Astron. Astrophys. 332, 314–324 (1998)

ASTRONOMY
AND
ASTROPHYSICS

Waves in a convective atmosphere: 1D periodical model

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Received 4 April 1997 / Accepted 27 November 1997

Abstract. This paper treats a one-dimensional model of stationary convection. In this model I derive the equation governing acoustic waves in an atmosphere that is structured by hot and cold flows. An exact solution is obtained in terms of an infinite Hill determinant. The physics of acoustic waves in a convective atmosphere and in a crystal lattice are similar, and some of the concepts of solid state physics are generalized to the current problem. It is shown that there are three basic wave modes, namely, acoustic waves, vibrational waves and turbulent sound, which are all different from acoustic waves in a uniform atmosphere. The vibrational waves could appear due to local oscillations of the convective elements. The turbulent sound is driven only by the dynamical pressure. The temperature and velocity fluctuations in a convective atmosphere are responsible for the appearance of Brillouin zones. Waves with frequencies within the band gap of the convective atmosphere undergo reflection at the edge of the Brillouin zones, where they meet a potential barrier. The application of the model to a turbulent atmosphere is discussed. Some effects relevant to helioseismology are outlined.

Key words: convection – hydrodynamics – turbulence – waves – Sun: oscillations – stars: oscillations

1. Introduction

The exploration of hydrodynamical waves in a convective atmosphere is initiated especially by helioseismology. The accuracy of the measured frequencies and line profiles of p modes is sufficient to reveal the effects of convection on the waves. A general theory of hydrodynamical waves in a turbulent convective atmosphere does not exist so far. There are several ways of attack, connected with different aspects. Wave generation by turbulence, wave absorption due to turbulent viscosity, the effect of turbulence on the effective phase velocity, the interaction of the wave turbulence and turbulence itself belong to these aspects. All these problems are coupled physical effects; their separate treatment is a causal approach. Conceptually, the limiting cases of slow and fast turbulence are used to simplify the problem. The approximation of fast *fast* turbulence is valid when the lifetime or turnover time of the turbulent elements is small enough in comparison with the wave period so that time averaging becomes possible. The simplest approach of this kind is to consider the effect of turbulent pressure on the sound velocity (Kosovichev 1995, Rosenthal 1997). Mean-field theory permits an easy treatment of the turbulent viscosity (Stix et al. 1993) and of the corrections to the sound velocity (Rüdiger et al 1997). In the limiting case of *slow* turbulence the waves are affected only by the space pattern of the random velocity and temperature fluctuations, which are so slow that the waves can follow the changes of the pattern. This approach has been used by Murawski & Roberts (1993a,b), Murawski & Goossens (1993) for f modes and by Rosenthal et al. 1995a,b for p modes. Their treatment of the f and p modes was based implicitly on the assumption that there are only slight changes of the waves that are typical for a uniform atmosphere. Consequently, perturbation theory has been applied to the problem. Such an approach is sound but not universally true.

A first attempt of an exact derivation of the effects of the velocity and temperature fluctuations on acoustic waves in an atmosphere has been undertaken by Zhugzhda & Stix (1994, hereafter Paper 1). In their simple 1D model of stationary convection a variety of wave modes has been found. These new effects can be understood in terms of solid state physics. A good example is the difference in the behaviour of electrons in vacuum and in the potential field of a metal lattice. In the latter case the motion of the electrons is described in terms of Bloch wave propagation (Bloch 1928), which is not a perturbation of the free electron state. Acoustic waves in a structured atmosphere are similar: They are affected by the fluctuations in temperature and velocity in such a way that their properties qualitatively differ from the properties of waves in a uniform atmosphere. Moreover, as will be shown, new wave modes appear in the structured atmosphere.

The analysis of Zhugzhda & Stix had been restricted to longitudinal waves, and the clues provided by solid state physics had not been appreciated. In the present paper the theory is developed in order to understand some of the properties of waves in a nonuniform atmosphere in terms of solid state concepts such as Brillouin zones, vibrational waves and band gaps. Although this terminology is used, it will be clear that the analogy to, for example, the theory of acoustic waves in a crystal lattice can be applied only in a general sense. The governing equation (Eq. 15 below) is substantially more complicated than, for example, the equation for electrons in a periodic potential, which appears in the theory of Bloch waves. This precludes the use of the theory of Mathieu functions. Instead, the exact solution of the problem will be obtained by means of the general theory of equations with periodic coefficients. In this paper I shall treat the simplest cases step by step in order to introduce the concepts that so far have not been discussed in the astrophysical literature.

The stationary 1D model of regular hot and cold flows could be applied to waves propagating in a stationary convective atmosphere. It is discussed whether the same effects appear in a turbulent convective atmosphere with random temperature and velocity fluctuations. Finally, the effects of convection on p modes are outlined.

2. Basic equations

An equilibrium atmosphere with vertical hot upflows and cold downflows is considered. The equilibrium pressure p_0 is constant in the entire atmosphere, while the equilibrium values of temperature T_0, density ρ_0 and vertical velocity $\boldsymbol{V} = (0, 0, V)$ are arbitrary functions of x. All equilibrium variables are independent of the vertical coordinate z. The set of linearized hydrodynamical equations is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\nabla p, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \mathrm{div}\boldsymbol{v} + \boldsymbol{V} \cdot \nabla\rho + \boldsymbol{v} \cdot \nabla\rho_0 = 0, \quad (2)$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T_0 + \boldsymbol{V} \cdot \nabla T + (\gamma - 1)T_0 \mathrm{div}\boldsymbol{v} = 0, \quad (3)$$

$$\frac{p}{p_0} = \frac{T}{T_0} + \frac{\rho}{\rho_0}. \quad (4)$$

The time derivative of the momentum equation is

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + (\boldsymbol{V} \cdot \nabla)\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\frac{\partial p}{\partial t}. \quad (5)$$

The substitution of Eqs. (2) and (3) into the time derivative of the linearized equation of state (4) gives

$$\frac{\partial p}{\partial t} = p_0\left[\frac{1}{T_0}\frac{\partial T}{\partial t} + \frac{1}{\rho_0}\frac{\partial \rho}{\partial t}\right] = -\gamma p_0 \mathrm{div}\boldsymbol{v} - \boldsymbol{V} \cdot \nabla p. \quad (6)$$

After substitution of ∇p from (1) into (6) and of (6) into (5) the set of Eqs. (1)-(4) is reduced to one vector equation for the velocity $\boldsymbol{v}$

$$\rho_0\left(\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + (\boldsymbol{V} \cdot \nabla)\frac{\partial \boldsymbol{v}}{\partial t}\right) =$$

$$\gamma p_0 \nabla \mathrm{div}\boldsymbol{v} - \nabla\left[\rho_0\left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{V} \cdot (\boldsymbol{V} \cdot \nabla)\boldsymbol{v}\right)\right]. \quad (7)$$

Now take the divergence of both sides of Eq. (7)

$$\rho_0\left(\frac{\partial^2 u}{\partial t^2} + V\frac{\partial^2 u}{\partial t \partial z}\right) + \nabla\rho_0 \cdot \left(\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + (\boldsymbol{V} \cdot \nabla)\frac{\partial \boldsymbol{v}}{\partial t}\right) + \quad (8)$$

$$\rho_0\frac{\partial V}{\partial x}\frac{\partial^2 v_x}{\partial z \partial t} = \gamma p_0 \Delta u - \Delta\left[\rho_0\left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{V} \cdot (\boldsymbol{V} \cdot \nabla)\boldsymbol{v}\right)\right],$$

where the notation $u = \nabla \cdot \boldsymbol{v}$ is used. The substitution of (7) into (8) gives

$$\rho_0\left(\frac{\partial^2 u}{\partial t^2} + V\frac{\partial^2 u}{\partial t \partial z}\right) + c_0^2 \nabla\rho_0 \cdot \nabla u - \quad (9)$$

$$-\frac{1}{\rho_0}\nabla\rho_0 \cdot \nabla\left[\rho_0\left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{V} \cdot (\boldsymbol{V} \cdot \nabla)\boldsymbol{v}\right)\right] +$$

$$\rho_0\frac{\partial V}{\partial x}\frac{\partial^2 v_x}{\partial z \partial t} = \gamma p_0 \Delta u - \Delta\left[\rho_0\left(\boldsymbol{V} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{V} \cdot (\boldsymbol{V} \cdot \nabla)\boldsymbol{v}\right)\right].$$

