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Chinese syzygy calculation established in the 13th century

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Abstract. In the 13th century, a new type of Chinese astronomical calculation was put into use, and it was almost entirely included in the calendar of Shoushi (formulated by Guo shoujing and his collaborators, epoch AD 1281, adopted from AD 1281−1644). Afterwards, this calendar has been thought of as one of the most accurate in the history of China. The perfectly systematic theoretical models at that time had been evolved since thousands of years. Nearly every part has its correspondences in modern astronomical year books, e.g., the syzygy calculation. By research on the origin of the Shoushi Calendar, we express the detailed and integral syzygy computing methods with several mathematical formulas, to simplify the complex reading and understanding of this calendar. The corrections of solar and lunar motions of this calendar for computing syzygys were compared with Newcomb's solar motion theory and the lunar ones of Chapront-Touzé and Chapront. The results show that they have good correspondence. This calendar is of high precision, and the mean error summed of the syzygy calculation is less than 21 minutes in its publishing periods.

Key words: Ephemerides – history and philosophy of astronomy

1. Introduction

The calendar of Shoushi is one of the most famous calendars in ancient China, which was recorded in Lizhi (ancient calendar book of China) of Yuanshi (annals of Yuan Dynasty AD 1279−1367). The mathematical method adopted in this calendar, far ahead of the foundation of Newton's mechanical system, summed up voluminous real measuring results. Now the computations of Chinese ancient calendars always puzzle us. The main problem is that the original is hard to read and understand. In this paper, we only try to put the whole calculations in order, so that the systematic mathematical models were reduced from the original, such as the calculations of mean and real syzygys, the corrections of solar and lunar motions. Then these computing models were compared with present theories of the solar and lunar motions. Finally, we give the accuracy of real new Moon computations in its publishing periods (Li and Zhang 1996a, b & c; Li 1997). In the system of ancient Chinese calendars, the researches on the calculations of other parts of the calendars, such as eclipses, were all based upon syzygy computations. Because ancient calendars, just as Shoushi, are also thought of as one kind of original materials that include a great number of actual ancient celestial records, these observations could help to identify the reality of historical events, to recover even some omitted data. If we could obtain these records again, the data would be very useful to contemporary studies, especially to the secular variation of the Earth's rotation.

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2. Mean syzygy calculation

The Chinese had their own traditional method to record dates, named Ganzhi, the same as Shoushi Calendar. In ancient China, the period of "Ganzhi" (i.e. the Heavenly Stems and Earthly Branches) is 60 days (from 0 to 59 in this paper). It is an independent method from very ancient times, day by day, one period by one period. Each day in one period has its own name, the name is also called one Ganzhi. Generally the data (Ganzhi number) is integer, but in this paper, due to our calculation: INT(Ganzhi) is as the date number, its decimal part (Ganzhi − INT(Ganzhi)) is the time (Beijing's apparent solar time, long. $116^\circ.4$ E, lat. $40^\circ.0$ N), and the unit is the day. So Ganzhi is also a kind of date recording method.

According to the Shoushi Calendar, the Ganzhi of mean new Moon, d_{MNM}, just before the Winter Solstice (AD 1280) is

d_{MNM} = QY − RY = 55.0600 − 20.2050 = $34^d.8550$,

with QY, named Qiying, days from the epoch to the first day of the period of Ganzhi which includes it, QY=55.0600. The "epoch" is the time of Winter Solstice in AD 1280. The number of Ganzhi is 55, the decimal part is the time, beginning at midnight, the unit is the day; RY, named Runying, days from the epoch to the new Moon which is just before it, RY=20.2050. The earlier adopted value is 20.1850 in the Shoushi Calendar. The name of Ganzhi is Wuxu (the number of Ganzhi is 34, the Gregorian date is 23 Dec. 1280), the time is 20:31 (Beijing's apparent solar time). This result is due to the real measurement of that time. The time of other mean new Moons equals to this value plus several lunar months B (named Shuoshi, B=$29^d.530593$).

