \epsilon_1 \frac{[b^2 R_1(\lambda) + b^4 R_2(\lambda)]}{4(\cos 2\lambda - 2)} \tag{30}$$

where

$$\begin{aligned} R_1(\lambda) &= 22 - 29\cos 2\lambda + 4\cos 4\lambda + 5\cos 6\lambda - 2\cos 8\lambda \quad , \\ R_2(\lambda) &= -12\cos 2\lambda + 6\cos 6\lambda + 6\cos 10\lambda \quad . \end{aligned} \tag{31}$$

With the aid of (14) we find that the region on the $\lambda - b$ plane where the family of orbits (15) exists is given by

$$\begin{aligned} &\frac{2}{\epsilon_1}(2 - \cos 2\lambda) \geq 5 - 6\cos 2\lambda + \cos 4\lambda + \\ &6b^2(\cos 6\lambda - \cos 2\lambda) \quad . \end{aligned} \tag{32}$$

It is evident that inequality (32) is always true for small values of ϵ_1 and b ($\epsilon_1 \leq 0.1 \quad , b \leq 0.2$). Fig. 2 shows a figure-eight orbit found by numerical integration of the equations of motion in potential (29) when $b = 0.1$, $\epsilon_1 = 0.03$. We choose $\lambda = 0$ so that the orbit starts at ($\lambda = 0$) $\quad x_0 = 1, \quad y_0 = 0$, with velocity $v_{0x} = 0$ while v_{0y} is found from the energy integral. Note that for the above initial conditions the initial value of energy is always $E = 1/2{+}2b^2$. The integration time was 60 time units and the test particle makes about ten revolutions during this time period.Comparing this orbit with the corresponding orbit in Fig. 1 we see that potential (29) produces very well the figure-eight orbits.

5. Discussion

In this work we have presented a potential for the figure-eight orbits the existence of which has been established by N-body simulations in the central parts of a barred galaxy model. The solution for the potential was given as a power series in the small parameter b characterising the members of the given family. It was found that the general solution contains an arbitrary function and its first derivative. Choosing the arbitrary function to be of form (27) and for appropriate values of the other constants we obtained the perturbed harmonic oscillator potential (29) which is of significant astronomical interest. This is justified by the fact that expansions of mass model potentials near an equilibrium point give potentials of this form (see also Caranicolas and Innanen 1991)

It is evident that solutions (26) and (28) are approximate solutions of Eq. (10). Due to this fact formula (30) and (32) are also periodic functions of λ. It is interesting to note that, in the case where $\epsilon_1 = 0$, we have an exact solution of Eq. (10) giving the potential of a two dimensional harmonic oscillator with ratio of frequencies 1:2. Of course in this particular case formula (30) becomes function of b only while inequality (32) reduces to $1 \geq 0$. Therefore one can conclude that, if one chooses as an arbitrary function an harmonic oscillator potential, then the result (for appropriate values of the constants involved) will be a two dimensional harmonic oscillator potential. If one includes a perturbation ($\epsilon_1 \neq 0$) then the result will be a perturbed two dimensional harmonic oscillator.

Using numerical integration we see that potential (29) reproduces very well the figure-eight orbits. The results are good for small values of ϵ_1 (say $\epsilon_1 \leq 0.1$) when $b \leq 0.2$. The numerical integration was made by a Bulirsh-Stoer method in double precision and the accuracy of the calculations was checked by the constancy of the energy integral which was conserved up to the twelfth significant figure.

Finally, the author would like to make clear that the results of this work do not apply to barred galaxies but to a non-rotating isolated bar. It is well known that a barred galaxy is always surrounded by a massive disk and the dynamics of its bar must be understood as the one of a rapidly tumbling object. On the other hand, we believe that this work is of interest, because it makes an attempt to go, from stellar orbits to galactic potentials and it can be considered as a first step in the connection of the inverse problem with Galactic Dynamics.

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