Now a harmonic dependence of the variables on t and z is introduced

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} \exp i(\omega t - k_z z), \quad u = \tilde{u} \exp i(\omega t - k_z z),$$

and the components of Eq. (7) and Eq. (9) are rewritten as

$$\omega(k_z V - \omega)v_x = c_0^2\frac{du}{dx} - \frac{i}{\rho_0}\frac{d}{dx}[\rho_0 V(\omega - k_z V)v_z], \quad (10)$$

$$ik_z c_0^2 u = (\omega - k_z V)^2 v_z, \quad (11)$$

$$\rho_0\omega(k_z V - \omega)u + c_0^2\frac{d\rho_0}{dx}\frac{du}{dx} - \frac{i}{\rho_0}\frac{d\rho_0}{dx}\frac{d}{dx}\left[\rho_0(\omega - k_z V)V v_z\right]$$

$$+\omega k_z \rho_0\frac{dV}{dx}v_x = \gamma p_0 \Delta u - i\Delta[\rho_0(\omega - k_z V)V v_z], \quad (12)$$

where the tilde has been omitted. After elimination of v_z and v_x from Eqs. (10)–(12) a second-order equation for u is obtained

$$(k_z V - \omega)u + c_0^2\frac{d\ln\rho_0}{dx}\frac{d}{dx}\left[\frac{u}{\omega - k_z V}\right]$$

$$+\frac{c_0^2 k_z}{k_z V - \omega}\frac{dV}{dx}\frac{d}{dx}\left[\frac{u}{\omega - k_z V}\right] = c_0^2\Delta\left[\frac{u}{\omega - k_z V}\right], \quad (13)$$

which in terms of the new dependent variable

$$y = \frac{u}{\omega - k_z V} \quad (14)$$

reads

$$\frac{d}{dx}\left[\frac{c_0^2}{V_{ph} - V}\frac{dy}{dx}\right] + k_z^2\left[V_{ph} - V - \frac{c_0^2}{V_{ph} - V}\right]y = 0, \quad (15)$$

where $V_{ph} = \omega/k_z$ is a phase velocity. Eq. (15) defines the dependence on x of the amplitude of an acoustic wave with frequency ω and vertical phase velocity V_{ph}. In the limit of a uniform atmosphere, where $V(x)$ = const and $c_0(x)$ = const, the equation is reduced to the dispersion relation $\omega^2 = (k_z^2 + k_\perp^2)c_0^2$, where $k_\perp$ is the "horizontal" wavenumber. *In the general case of a structured atmosphere the dispersion equation for acoustic waves could be defined by the condition that the wave amplitude is bounded at infinity.* This condition corresponds to the requirement that $k_\perp^2 \geq 0$ in the case of the uniform atmosphere.

In this paper I treat only the case where the equilibrium temperature, density and vertical velocity are periodic functions of x, with spatial period $2d$:

$$T_0(x+2d) = T_0(x),\ \rho_0(x+2d) = \rho_0(x),$$
$$V(x+2d) = V(x). \tag{16}$$

Eq. (15) belongs to the class of equations with periodic coefficients. The treatment of such equations has been developed especially for the Mathieu and Hill equations, which often appear in physical problems (Ince 1944). However, Eq. (15) is more general than the Mathieu and Hill equations, and a more detailed treatment is necessary.

For convenience we introduce dimensionless variables

$$\xi = k_s x, \quad \text{where } k_s = \frac{\pi}{2d}$$
$$\hat{k}_z = \frac{k_z}{k_s},\ \hat{\omega} = \frac{\omega}{k_s \hat{c}},\ \hat{V} = \frac{V}{\hat{c}},\ \hat{V}_{ph} = \frac{V_{ph}}{\hat{c}},\ \hat{c}_0^2 = \frac{c_0^2}{\hat{c}^2}, \tag{17}$$
$$\text{where} \qquad \hat{c}^2 = \left\langle c_0^2 \right\rangle \tag{18}$$

is the mean of the sound velocity over the space period $2d$. The reason for the unusual definition of the spatial wavenumber k_s is that in this way Eq. (15) is reduced to the form used for the Mathieu and Hill equations. It is thus easier to refer to the properties of equations with periodic coefficients. In terms of the dimensionless variables Eq. (15) reads

$$\frac{d}{d\xi}\left[\frac{\hat{c}_0^2}{\hat{V}_{ph} - \hat{V}}\frac{dy}{d\xi}\right] + \hat{k}_z^2\left[(\hat{V}_{ph} - \hat{V}) - \frac{\hat{c}_0^2}{\hat{V}_{ph} - \hat{V}}\right]y = 0. \tag{19}$$

3. Vertical wave propagation

Before considering the general case of oblique wave propagation we first treat the special case of propagation along the vertical layers of a periodic atmospheric structure. This case serves to explain the approach and to introduce some definitions. In addition, in order to separate the diverse effects, we first investigate atmospheres having either temperature or velocity fluctuations.

3.1. Thermally structured atmosphere

A motionless atmosphere ($V = 0$), structured only by fluctuations of temperature, hereafter will be called a thermally structured atmosphere. With $V = 0$ Eq. (19) reads

$$\frac{d}{d\xi}\left(\hat{c}_0^2\frac{dy}{d\xi}\right) + \hat{k}_z^2(\hat{V}_{ph}^2 - \hat{c}_0^2)y = 0. \tag{20}$$

If the sound speed is a sinusoidal function of the horizontal coordinate,

$$\hat{c}_0^2 = 1 + 2\delta\cos 2\xi, \tag{21}$$

Eq. (20) reduces to

$$\frac{d}{d\xi}\left[(1 + 2\delta\cos 2\xi)\frac{dy}{d\xi}\right] + \hat{k}_z^2(\hat{V}_{ph}^2 - 1 - 2\delta\cos 2\xi)y = 0, \tag{22}$$

which will be solved for arbitrary values of δ. This equation is not the Mathieu equation, because the second and the first derivatives have periodic coefficients. According to the Floque theorem (Ince 1944) Eq. (22) has four different solutions

$$y = \sum_{m=0}^{\infty} A_{2m+p}\cos(2m+p)\xi, \quad p = 0 \text{ or } 1 \tag{23}$$

$$y = \sum_{m=0}^{\infty} B_{2m+p}\sin(2m+p)\xi, \quad p = 0 \text{ or } 1. \tag{24}$$

For $p = 0$ the solution has the period π, which coincides with the period of the atmospheric structure defined by (21), while for $p = 1$ the period is 2π, twice that period. An important difference between the even and odd solutions is the constant term A_0. Notice that the the individual terms of (23) and (24) are *not* solutions of (22). This is related to the reason why the perturbation method does not work for Eq. (15).

The even solution (23) of period π is considered first. Dispersion equation (A4) (see the derivation in the Appendix) has an infinite number of roots $\hat{\omega}^2$, which in the limit of a uniform atmosphere ($\delta \to 0$) are

$$\hat{\omega}^2 = \hat{k}_z^2 + (2m)^2, \quad m = 0, 1, 2 \ldots, \tag{25}$$

while the eigenfunctions (23) are

$$y = A_{2m}\cos 2m\xi. \tag{26}$$

The root with $m = 0$ corresponds to an acoustic plane wave propagating in the z-direction. For $m = 1, 2 \ldots$ there are acoustic waves propagating vertically, but standing horizontally. Discrete values of the horizontal wavenumber appear because here the uniform atmosphere is considered as a limiting case of the structured atmosphere. The periodicity of the eigenfunctions $\cos 2m\xi$ is imposed by the periodic structure of the non-uniform case.

For $\delta \ll 1$ the first two roots of the dispersion equation (A4) are approximately

$$\hat{\omega}^2 \approx \hat{k}_z^2 - 0.5\delta^2\hat{k}_z^4, \tag{27}$$
$$\hat{\omega}^2 \approx \hat{k}_z^2 + 4 + 0.5\delta^2\hat{k}_z^4. \tag{28}$$

The vertically propagating wave with $m = 0$ has dispersion; its phase velocity is less than the mean sound speed, which is unity in our dimensionless variables. The slow-down of this acoustic wave increases with wavenumber k_z and with the amplitude δ of the temperature fluctuation. The coefficients of the eigenfunction (23) are approximately equal to $A_{2m} \approx \delta^m A_0$. Relation (28) for $m = 2$ shows the dispersion of the waves. For this case the eigenfunction (23) contains all terms of the sum. The difference to the case $m = 0$ is that $A_{2m} \approx \delta^{|m-2|}A_2$. The wave mode (28) has a cut-off frequency and becomes evanescent, i.e. $\hat{k}_z^2 < 0$, for $\hat{\omega} < 2$. The occurrence of a cut-off frequency becomes clear if the structured atmosphere is considered as a multilayer wave guide, consisting of a set of layer waveguides of thickness $2d$. The cut-off frequency is the same as in a uniform waveguide with sound speed $\hat{c}_0$ and thickness $2d$, and with $k_z = (\omega/c_0)\sqrt{1-(\omega_{off}/\omega)}$ (Ingard 1988). The cut-off frequency corresponds the case where an integer number of half wavelengths fits into the thickness $2d$. For $\omega < \omega_{off}$ the wavenumber k_z is imaginary, and the wave is evanescent.