The theory is very convenient, but the calculation is very complicated.

Generally, the Ganzhi of the Winter Solstice of the year before, d_{WS}, is given by
$d_{WS} = \mathrm{MOD}(N \times A' + QY, P)$.
In this formula, the function MOD(X, Y) = mod(X, Y), when mod(X, Y)$\geq$0; MOD(X, Y)=(mod(X, Y)+Y), when mod(X, Y)$<$ 0; N=year-1281 (named Jusuan or Jilian); $A' = A + 10^{-4} \times \mathrm{INT}(\frac{N}{100}$, (named Suishi, days of tropical year, A is the mean value, A=$365^{d}.2425$); P = 60, (named Xunzhou or Jifa, period of Ganzhi).

So R_Y (days from the Winter Solstice of the year before to the mean new Moon just before it) is
$R_Y = \mathrm{MOD}(N \times A' + RY, B)$.

The Ganzhi of every mean new Moon in a given year could be calculated as

$$d_{MNM} =$$

$$\mathrm{MOD}((N \times A' + QY - \mathrm{MOD}(N \times A' + RY, B) + n \times B), P), \quad (1)$$

with n, the numbers of the syzygys from the mean syzygy of just before the last Winter Solstice (beginning at n = 0).

The Ganzhi of every mean solar term in the year, d_{ST}, is given by

$$d_{ST} = \mathrm{MOD}((d_{WS} + n \times \frac{1}{24}A'), P) =$$

$$\mathrm{MOD}((N \times A' + QY + n \times \frac{1}{24}A'), P). \quad (2)$$

In these two equations, the result represents the number of Ganzhi (integral part) and its decimal, the unit is days, beginning at midnight. When n = 0, the d_{ST} represents the Ganzhi of the Winter Solstice just before the year, and the d_{MNM} is the Ganzhi of the mean new Moon just before the Winter Solstice.

If the parameters N (year−1281) and n are given, according to Eq. (1) and (2), we could compute the Ganzhi of mean new Moon and that of mean solar term. This just is the method of the Shoushi Calendar to reckon mean syzygys and solar terms.

The key to this part is to get the beginning point of the mean syzygy and solar term of the year for calculations. The other mean values of this year are equal to these values in addition to several (n) B for syzygys or several (n) $\frac{1}{24}$ A' for solar terms.

3. Corrections of solar motion

We have considered the Sun's average motion (mentioned above), but to get the real syzygy, it is necessary to understand the circumstance of the Sun's real motion in the period of a tropical year (A'). The functions M(C) and N(C) in the Shoushi Calendar were established for this purpose. The correction of solar motion is called Richan (i.e. solar equation). According to this calendar, M(C) and N(C) are defined as
$M(C) = 10^{-8} \times 5133200 - (31 \times C + 24600) \times C) \times C$;
$N(C) = 10^{-8} \times 4870600 - (27 \times C + 22100) \times C) \times C$,

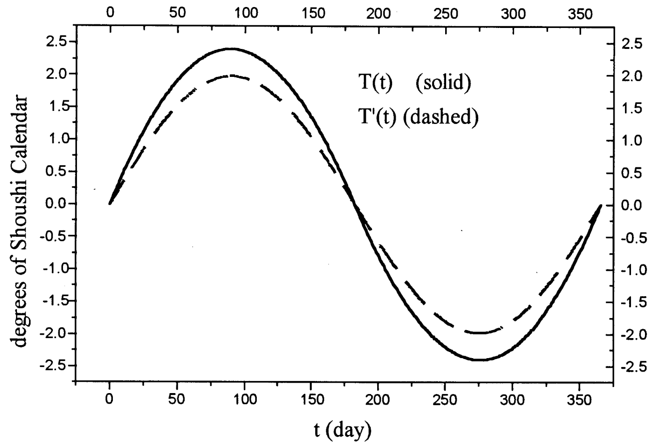

Fig. 1. The solar corrective function T(t) (solid) of Shoushi Calendar and the approximate solar model of Newcomb for computing the fluctuation of the longitude T'(t) (dashed) within one period (A') from Winter Solstice of 1280. The unit is degree of Shoushi.

where C is named as Chuxian or Moxian, and the unit of the values is the degree (but the Shoushi Calendar takes the circle as $365°.2575$, so at that time 1 degree equals to $0°.9856$ now).