In the case $\hat{k}_z = 0$ the cut-off frequencies $\hat{\omega} = 2m$ must be considered as eigenfrequencies of the structured atmosphere. This is similar to the vibrations of a one-dimensional lattice. But whereas in the latter case only propagation along the 1D lattice is possible we may consider two-dimensional wave propagation in the case of a structured atmosphere. We shall return to this point below in the context of oblique propagation. At present, the solutions of (A4) for $m = 2, 4\ldots$ can be considered as waveguide modes of a multilayer waveguide for $k_z \neq 0$, and as eigenmodes of the structured atmosphere for $k_z = 0$. Unfortunately the analogy with the waveguide is reasonable only for the 1D model, which we merely use in order to avoid the difficulties of the 3D treatment. To follow the analogy with lattice vibrations we shall denote the wave modes (25) vibrational waves. These modes were revealed for the first time in Paper 1, where they had been called high-frequency modes.

For the even solution with period 2π the dispersion relation is defined by Eq. (A6). In the limit $\delta \to 0$ this dispersion equation has the solutions

$$\hat{\omega}^2 = \hat{k}_z^2 + (2m+1)^2, \quad m = 0, 1, 2 \ldots . \tag{29}$$

For $\delta \ll 1$ we obtain vibrational waves with cut-off frequencies $\omega = 1, 3\ldots$. For example, the first root of the dispersion equation (A6) is

$$\hat{\omega}^2 = \hat{k}_z^2 + 1 - \delta + \delta\hat{k}_z^2. \tag{30}$$

The treatment of the odd solutions (24) with spatial periods π and 2π is similar to the foregoing analysis of the solution (23). All odd modes are vibrational waves, corresponding to $m = 1, 2, 3, 4\ldots$ (see (25) and (29)), because of $B_0 \equiv 0$.

3.2. Moving isothermal structured atmosphere

For an atmosphere with constant temperature, but with vertical flows

$$\hat{V} = 2\Delta\cos 2\xi \tag{31}$$

In this case Eq. (19) reduces to Eq. (A7). The first two roots of this dispersion equation (A11) are approximately

$$\hat{V}_{ph}^2 \approx 1 + 2\Delta^2 - 2\Delta^2\hat{k}_z^2, \tag{32}$$

$$\hat{\omega}^2 \approx 4 + (1+3\Delta^2)\hat{k}_z^2. \tag{33}$$

The dispersion equation (32) is valid for acoustic waves in an atmosphere consisting of layers with flows (31) of opposite direction. It differs essentially from the dispersion equation (28) for acoustic waves in a thermally structured atmosphere. In the low-frequency limit, $\hat{k}_z \to 0$, the phase velocity tends to $\hat{V}_{ph} = 1 + \Delta^2$ in the atmosphere with flows, i.e. to a limit that exceeds the sound speed; in the same limit the phase velocity of waves in the motionless thermally structured atmosphere is just the sound speed. In both cases the slow-down of the acoustic waves increases with increasing frequency.

3.3. Turbulent sound

In the moving atmosphere there exists another mode that is absent in the motionless atmosphere. The phase velocity of this mode is approximately given by

$$\hat{V}_{ph}^2 \approx 4\Delta^2. \tag{34}$$

This mode was revealed for the first time in Paper 1. In order to explore its properties we determine the fluctuations of the pressure and of the vertical and horizontal velocity components for the solution (23),

$$p = i\gamma p_0 y, \tag{35}$$

$$v_z = \frac{ic_0^2 y}{\hat{c}(\hat{V}_{ph} - 2\Delta\cos 2\xi)}, \tag{36}$$

$$v_x = -\frac{c_0^2}{\hat{c}(\hat{\omega} - 2\Delta\hat{k}_z\cos 2\xi)}\frac{dy}{d\xi}. \tag{37}$$

If the coefficient A_0 in (23) is zero or small in comparison with the other coefficients A_{2m}, the mean gas pressure fluctuation, calculated by integration of (35) over the space period, will to be small or zero. The average over $2d$ of the vertical velocity (36) does not vanish. Nor does the mean large-scale fluctuation $\overline{v_z V}$ of the dynamic pressure vanish; it gets rather large if the phase velocity $V_{ph} \approx 2\Delta$ as it is the case for this slow mode. It turns out that the gradient of the mean dynamical pressure indeed drives this mode, while the gradient of the mean gas pressure is absent. The mean horizontal velocity $\overline{v}_x$ is finite along with the mean vertical velocity $\overline{v}_z$. Nevertheless, the mean div $\boldsymbol{v}$ is zero. *Since the slow mode is driven by the gradient of the mean dynamical pressure, we consider it as turbulent sound.* The turbulent sound does not exist in an incompressible atmosphere, in spite of its solenoidal mean in the compressible case. Small-scale pressure and density fluctuations are a necessary condition for its existence.

The turbulent sound is not a unique mode. There is a number of similar modes, equal to the number of vibrational wave modes. They are distinguished in that one of the coefficients A_m in the solution (23) is much larger than all others. Moreover, there are turbulent sound modes corresponding to solutions (23) and (24) with period 2π, which have exactly constant mean gas pressure because there is no A_0 term for $p = 1$. Turbulent sound modes require a special exploration, in particular because they produce strongly sheared flows. Even singularities of the vertical velocity (36) may appear, if $V_{ph} < 2\Delta$; in this case the temperature and density fluctuations also become singular. We shall resume the discussion of the turbulent sound below after oblique propagation has been considered.

3.4. General case of a structured atmosphere

In the general case the structured atmosphere consists of hot upflows and cold downflows. The sound speed and the velocity of the flows are given by

$$\hat{c}_0^2 = 1 + 2\delta \cos 2\xi, \quad \hat{V} = V_m + 2\Delta \cos 2\xi. \tag{38}$$

Eq. (19) is reduced to Eq. (A12). The approximate solutions of the dispersion equation (A16) are

$$\hat{V}_D^2 \approx 1 + 2\Delta(\Delta + \delta) - 0.5(2\Delta + \delta)^2 \hat{k}_z^2, \tag{39}$$

$$(\hat{\omega} - \hat{k}_z)^2 \approx 4 - \Delta(11\Delta + 14\delta) + \\ +(1 + 6.125\Delta^2 + 4\delta\Delta + 0.5\delta^2)\hat{k}_z^2, \tag{40}$$

$$V_D^2 \approx 4\Delta^2. \tag{41}$$

The joint action of the temperature and velocity fluctuations increases the dispersion of the acoustic and vibrational modes, (39) and (40). In the low-frequency limit, $\hat{k}_z \to 0$, the phase velocity of acoustic waves tends to $\hat{V}_{ph} = 1 + 2\Delta(\Delta + \delta)$; temperature fluctuation enhances the effect of flow on the phase velocity in this limit. The cut-off frequency of the vibrational modes decreases in comparison to the cases where only temperature or velocity fluctuations exist. The turbulent sound mode (41) depends on the temperature fluctuation only in higher orders of the expansion in terms of Δ and δ.

4. Oblique propagation

In the case of inclined wave propagation there are solutions of Eq. (22) with period π ($p = 0$) and 2π ($p = 1$)

$$y = e^{i\hat{k}_\perp \xi} \sum_{-\infty}^{\infty} C_{2m+p} e^{i(2m+p)\xi}, \quad p = 0, 1. \tag{42}$$

The dispersion equations for $p = 0, 1$ are Eqs. (A19) and (A20), respectively, in terms of the Hill determinant. If we truncate the Hill determinant in Eq. (A19), retaining three rows centered around the row that contains q_0, an approximate dispersion equation for the first three roots is obtained. For the motionless structured atmosphere, where $\Delta = 0$, this approximate dispersion equation is

$$(\hat{\omega}^2 - \hat{k}_0^2)[(\hat{\omega}^2 - 4 - \hat{k}_0^2)^2 - 16\hat{k}_\perp^2] = \\ 2\delta^2[(\hat{\omega}^2 - 4 - \hat{k}_0^2)(\hat{k}_0^4 + 4\hat{k}_\perp^2) + 16\hat{k}_0^2\hat{k}_\perp^2], \tag{43}$$

where the squared full wavenumber is $\hat{k}_0^2 = \hat{k}_z^2 + \hat{k}_\perp^2$. For $\delta^2 \ll 1$ (43) has a root that corresponds to $\hat{\omega}^2 \simeq \hat{k}_0^2$, up to a term of order δ^2. In this case the distortion due to the atmospheric structure yields

$$\hat{\omega}^2 = \hat{k}_0^2 - \frac{\delta^2}{2} \frac{\hat{k}_0^4 - 4\hat{k}_0^2\hat{k}_\perp^2 + 4\hat{k}_\perp^2}{1 - \hat{k}_\perp^2}. \tag{44}$$

This solution is valid only if $|\hat{k}_\perp^2| - 1 \gg \delta^2$. If $|\hat{k}_\perp^2| \simeq 1 + \delta^2$, then Eq. (43) should be used. When $\hat{k}_\perp$ tends to zero, the approximate solution (44) reduces to the case of vertical propagation, Eq. (28). For horizontal propagation, in the low-frequency limit, relation (44) reduces to

$$\hat{\omega}^2 = (1 - 2\delta^2)\hat{k}_\perp^2. \tag{45}$$

In this limit the phase velocity turns out to be less than the sound speed. We recall that in the low-frequency limit the phase velocity of vertically propagating acoustic waves in the thermally structured atmosphere did not differ from the sound speed. For the moving atmosphere ($\Delta \neq 0, \delta = 0$) the phase velocity of horizontally propagating waves ($k_z = 0$) does not differ from the sound speed; this can be seen directly from Eq. (15), if one replaces V_{ph} by ω/k_z. At the same time the phase velocity of vertically propagating acoustic waves in a moving isothermal atmosphere in the low-frequency limit $\hat{k}_z \to 0$ appears to be larger than the sound speed, see (32).

In the case of oblique propagation each vibrational wave mode splits into two modes. For $\delta \to 0$ and $m = 2$ their frequencies are given by

$$\hat{\omega}_{1,2}^2 = 4 + \hat{k}_0^2 \pm 4\hat{k}_\perp, \tag{46}$$

which follows from (43). The first mode is a waveguide mode with a low-frequency cut-off. The second mode is a classical wave mode in a 1D lattice (Ingard 1988); this mode has a high-frequency cut-off.

5. Brillouin zones and vibrational waves

Even the simplest approximate dispersion equations for oblique modes are cumbersome due to the large number of modes: distorted acoustic modes, vibrational modes, and the turbulent sound mode. Therefore we present here numerical solutions of the dispersion equation (A19).The results are presented in the form $(\hat{k}_\perp, \hat{\omega})$-diagrams, which are divided into so-called Brillouin zones separated by $\hat{k}_\perp = 0, \pm 1, \pm 2 \ldots$. The reason of this separation is seen from the solution (42), which changes form one type to another when $\hat{k}_\perp$ is changed by 1. Therefore the first Brillouin zone already reveals all relevant physical effects.

5.1. Thermally structured atmosphere

A rather simple $(\hat{k}_\perp, \hat{\omega})$-diagram is obtained in the case of a thermally structured atmosphere with $\delta = 0.1$. It is well described by the approximate dispersion equation (43), and is presented in Fig. 1a. The solid lines correspond to horizontal propagation, $\hat{k}_z = 0$. The solid line passing through the origin is the acoustic mode that in the uniform atmosphere ($\delta = 0$) is the straight line $\hat{\omega} = \hat{k}_\perp$. In the structured atmosphere, the behaviour of the acoustic branch changes drastically near the point where $\hat{\omega} = 1$ and $\hat{k}_\perp = 1$. Beyond this point the phase velocity decreases with increasing $\hat{k}_\perp$, and the directions of the group and phase velocities are opposite. The second solution of the dispersion equation is a vibrational mode with cut-off frequency $\hat{\omega}_{off} = 2$; it is shown by the solid curve passing through the point $\hat{\omega} = 2$ on the ordinate axis. In the limit of a uniform atmosphere this curve is the straight line $\hat{\omega} = 2 - \hat{k}_\perp$, which follows from (46)

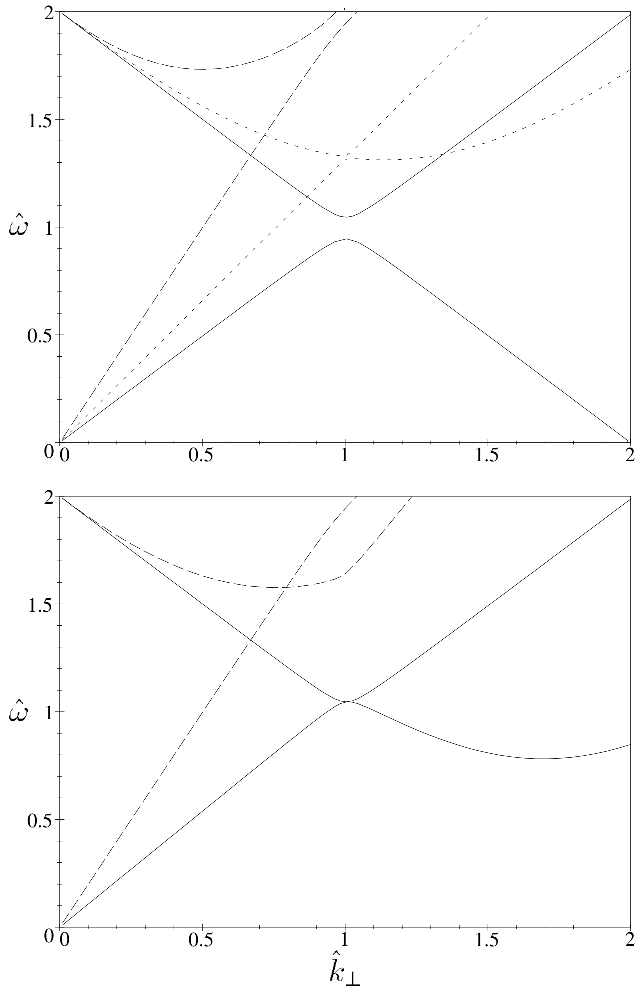

Fig. 1. The $(\hat{k}_\perp, \hat{\omega})$-diagrams for acoustic and vibrational modes in a thermally structured atmosphere (δ = 0.1). Upper diagram: Modes propagating at $\phi = 0°$ (solid), 41°(dotted), and 60°(dashed). Lower diagram: Acoustic modes with ϕ = 23° (solid) and 60° (dashed); vibrational modes with ϕ = 0° (solid) and 52° (dashed).

for the case of horizontal propagation, $\hat{k}_0 = \hat{k}_\perp$. In the present case of a structured atmosphere ($\delta \neq 0$) the curve changes its direction near the point $\hat{\omega} = 1$, $\hat{k}_\perp = 1$; its second part follows exactly the route taken by the acoustic branch in the case of a uniform atmosphere. Thus, the second parts of both the acoustic and vibrational modes are mutual extensions of the first parts of the respective other branch in the solution of the dispersion equation. There is a frequency gap between the two curves, where horizontal wave propagation is impossible. Waves with frequencies within this gap are evanescent. Such behaviour is inherent to waves in a periodic structure. The classical example is the behaviour of electrons in the periodic potential of a crystal lattice. The $(\hat{k}_\perp, \hat{\omega})$-diagram of Fig. 1a for the horizontal wave propagation is the conventional Brillouin zone diagram, and $\hat{k}_\perp = 1$ marks the border between the first and second Brillouin zones; the frequency gap between the two solids curves corresponds to the energy barrier which can be passed only by means of the tunnel effect.

The analogy with the Brillouin zones is exact for the case of horizontal wave propagation. To explore the general case of oblique propagation let us consider waves with a fixed direction of propagation. The dotted and dashed lines in Fig. 1a correspond to waves with $\phi = \arctan(\hat{k}_z/\hat{k}_\perp)$ equal to 41° and 60°, respectively. The angle 41° is peculiar for the chosen value of δ, because the curves of acoustic and vibrational modes touch each other; both pass continuously through the point $\hat{k}_\perp = 1$. The frequency gap disappears, and acoustic waves pass through the border of the Brillouin zones without reflection.