The parameter t, days from the mean syzygy to the Winter Solstice just before it (t$\leq$ A'), is reckoned by

$$t = \mathrm{MOD}((A' - R_Y + n \times B), A') =$$

$$\mathrm{MOD}((A' - \mathrm{MOD}(N \times A' + RY, B) + n \times B), A'). \quad (3)$$

A tropical year (A') was divided into two parts, Yingli (0$\leq$t$<$0.5A', expanding area in one period) and Suoli (0.5A' $\leq$ t $\leq$ A', contracting area in one period). In this paper, the solar corrective function is T(t), which was defined by the calendar (the unit is degree),

$$T(t) = \begin{cases} M(C), \mathrm{in Yingli, Chuxian}, & C = t, \\ \quad (0 \leq t < 0.25A' - 2.4014); & \\ N(C), \mathrm{in Yingli, Moxian}, & C = 0.5A' - t, \\ \quad (0.25A' - 2.4014 \leq t < 0.5A'); & \\ -N(C), \mathrm{in Suoli, Chuxian}, & C = t - 0.5A', \\ \quad (0.5A' \leq t < 0.75A' + 2.4014); & \\ -M(C), \mathrm{in Suoli, Moxian}, & C = A' - t, \\ \quad (0.75A' + 2.4014 \leq t \leq A'). & \end{cases} \quad (4)$$

From Fig. 1, the plot of T(t) resembles that of a trigonometric function. The amplitude is $2°.4014$. The t represents the days from the beginning of the tropical year. When t=0 (Winter Solstice), 0.5A' (Summer Solstice), A' (Winter Solstice), T(t)=0, the Sun is at its mean position, and there are no fluctuations. When $t = 0.25A' - 2.4014$, T(t)= $2°.4014$ (maximum); when $t = 0.75A' + 2.4014$, T(t)= $-2°.4014$ (minimum).

Eq. (4) is roughly the same as Newcomb's solar motion theory (1898). According to our earlier work on Newcomb's Tables of the Sun (Li and Xu 1995), the fluctuations ($> 10''$) of the solar longitude are collected to compare with T(t). All of them include the equation of the center (period of year and half

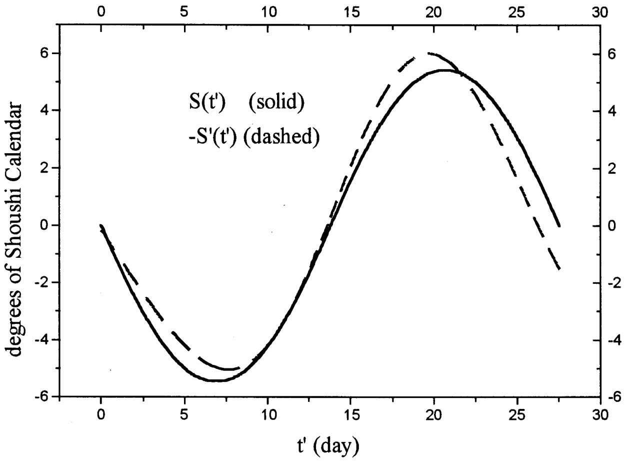

Fig. 2. The lunar corrective function S(t') (solid) of Shoushi Calendar and contemporary approximate lunar model for computing the fluctuation of the longitude S'(t') (shows - S'(t') in this figure, dashed) within one period (B') from perigee just before Winter Solstice of 1280. The unit is degree of Shoushi.

year) and the main nutation term (period of 18.6 year). Due to this consideration, we give the main solar longitude fluctuation of the epoch (from Winter Solstice of 1280, Nov. 14, $1^h 49^m$, 120° E zone time) within one tropical year:

$$\mathrm{T}'(\mathrm{t}) \approx 1^\circ.9774\sin(0^\circ.9856t) + 0^\circ.0204\sin(1^\circ.9712t) - 0^\circ.0049\sin(0^\circ.05295\mathrm{t} + 190^\circ),$$

where t is the same as that in T(t), the unit of T'(t) is the degree of Shoushi.

Fig. 1 shows both functions of T(t) and T'(t). Comparing them, the two models are in reasonable correspondence, almost the same amplitude and phase angle. According to the Shoushi Calendar, the formulae T(t) applied then by ancient Chinese astronomers in the 13th century approximately represents the real motions of the Sun.

4. Corrections of lunar motion

To understand the details of the Moon's real motion in one anomalistic month (B', named Zhuanzhong, B'= $27^d.5546$), the function P(D) in the Shoushi Calendar was made up for this purpose. The correction of lunar motion is called Yueli (i.e. lunar equation). According to this calendar, the P(D) was defined as
$\mathrm{P(D)} = 10^{-8} \times (11110000 - (325 \times \mathrm{D} + 28100) \times \mathrm{D}) \times \mathrm{D}$,
where D is also named as Chuxian or Moxian, the unit of P(D) is the degree.

The parameter t', days from the mean syzygy to the start of the anomalistic month (perigee) just before it (t' $\leq$ B'), is obtained by

$$\mathrm{t}' = \mathrm{MOD}((\mathrm{N} \times \mathrm{A}' + \mathrm{ZY} - \mathrm{R_Y} + \mathrm{n} \times \mathrm{B}), \mathrm{B}') =$$

$$\mathrm{MOD}((\mathrm{N} \times \mathrm{A}' + \mathrm{ZY} - \mathrm{MOD}(\mathrm{N} \times \mathrm{A}' + \mathrm{RY}, \mathrm{B}) + \mathrm{n} \times \mathrm{B}), \mathrm{B}'), \quad (5)$$

ZY, named Zhuanying, days from the epoch to the start of the anomalistic month which is just before it, ZY = 13.0205. The earlier adopted value is 13.1904 in the Shoushi Calendar.

One anomalistic month (B') was divided into two parts, Jili ($0 \leq \mathrm{t}' \leq 0.5\mathrm{B}'$, expanding area in one period) and Chili ($0.5\mathrm{B}' \leq \mathrm{t}' \leq \mathrm{B}'$, contracting area in one period). In this paper, the lunar corrective function is S(t'), which was defined by the calendar, the unit is the degree, the coefficient K = $1/0.082$,

$$\mathrm{S}(\mathrm{t}') = \begin{cases} -\mathrm{P(D), inJili, Chuxian}, & \mathrm{D} = \mathrm{Kt}', \\ \quad (0 \leq \mathrm{t}' < 0.25\mathrm{B}'); & \\ -\mathrm{P(D), inJili, Moxian}, & \mathrm{D} = \mathrm{K}(0.5\mathrm{B}' - \mathrm{t}'), \\ \quad (0.25\mathrm{B}' \leq \mathrm{t}' < 0.5\mathrm{B}'); & \\ \mathrm{P(D), inChili, Chuxian}, & \mathrm{D} = \mathrm{K}(\mathrm{t}' - 0.5\mathrm{B}'), \\ \quad (0.5\mathrm{B}' \leq \mathrm{t}' < 0.75\mathrm{B}'); & \\ \mathrm{P(D), inChili, Moxian}, & \mathrm{D} = \mathrm{K}(\mathrm{B}' - \mathrm{t}'), \\ \quad (0.75\mathrm{B}' \leq \mathrm{t}' \leq \mathrm{B}'). & \end{cases} \quad (6)$$

The function S(t') also resembles a trigonometric function. The amplitude is 5°.4289. The t' are the days from the beginning of the anomalistic month. When t'=0 (perigee), 0.5B' (apogee), B' (perigee), S(t')=0, the Moon is at its mean position, and there are no fluctuations. When t' is around 0.25B', S(t') = $-5^\circ.4289$ (minimum); when t' is about 0.75B', S(t')=5°.4289 (maximum).