For angles $41° < \phi < 23°$ it is possible to continuously prolong the acoustic and vibrational branches by choosing different angles of propagation for the two branches. The case where the vibrational mode propagates horizontally, while the acoustic wave is running at an angle $\phi = 23°$, is shown by the solid curves of Fig. 1b. In this case the acoustic wave does not meet the potential barrier at the border between the Brillouin zones, but changes from oblique propagation in the first Brillouin zone to horizontal propagation in the second Brillouin zone. Partial reflection occurs due to refraction at the border of the Brillouin zones. If in the first Brillouin zone the propagation angle of the acoustic waves is smaller than 41° but larger than 23°, then the frequency gap between the acoustic and vibrational modes is removed by the change in the direction of propagation. Acoustic waves that propagate at angles $\phi < 23°$ tunnel through the border between the Brillouin zones and are transformed to horizontally propagating waves in the second Brillouin zone; in this way they meet the smallest possible potential barrier. The special angle $\phi = 41°$ does not depend noticeably on the value of δ. In the solar convection zone we should have $\delta < 0.3$.

The width of the potential barrier for the horizontally propagating waves is about δ in terms of the dimensionless frequency $\hat{\omega}$. Horizontally propagating acoustic waves passing through the Brillouin zone border undergo the strongest reflection, because the potential barrier is largest in this case. The reflection decreases with increasing angle of propagation as the potential barrier becomes narrower and disappears at $\phi = 23°$. However, reflection does not entirely cease at this angle, because the effect of changing the direction of propagation precludes the free passage of the Brillouin zone border. Only when the propagation angle reaches 41° the acoustic wave crosses the Brillouin zone without reflection. Thus, the effects of the potential barrier and the refraction at the Brillouin zone border favour waves with frequencies $0.94 < \hat{\omega} < 1.41$.

5.2. Vibrational waves

In order to explain the upper part, $\hat{\omega} > 1.41$, of the $(\hat{k}_\perp, \hat{\omega})$-diagram the analogy with solid state physics will be used. The microscopic theory of acoustic waves in a crystal lattice shows that along with acoustic waves a branch of vibrational waves exists. The vibrational phonons are elastic vibrations of the atoms around their equilibrium sites in the lattice. The propagation of these vibrations is considered as the vibrational waves. In the

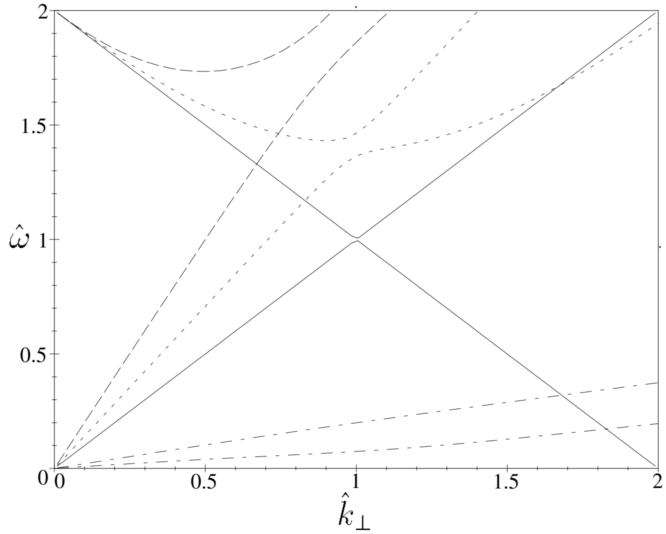

Fig. 2. The $(\hat{k}_\perp, \hat{\omega})$-diagram for a moving atmosphere ($\Delta = 0.1$) for acoustic and vibrational modes with $\phi = 0°$ (solid), 45° (dotted), 60° (dashed), and for the turbulent sound mode with $\phi = 45°$ (dash-dotted).

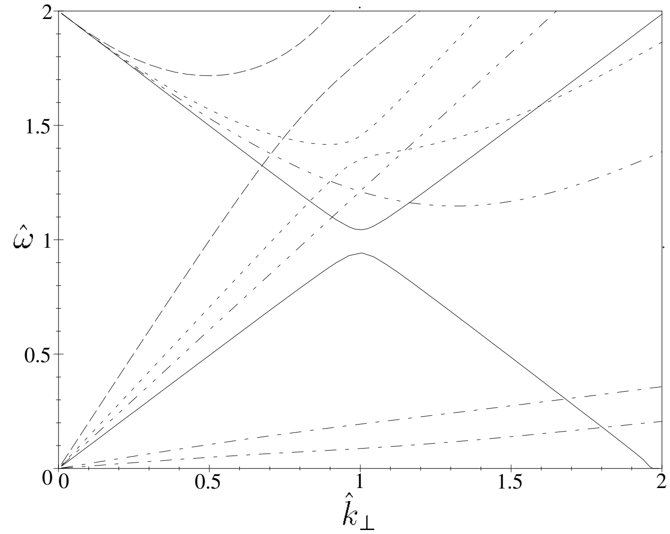

Fig. 3. The $(\hat{k}_\perp, \hat{\omega})$-diagram for a moving thermally structured atmosphere ($\delta = 0.1$, $\Delta = 0.1$) for acoustic and vibrational modes with $\phi = 0°$ (solid), 35° (dash-dot-dotted), 45° (dotted), 60° (dashed), and for turbulent sound branches (dash-dotted).

case of a continuous periodically structured atmosphere similar branches appear, as was shown in the preceding section. This means that acoustic energy can be transported in a structured atmosphere not only by acoustic waves but also by vibrational waves. In the lower part of the $(\hat{k}_\perp, \hat{\omega})$-diagram ($\hat{\omega} < 1.41$) the vibrational waves are very different from the acoustic waves, because the directions of their phase and group velocities are opposite. Fig. 1a shows the branches of acoustic and vibrational waves for $\phi = 60°$. The behaviour of the two modes is typical for coupled oscillators. These curves are drawn for an imposed value of the propagation angle.

Fig. 1b shows the acoustic branch for $\phi = 60°$ and the vibrational branch for $\phi = 52°$. The latter turns upwards after crossing the acoustic curve, so the behaviour of the two branches is similar after the crossing. For a prescribed point of the acoustic branch we may determine the angle for which the vibrational branch crosses. If the crossing happens on the ascending part of the vibrational branch, then coupling and exchange of energy between the two wave modes are possible. The boundary of the region where the coupling of acoustic and vibrational phonon branches occurs is defined by the points where the acoustic branches cross the minimum of the vibrational curve. The border line connects the points $(0, 2)$ and $(1, 1.41)$ in the $(\hat{k}_\perp, \hat{\omega})$-diagram of Fig. 1a. The coupling of acoustic and vibrational modes as well as the crossing of the Brillouin zone border needs a special treatment, because all modes are anisotropic and, consequently, the directions of the phase and group velocities do not coincide. But this is beyond the scope of this paper.

To give a comprehensive picture of the wave branches in a structured atmosphere the modes with space period 2π must be considered as well. This will be done for the moving atmosphere. One should not go too far with the analogy between pressure waves in the structured atmosphere and elastic oscillations of the crystal lattice, because the physics and the governing equations are rather different. But it is worth mentioning that the frequency $\hat{\omega} = 1$ plays a similar role as the Debye frequency, and that k_s corresponds to the translation vector of the lattice.

5.3. Moving atmosphere

Fig. 2 shows the $(\hat{k}_\perp, \hat{\omega})$-diagram for the moving atmosphere with $\Delta = 0.1$ and $\delta = 0$. The diagrams of Fig. 1a and Fig. 2 show many similarities, but also a number of differences. For horizontal propagation there is no frequency gap between the vibrational and acoustic branches of Fig. 2. This is because flows that are perpendicular to the direction of wave propagation have no influence, and the coefficients in Eq. (15) do not depend on V. The frequency gap does exist for $\phi > 0$. But this does not mean that the waves meet a potential barrier at the Brillouin zone border; they only undergo refraction in the same way as in a thermally structured atmosphere for $\phi > 23°$ and $\delta = 0.1$. The only difference is that in the moving atmosphere the refraction at the Brillouin zone border increases with the propagation angle ϕ, while in a thermally structured atmosphere it decreases and disappears for $\phi = 45°$.

The general case of a moving thermally structured atmosphere ($\delta = 0.1$, $\Delta = 0.1$) is shown in Fig. 3. The potential barrier for horizontally propagating waves appears due to the temperature structure, while the refraction for $\phi = 45°$ results from the flows. The potential barrier and the refraction at the Brillouin zone border do not appear for a propagation angle $\phi = 35°$. All three diagrams of Figs. 1a, 2, and 3 show the same behaviour of the acoustic and vibrational branches for propagation angles $\phi > 45°$, where linear coupling of these modes occurs. Fig. 4 shows a $(\hat{k}_\perp, \hat{\omega})$ – diagram for vibrational branches with period 2π, defined by the dispersion equation

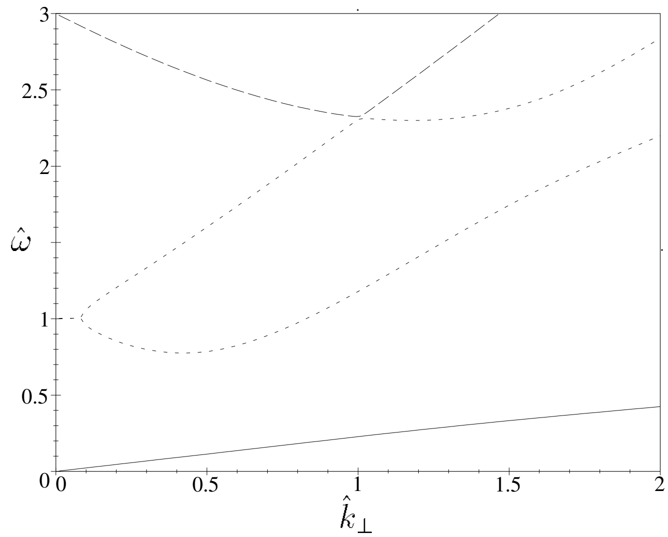

Fig. 4. The $(\hat{k}_\perp, \hat{\omega})$-diagram for vibrational modes period 2π with $\hat{\omega}_g$ = 1(dotted), 3(dashed), and a turbulent sound mode (solid) with $\phi = 50°$ in a moving thermally structured atmosphere ($\delta = 0.1$, $\Delta = 0.1$).

(A20). The branches start at $n\omega_D (n = 1, 3, 5 \ldots)$, where the central frequency of the first band gap, ω_g, is

$$\omega_g = k_s \hat{c}. \tag{47}$$

There is no acoustic branch in this diagram, because acoustic waves, by definition, correspond to $A_0 \neq 0$ in (42), that occur for solutions with space period π only. The vibrational waves with period 2π show coupling at the borders of the Brillouin zones.

5.4. Turbulent sound

Turbulent sound branches appear in the $(\hat{k}_\perp, \hat{\omega})$-diagrams of wave modes with space period π (Figs. 2 and 3) and 2π (Fig. 4). Eqs. (35)-(36) are valid for oblique propagation as well, but Eq. (37) should be replaced by

$$\hat{k}_\perp v_x = -\hat{k}_z \left[(\hat{V}_{ph} - 2\Delta \cos 2\xi)^2 + 1 \right] v_z, \tag{48}$$

where $\hat{V}_{ph}$ is still the component of the phase velocity along the direction of the flows: $\hat{V}_{ph} = \hat{\omega}/\hat{k}_z$. After averaging over the space period $2d$ Eq. (48) can be replaced by the approximate relation $\hat{k}_\perp v_x \approx -\hat{k}_z v_z$; this can be rewritten as a vector condition $\hat{\mathbf{k}}_0 \cdot \boldsymbol{v} = 0$ that characterizes transversal waves. Thus, the turbulent sound is an almost transversal wave for $\hat{V}_{ph} \approx 2\Delta$. We recall that the average of div $\boldsymbol{v}$ is zero. The important issue for turbulent sound, as disclosed above, is whether the phase velocity along the flows is less than 2Δ. Figs. 5a,b show the dependence of the vertical phase velocity on the parameters δ and Δ, and on the propagation angle. The vertical phase velocity of the turbulent sound can be larger or smaller than 2Δ, depending on the parameters. Singularities in the vertical and horizontal velocity components appear as a consequence of the inviscid treatment. If viscosity is taken into account, the frequencies become complex and the singularities are removed. However, in the case of weak viscosity the wave energy is concentrated in thin layers, and small-scale shear flows appear near the locations where the singularities occur in the inviscid case. This means that even for a rather small mean wave amplitude the nonlinear effects should be taken into account. It seems that real turbulence has to generate turbulent sound among other modes. But the turbulent sound produces small-scale shear flows, which do not arise from other wave modes. This small-scale shear may in turn generate small-scale turbulence. Thus, turbulent sound is a likely mechanism of generating small-scale turbulence from large-scale turbulence.

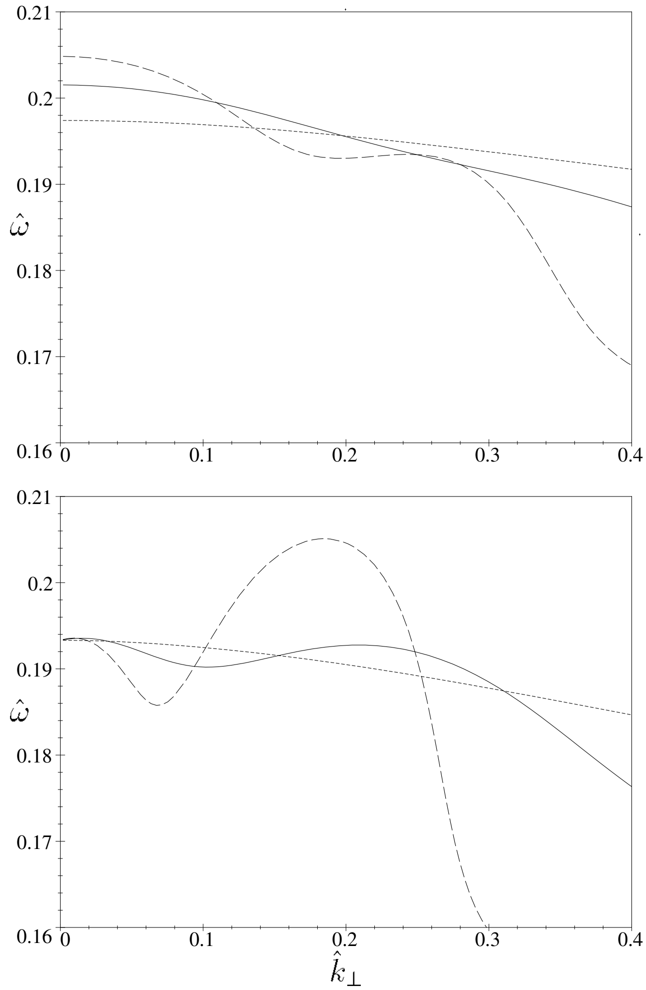

Fig. 5. The vertical phase velocity $\hat{V}_{ph}$ for turbulent sound in a moving thermally structured atmosphere with $\delta = 0.1$ and $\Delta = 0.1$. Upper diagram for even solutions, lower diagram for odd solutions; $\phi = 90°$(dotted), $45°$(solid), $30°$(dashed).

6. Discussion

The main distinguishing feature of turbulent convection is a random pattern of the temperature and velocity fluctuations. The size of the turbulent elements varies from the largest, defined by forcing, to the smallest, defined by viscous dissipation. However, the effect of the fluctuations on waves declines with the decrease of the size of the turbulent elements. Thus, the main effect of the fluctuations could be assigned to the largest turbulent elements. Unfortunately, the largest convective cells (for example supergranulation on the Sun) do not in general form a perfect lattice. The question arises of whether the effects of a regular convective pattern occur as well in the case of slow turbulent convection.

The effect of sound dispersion appears due to finite scale $2d$ in the model. In the turbulent atmosphere the sound dispersion is governed mostly by the largest turbulent elements. The method developed here allows the exact calculation of this dispersion. It probably overestimates the dispersion due to the use of a regular pattern, but underestimates it as the effect of the smaller elements is neglected. In any case there seems to be no other method to calculate the effect.

The present model gives the properties of the vibrational waves, which are nothing else but sound turbulence. Solution (42) shows that these waves are not simple acoustic plane waves, as assumed so far. The physics is clear: the wave functions of the vibrational waves follow the pattern of the temperature and velocity fluctuations. They will have this property also in the case of random fluctuations. The question is whether the banded structure of the frequency spectrumof the vibrational waves survives when the wavenumber spectrum of the fluctuations has a finite lower bound.

The advantage of the Brillouin zone diagram is that it demonstrates the coupling between the acoustic and vibrational waves at $\hat{k}_\perp = 1$. An indication of the Brillouin zones appeared in the treatment of the f mode in an atmosphere with random density fluctuations (Murawski & Goossens 1993). The wave branch in the $(k_\perp, \omega)$−diagram (Fig. 5 of that paper) shows a peculiar behaviour for a wavelength about the correlation length, which can be considered as the size of the largest turbulent element. This indication of Brillouin zones appeared in spite of the use of the Born approximation, which is not valid at the border of the Brillouin zones. Thus, there are reasons to believe that the effect of Brillouin zones will appear in the case of random fluctuations as well.