Eq. (6) has correspondence to contemporary lunar motion model. We compare the computing model of lunar longitude of Chapront-Touzé and Chapront (1983 & 1988) with S(t'). In this paper, we only discuss the main terms of this model (amplitude>1000". There are three terms. The largest is the equation of the center in the same way as for the Sun. Due to this consideration, the approximate fluctuation of lunar longitude of the epoch (from the perigee just before Winter Solstice of 1280) within one anomalistic month is:

$$\mathrm{S}'(\mathrm{t}') \approx 6^\circ.3807\sin(13^\circ.0650t') + 1^\circ.2926\sin(11^\circ.3165t' + 175^\circ.5) + 0^\circ.6679\sin(24^\circ.3815\mathrm{t}' + 175^\circ.5),$$

where t' is the same as that in S(t'), the unit of S'(t') is the degree of Shoushi.

Fig. 2 plots both functions of S(t') and -S'(t'). By comparing them, the two models are of good correspondence, almost the same amplitude, but they have phase reversal. Due to Shoushi, the results calculated by S(t') are reversed to the real motion of the Moon, because the function S(t') given in the Shoushi Calendar is just for counting the real syzygys. This is why both phases are reversed. The function -S(t') used by ancient Chinese astronomers at that time approximately shows the real motions of the Moon.

5. Real syzygy calculations

Because the Moon's average motion is 13°.36875 per day, this calendar gets 1 Xian = 0.082 day, so one anomalistic month 1B' = 336 Xian, within this period, the Moon moves:
$0.082 \times 13^\circ.36875 = 1^\circ.0962$ per Xian.

To solve the real syzygy, we must first get the values of V(t'). The V(t') is defined as the velocity of the lunar motion, the unit is degree per Xian. Because this calendar had not detailed information about how to compute and obtain the V(t') value, we could not get the V(t') value from the Shoushi Calendar directly, but it could be read in the Datong Calendar (another Chinese ancient calendar in the Ming dynasty, which has almost the same fundamental constants and calculating methods as Shoushi, epoch of AD 1304, so it is thought that this calendar is roughly little different from Shoushi). There is only one value of V(t') per Xian. By analysing the Datong Calendar and its V(t') table, we give one method to compute V(t') values, so it is not necessary to read them from the table inconveniently (Li and Zhang 1996a).

The function Q(g) is given as

$Q(g) = 0^\circ.11081575 - 0^\circ.0005815g - 0^\circ.00000975g \times (g-1)$, where g is another parameter, and its unit is Xian. The unit of Q(g) is the degree, the unit of V(t') is degree per Xian. So

$$V(t') = \begin{cases} 1^\circ.0962 + Q(g),\ g = \\ \quad Kt', \quad (0 \le t' < 6.6420); \\ 1^\circ.0962 + Q(g) + 0.002 \times (g-81),\ g = \\ \quad Kt', \quad (6.6420 \le t' < 7.0520); \\ 1^\circ.0962 - Q(g),\ g = \\ \quad K(0.5B' - t') - 1, \quad (7.0520 \le t' < 0.5B'); \\ 1^\circ.0962 - Q(g),\ g = \\ \quad K(t' - 0.5B'), \quad (0.5B' \le t' < 20.4193); \\ 1^\circ.0962 - Q(g) - 0.002 \times (g-81),\ g = \\ \quad K(t' - 0.5B'), \quad (20.4193 \le t' < 20.8293); \\ 1^\circ.0962 + Q(g),\ g = \\ \quad K(B' - t') - 1, \quad (20.8293 \le t' \le B'). \end{cases} \tag{7}$$

The difference between the value computed by this method and the one read from the table is about 10^{-4} degree of Shoushi.