The model with periodic fluctuations seems to be a necessary step towards the more general treatment of random fluctuations (Zhugzhda, in progress). The periodic model gives a clear physical picture of the wave phenomena in an atmosphere with fluctuations and allows to calculate the effects of convection on waves.

7. P modes and convection

The current study was undertaken in connection with the theory of p modes in the solar convection zone. As a sequel to Paper 1, a new approach has been developed, but specific problems will require separate studies. In this section we just outline the basic effects that in our view are essential for helioseismology. The slow turbulence limit, which is used in the current study is valid

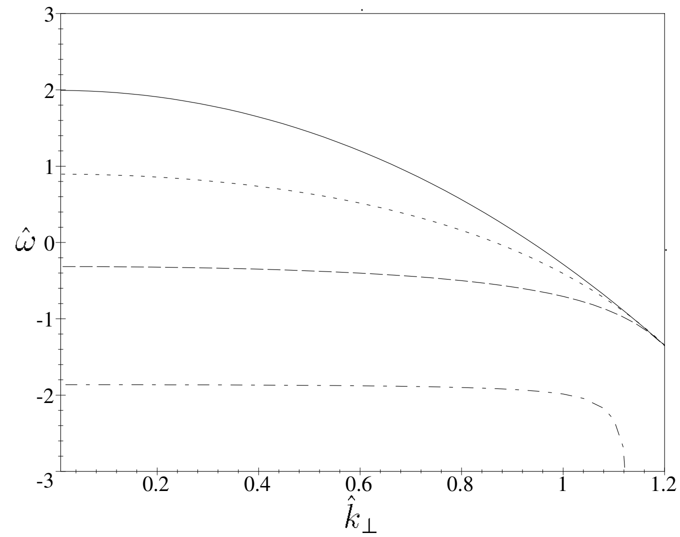

Fig. 6. Difference between the vertical phase velocities in a moving thermally structured (δ = 0.1, Δ = 0.1) and a uniform atmosphere, $\hat{V}_{ph} - \sin^{-1}\phi$, in units of $\delta^2 = 0.01$, for ϕ = 90° (solid), 60°(dotted), 45°(dashed), 30°(dash-dotted).

for p modes except upper thin layer of the atmosphere, where the turnover time of convective cells 2–3 times less than the wave period. The fast turbulence limit, which has been developed so far only for the case of $l = 0$ (Rüdiger et al 1997), is not valid for p modes in this layer as well. The reason to believe, that the applications of the developed methods to the treatment of p modes (Zhugzhda 1994, Rüdiger et al 1997) are reasonable, is that the effect of the sound dispersion appears in both of the limiting cases of slow and fast turbulence.

Dispersion of acoustic waves and the eigenfrequencies of p modes. For a given model of the convection zone, the phase velocity of acoustic waves in a convective atmosphere depends on frequency and degree l. For $l = 0$ it was shown in Paper 1 that the corrections to the eigenfrequencies due to the dispersion of the acoustic waves are essential and have the right sign and order of magnitude to remove the discrepancy between theory and observation. The current treatment allows to calculate the phase velocity of these waves for any value of l. Fig. 6 shows the relative correction to the vertical phase velocity as a function of $\hat{\omega}$ for different values of ϕ. The phase velocity decreases with decreasing ϕ for $\hat{\omega} \ll 1$ because for the case of vertical propagation it exceeds the sound velocity in the low frequency limit by approximately $2\Delta^2$ due to the flow effect as expressed by Eq. (32); in the case of horizontal propagation the phase velocity is smaller than the sound speed by $2\delta^2$, cf. Eq. (45), due to the temperature fluctuations. The sharp increase of the dispersion for $\hat{\omega} \approx 1$ and $\phi \leq 30°$ is due to the approach to the Brillouin zone border, where the acoustic branches in the $(\hat{\omega}, \hat{k}_\perp)$-diagram change their direction (Figs. 1,2,3).

Vibrational waves. The vibrational waves, which appear as an exact analytical solution of Eq. (15), are local acoustic oscillations in a convective atmosphere whose frequencies fall within band above ω_g (47). But the "eigenoscillations" of adjacent turbulent elements are strongly coupled, like the oscillations of the atoms in a crystal lattice. This enables these local oscillations to propagate, which is why they are called vibrational waves. Vibrational waves in the convection zone transport acoustic energy, while vibrational waves (phonones) in a solid are responsible for heat transfer.

Brillouin zones. Partial reflection occurs when the waves meet a Brillouin zone border on their path in the convection zone. This additional partial scattering at certain levels in the convection zone exists only for waves of a definite frequency, which depends on the size of the convective elements and on the sound velocity. Such selective reflection could produce special features in the spectrum of p modes, which may hardly be explained by peculiarities in the temperature gradient at that level.

Acknowledgements. I am grateful to the Alexander von Humboldt Foundation for support of this research and to Prof. M. Stix for valuable discussions and for reading the manuscript. I thank an anonymous referee for comments, which helped to improve the paper.

Appendix A: derivation of dispersion equation

A.1. Vertical propagation

Thermally structured atmosphere Substituting the solution (23) into Eq. (22) and collecting the terms with the same space periods we find the recurrence relations for the coefficients A_{2m}:

$$a_0 A_0 - \delta b_0 A_2 = 0,\ a_2 A_2 - \delta(2b_0 A_0 + b_2 A_4) = 0 \quad \text{(A1)}$$

$$a_m A_m - \delta(b_m A_{m+2} + b_{m-2} A_{m-2}) = 0, m = 4, 6 \ldots, \quad \text{(A2)}$$

$$a_m = \hat{\omega}^2 - \hat{k}_z^2 - m^2,\ \ b_m = m(m+2) + \hat{k}_z^2. \quad \text{(A3)}$$

This infinite set of equations completely defines the solution. A nontrivial solution exists if the infinite Hill determinant is zero:

$$\begin{vmatrix} a_0 & -\delta b_0 & 0 & 0 & \ldots \\ -2\delta b_0 & a_2 & -\delta b_2 & 0 & \ldots \\ 0 & -\delta b_2 & a_4 & -\delta b_4 & \ldots \\ 0 & 0 & -\delta b_4 & a_6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \quad \text{(A4)}$$

This condition is the dispersion equation for waves with spatial period π and even eigenfunction. It is known that an infinite Hill determinant written in the above form is convergent.

For the even solution with period 2π the recurrence relation for the coefficients A_{2m+1}

$$a_1 A_1 - \delta(b_{-1} A_1 + b_1 A_3) = 0 \quad \text{(A5)}$$

along with (A2) defines the coefficients of the solution (23). The dispersion equation is

$$\begin{vmatrix} a_1 - \delta b_{-1} & -\delta b_1 & 0 & 0 & \ldots \\ -\delta b_1 & a_3 & -\delta b_3 & 0 & \ldots \\ 0 & -\delta b_3 & a_5 & -\delta b_5 & \ldots \\ 0 & 0 & -\delta b_5 & a_7 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \quad \text{(A6)}$$

Moving isothermal structured atmosphere In this case Eq. (19) reads

$$(\bar{a}_1 + \bar{a}_2 \cos 2\xi)\frac{d^2 y}{d\xi^2} + \bar{b}\sin 2\xi \frac{dy}{d\xi} + (\bar{c}_1 + \bar{c}_2 \cos 2\xi +$$
$$+\bar{c}_3 \cos 4\xi + \bar{c}_4 \cos 6\xi)\, y = 0, \quad \text{(A7)}$$

$$\text{where } \bar{a}_1 = \hat{V}_{ph},\quad \bar{a}_2 = -2\Delta,\quad \bar{b} = -4\Delta,$$
$$\bar{c}_1 = (\hat{V}_{ph}^3 - \hat{V}_{ph} + 6\Delta^2 \hat{V}_{ph})\hat{k}_z^2,\ \ \bar{c}_3 = 6\hat{V}_{ph}\Delta^2 \hat{k}_z^2,$$
$$\bar{c}_2 = \Delta[2(1 - 3\hat{V}_{ph}^2) - 6\Delta^2]\hat{k}_z^2,\ \ \bar{c}_4 = -2\Delta^3 \hat{k}_z^2. \quad \text{(A8)}$$