Eq. (7) really represents the velocity of the Moon's motion, which also has correspondence to contemporary lunar motion model. We compared approximate computing models of lunar longitude (mentioned above) with V(t'). The lunar mean longitude is

$\omega_1(t') \approx 218^\circ18'59''.95571 + 1732559343''.73604t'$.

The parameter t' is TDB from J2000, and the unit is the Julian Century. We change the unit of t' to the day and ω_1(t') to degree of Shoushi. So the velocity (from the perigee just before Winter Solstice of 1280) is

$$\begin{aligned} V'(t') &= \frac{d\omega_1(t')}{dt'} + \frac{dS'(t')}{dt'} \\ &= 1^\circ.0962 + 0^\circ.1193\cos(13^\circ.0650t') \\ &\quad +0^\circ.0209\cos(11^\circ.3165t' + 175^\circ.5) \\ &\quad +0^\circ.0233\cos(24^\circ.3815t' + 175^\circ.5), \end{aligned}$$

where t' is the same as that in S(t'), the unit of V'(t') is degree of Shoushi per Xian.

Fig. 3 plots both functions of V(t') and V'(t'). By comparing them, we see that the two models are in good correspondence, almost the same amplitude and phase.

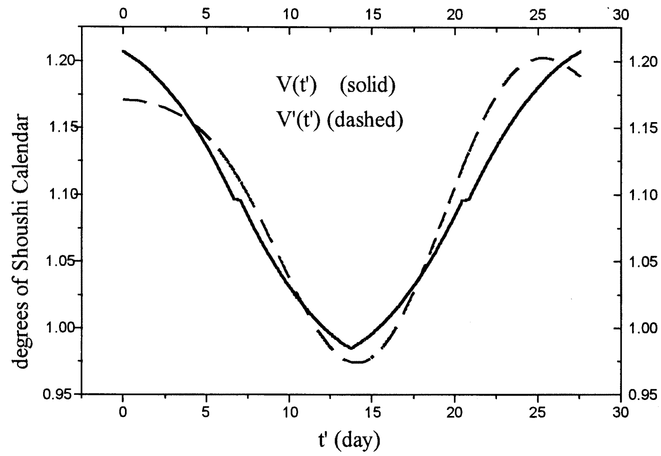

Fig. 3. The velocity function V(t') (solid) of lunar motion of Shoushi Calendar and contemporary approximate lunar model for computing the velocity of the longitude V'(t') (dashed) within one period (B') from perigee just before Winter Solstice of 1280, the unit is degree of Shoushi per Xian, 1 Xian = 0.082 day, when t' = 0 (perigee), V(t') = $1^\circ.2070$ (maximum); when t' = 0.5B' (apogee), V(t') = $0^\circ.9854$ (minimum).

According to the Shoushi Calendar, the Ganzhi numbers of the real new Moon, d_{RNM}, is calculated by

$$d_{RNM} = d_{MNM} + \frac{T(t) + S(t')}{K \times V(t')}, \tag{8}$$

the unit is the day.

In fact, since V(t') represents the velocity of lunar motion within one anomalistic month, and both models of T(t) and −S(t') just stand for the solar and lunar fluctuations, so the T(t) + S(t') is really the difference between the motions of the Sun and Moon. This is the crux of the matter for the Shoushi Calendar to calculate the real syzygys. The models of solar (Fig. 1) and lunar (Fig. 2) motions in the Shoushi Calendar and contemporary astronomical ephemerides are almost in agreement. As regards the calculating method, the results of Shoushi are by approximate calculation, the time equals to distance (T(t) + S(t')) divided by velocity (V(t')). The modern way is by iterative methods between longitude calculations of the Sun and Moon.

Now we have given all the formulas for calculating the real new Moon. However, how about the full Moon calculation? In these equations, if we take (n × B + 0.5B) in the place of (n × B), so we could get the values of the real full Moon.

6. Concluding remarks

Since knowing the data of the year, n, and Ganzhi calculated by the Shoushi Calendar system, it can thus readily be converted to the Gregorian Calendar. Now how to review this system? What about its accuracy? The makers wanted it to be used forever, but is that possible? Next, by the way of the Shoushi Calendar, we give the detailed precision of syzygys within its publishing period (Table 1 & Fig. 4). In fact, due to Shoushi, we can only compute the apparent solar time. In this paper, the precision

Table 1. The time precision of real new Moon calculated by the method of the Shoushi Calendar

Year	Data	Mean value (min.)	Mean value of absolute (min.)
1280-1645	4526	0.9	21.0
1280-1500	2722	-3.3	17.7
1280-1400	1486	-5.9	16.5

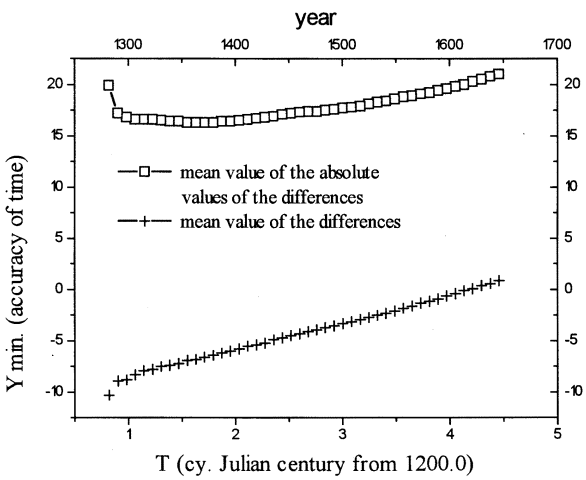

Fig. 4. The mean value distributions of the syzygy calculation by the method of Shoushi Calendar within AD 1280 to AD 1645. This figure shows the details of Table 1, the square stands for the mean value of the absolute values of the new Moons, the cross represents average value.

is obtained by comparing the difference of apparent solar time of syzygy calculated by Shoushi and mean solar time by the method of contemporary celestial mechanics.

In Table 1, "Mean value" is the mean value of the differences. "Mean value of absolute" is the mean value of the absolute values of the differences, which shows the deviations of the errors more accurately. By the method of the Shoushi Calendar, we computed all real new Moons from AD 1280 to 1645 and give their distributions of the mean values (Fig. 4). The average value of the differences is 0.9 minute and the mean value of the absolute is 21.0 minutes (Li and Zhang 1996a). By Table 1 and Fig. 4, the longer this calendar was used, the lower its precision. This is one of the important reasons why there were more than 100 calendars established in ancient China.

In fact, the real syzygy calculation is about the relationship among three models of T(t), S(t') and V(t'), all periodic functions. The accuracy of the calculations is based upon the corrections and precision of these models. The results computed by us show that the calculations of this calendar were very exact even in those times, but it could not be used forever, as other calendars published before.

Although this calendar was only used for about 350 years because of the problem of precision, many real historical surveying records fall in it. By the method of the Shoushi Calendar, to recover those data of celestial phenomena which were omitted has great potentialities, because it is very difficult to collect them by other ways. The longer the interval of the time, perhaps the more useful this method could be. So perhaps this calendar is of wide application. Now the contemporary research area of making full use of the ancient records of astronomical phenomena is very active. Our work on "secular variation of Earth's rotation: inferred from the Chinese ancient Shoushi Calendar (AD 1281)" belongs to this area (Li and Zhang 1997). Its prerequisite is how to discover those real ones at that time. According to the Shoushi Calendar's calculating methods, we make a try to get the parameter of times. By selecting these calculating data, some of them may be applied to identify or recover the original observing material of Earth's rotation.

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