Eq. (A7) again has the solutions (23) and (24). The even solution of period π is defined by the recurrence relations

$$\bar{q}_0 A_0 + \bar{l}_2^+ A_2 + \bar{r} A_4 + \bar{s} A_6 = 0,$$
$$2\bar{l}_0 A_0 + (\bar{q}_2 + \bar{r})A_2 + (\bar{l}_4^+ + \bar{s})A_4 + \bar{r} A_6 + \bar{s} A_8 = 0,$$
$$2\bar{r} A_0 + (\bar{l}_2^- + \bar{s})A_2 + \bar{q}_4 A_4 + \bar{l}_6^+ A_6 + \bar{r} A_8 + \bar{s} A_{10} = 0,$$
$$2\bar{s} A_0 + \bar{r} A_2 + \bar{l}_4^- A_4 + \bar{q}_6 A_6 + \bar{l}_8^+ A_8 + \bar{r} A_{10} + \bar{s} A_{12} = 0$$
$$\bar{q}_m A_m + \bar{l}_{m+2}^+ A_{m+2} + \bar{l}_{m-2}^- A_{m-2} + \bar{r}(A_{m+4} + A_{m-4})$$
$$+\bar{s}(A_{m+6} + A_{m-6}) = 0, m = 8, 10 \ldots, \quad \text{(A9)}$$

$$\text{where } \bar{q}_m = -m^2 \bar{a}_1 + \bar{c}_1, \bar{r} = 0.5\bar{c}_3,\quad \bar{s} = 0.5\bar{c}_4,$$
$$\bar{l}_m^{\pm} = 0.5(-m^2 \bar{a}_2 \mp m\bar{b} + \bar{c}_2). \quad \text{(A10)}$$

The recurrence relations (A9) define the Hill determinant. The dispersion equation is

$$\begin{vmatrix} \bar{q}_0 & \bar{l}_2^+ & \bar{r} & \bar{s} & 0 & \cdots \\ 2\bar{l}_0 & \bar{q}_2 + \bar{r} & \bar{l}_4^+ + \bar{s} & \bar{r} & \bar{s} & \cdots \\ 2\bar{r} & \bar{l}_2^- + \bar{s} & \bar{q}_4 & \bar{l}_6^+ & \bar{r} & \cdots \\ 2\bar{s} & \bar{r} & \bar{l}_4^- & \bar{q}_6 & \bar{l}_8^+ & \cdots \\ 0 & \bar{s} & \bar{r} & \bar{l}_6^- & \bar{q}_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \quad \text{(A11)}$$

General case of a structured atmosphere Eq. (19) is reduced to

$$(\bar{a}_1 + \bar{a}_2 \cos 2\xi + \bar{a}_3 \cos 4\xi)\frac{d^2 y}{d\xi^2} + \bar{b}\sin 2\xi \frac{dy}{d\xi} +$$
$$+(\bar{c}_1 + \bar{c}_2 \cos 2\xi + \bar{c}_3 \cos 4\xi + \bar{c}_4 \cos 6\xi)\, y = 0, \quad \text{(A12)}$$

where

$$\bar{a}_1 = V_D - 2\delta\Delta, \bar{a}_2 = 2(\delta V_D - \Delta), \bar{a}_3 = -2\delta\Delta, \quad \text{(A13)}$$
$$\bar{b} = -4(\delta V_D + \Delta), \bar{c}_1 = (V_D^3 - V_D + 6\Delta^2 V_D - \bar{a}_3)\hat{k}_z^2,$$
$$\bar{c}_2 = -(6V_D^2\Delta + 6\Delta^3 + \bar{a}_2)\hat{k}_z^2, \bar{c}_3 = (6V_D\Delta^2 - \bar{a}_3)\hat{k}_z^2,$$
$$\bar{c}_4 = -2\Delta^3\hat{k}_z^2, V_D = V_{ph} - \overline{V}, \overline{V} = \delta\Delta^{-1}(1 - \sqrt{1 - 4\delta^2}).$$

The mean flow $\overline{V}$ is introduced to remove mean plasma flux from the model. It causes a Doppler shift, which is eliminated by replacing the phase velocity in (A10) by $V_{ph} - \overline{V}$ in (A13); the parameter $\bar{r}$ is replaced by

$$\bar{r}_m = 0.5(-m^2 \bar{a}_3 + \bar{c}_3). \quad \text{(A14)}$$

The recurrence relations are

$$\bar{q}_0 A_0 + \bar{l}_2^+ A_2 + \bar{r}_4 A_4 + \bar{s} A_6 = 0,$$

$$
\begin{aligned}
&2\bar{l}_0 A_0 + (\bar{q}_2 + \bar{r}_2)A_2 + (\bar{l}_4^+ + \bar{s})A_4 + \bar{r}_6 A_6 + \bar{s}A_8 = 0,\\
&2\bar{r}_0 A_0 + (\bar{l}_2^- + \bar{s})A_2 + \bar{q}_4 A_4 + \bar{l}_6^+ A_6 + \bar{r}_8 A_8 + \bar{s}A_{10} = 0,\\
&2\bar{s}A_0 + \bar{r}_2 A_2 + \bar{l}_4^- A_4 + \bar{q}_6 A_6 + \bar{l}_8^+ A_8 +\\
&+\bar{r}_{10} A_{10} + \bar{s}A_{12} = 0\\
&\bar{q}_m A_m + \bar{l}_{m+2}^+ A_{m+2} + \bar{l}_{m-2}^- A_{m-2} + \bar{r}_{m+4} A_{m+4}\\
&+\bar{r}_{m-4} A_{m-4} + \bar{s}(A_{m+6} + A_{m-6}) = 0, m = 8, 10\ldots, \quad \text{(A15)}
\end{aligned}
$$

and the dispersion relation reads

$$
\begin{vmatrix}
\bar{q}_0 & \bar{l}_2^+ & \bar{r}_4 & \bar{s} & 0 & \cdots\\
2\bar{l}_0 & \bar{q}_2 + \bar{r}_2 & \bar{l}_4^+ + \bar{s} & \bar{r}_6 & \bar{s} & \cdots\\
2\bar{r}_0 & \bar{l}_2^- + \bar{s} & \bar{q}_4 & \bar{l}_6^+ & \bar{r}_8 & \cdots\\
2\bar{s} & \bar{r}_2 & \bar{l}_4^- & \bar{q}_6 & \bar{l}_8^+ & \cdots\\
0 & \bar{s} & \bar{r}_4 & \bar{l}_6^- & \bar{q}_8 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix} = 0. \quad \text{(A16)}
$$

A.2. Oblique propagation

The coefficients of the solution (42) satisfy the recurrence relations

$$
\begin{aligned}
&q_m C_m + l_{m+2}^+ C_{m+2} + l_{m-2}^- C_{m-2} + r_{m+4} C_{m+4} +\\
&+r_{m-4} C_{m-4} + s(C_{m+6} + C_{m-6}) = 0, \quad \text{(A17)}\\
&\text{with } \tilde{k}_{\perp,m} = m + \tilde{k}_\perp, q_m = -\tilde{k}_{\perp,m}^2 \bar{a}_1 + \bar{c}_1,\\
&l_m^\pm = 0.5(-\tilde{k}_{\perp,m}^2 \bar{a}_2 \mp \tilde{k}_{\perp,m}\bar{b} + \bar{c}_2), \quad \text{(A18)}\\
&r_m = 0.5(-\tilde{k}_{\perp,m}^2 \bar{a}_3 + \bar{c}_3), \quad s = 0.5\bar{c}_4,\\
&m = 0, 2, 4\ldots \text{ for } p = 0, m = 1, 3, 5\ldots \text{ for } p = 1,
\end{aligned}
$$

where $\bar{a}_n, \bar{b}, \bar{c}_n$ are defined by (A13). The recurrence relations define an infinite set of linear algebraic equations, which has a non-trivial solution if an infinite Hill determinant is zero. This defines the dispersion equation. For solutions with $p = 0$ this equation is

$$
\begin{vmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & q_{-4} & l_{-2}^+ & r_0 & s & 0 & \cdots\\
\cdots & l_{-4}^- & q_{-2} & l_0^+ & r_2 & s & \cdots\\
\cdots & r_{-4} & l_{-2}^- & q_0 & l_2^+ & r_4 & \cdots\\
\cdots & s & r_{-2} & l_0^- & q_2 & l_4^+ & \cdots\\
\cdots & 0 & s & r_0 & l_2^- & q_4 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix} = 0. \quad \text{(A19)}
$$

For solutions with $p = 1$ it reads

$$
\begin{vmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & q_{-3} & l_{-1}^+ & r_1 & s & \cdots\\
\cdots & l_{-3}^- & q_{-1} & l_1^+ & r_3 & \cdots\\
\cdots & r_{-3} & l_{-1}^- & q_1 & l_3^+ & \cdots\\
\cdots & s & r_{-1} & l_1^- & q_3 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix} = 0. \quad \text{(A20)}
$$

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