

Lensing by gravitational waves in scalar-tensor gravity: Einstein frame analysis

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Abstract. The amplification of a light beam due to intervening gravitational waves is studied. The previous Jordan frame result according to which the amplification is many orders of magnitude larger in scalar-tensor gravity than in general relativity does not hold in the Einstein conformal frame. Lensing by gravitational waves is discussed in relation to the ongoing and proposed *VLBI* observations aimed at detecting the scintillation effect.

Key words: gravitation – gravitational waves – relativity – gravitational lensing

1. Introduction

Among the proposed theories of gravity, a special position is occupied by scalar-tensor theories, which currently are the subject of great interest because they exhibit features that resemble those of string theories (Green, Schwarz & Witten 1987). First of all, a fundamental scalar field ϕ appears in scalar-tensor theories in addition to the metric tensor $g_{\mu\nu}$, and massless scalar fields coupled to gravity are an essential feature of string and supergravity theories. Second, the gravitational part of the scalar-tensor action,

$$S_g = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi \right], \quad (1)$$

exhibits a conformal invariance that mimics the conformal invariance of string theories at high energies (Cho 1992, 1994; Damour & Esposito-Farese 1992; Turner 1993; Kolitch & Eardley 1995; Brans 1997). Further motivation for the study of scalar-tensor theories comes from the extended and hyperextended inflationary scenarios of the early universe (La & Steinhardt 1989; Steinhardt & Accetta 1990; Kolb, Salopek & Turner 1990; Liddle & Wands 1992; Crittenden & Steinhardt 1992; Steinhardt 1993; Laycock & Liddle 1994). Given that the classical tests of gravity in the Solar System (e.g. Will 1993) tell us that gravity is very close to Einstein gravity today¹, any experi-

ment that allows one to discriminate between general relativity and an alternative theory of gravity with present technology is important. An astronomical effect with such a potentiality was pointed out recently (Faraoni 1996); by studying the propagation of a light beam through gravitational waves, it was shown that the time-dependent amplification induced in the beam is a first order effect in the gravitational wave amplitudes, in scalar-tensor theories. This is an improvement of several orders of magnitude over the case of general relativity, in which the effect is only of second order (Bertotti 1971). The study of this effect is particularly relevant for the *VLBI* observations presently carried out on the radio source 2022+171 (Pogrebenko et al. 1994a,b, 1996) or proposed by Labeyrie (1993) (see also Bracco 1997) in order to detect the scintillation induced by gravitational waves. In recent years, many theoreticians have devoted their attention to the action of gravitational waves as lenses (Braginsky et al. 1990; Faraoni 1992a,b, 1993, 1996, 1997; Fakir 1993, 1994a,b, 1995, 1997; Durrer 1994; Marleau & Starkman 1996; Kaiser & Jaffe 1997; Gwinn et al. 1997), or as perturbations of conventional gravitational lenses (McBreen & Metcalfe 1988; Allen 1989, 1990; Kovner 1990; Frieman, Harari & Surpi 1994; Bar-Kana 1996).

The analysis of the amplification effect in (Faraoni 1996) was performed in the Jordan conformal frame and was based on the propagation equations for the optical scalars² (Sachs 1961)

$$\theta = \frac{1}{2} k^\alpha{}_{;\alpha}, \quad (2)$$

$$\sigma = \frac{1}{\sqrt{2}} \left[k_{(\alpha;\beta)} k^{\alpha;\beta} - \frac{1}{2} (k^\alpha{}_{;\alpha})^2 \right]^{1/2}, \quad (3)$$

in the era of matter domination (Damour & Nordvedt 1993a,b). If this is the case, the possibility of testing relativistic gravity at high redshifts is even more attractive.

² The metric signature is $-+++$, the speed of light and Newton's constant are set equal to unity, a colon and a semicolon denote, respectively, ordinary and covariant differentiation, ∇_μ is the covariant derivative operator. Round and square brackets around indices denote, respectively, symmetrization and antisymmetrization. The Ricci tensor is given in terms of the Christoffel symbols $\Gamma_{\alpha\beta}^\delta$ by $R_{\mu\rho} = \Gamma_{\mu\rho,\nu}^\nu - \Gamma_{\nu\rho,\mu}^\nu + \Gamma_{\mu\rho}^\alpha \Gamma_{\alpha\nu}^\nu - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\mu}^\nu$. A tilde denotes quantities defined in the Einstein conformal frame.

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¹ It is possible that gravity was described by a scalar-tensor theory early in the history of the universe, and converged to general relativity

$$\omega = \frac{1}{\sqrt{2}} [k_{[\alpha;\beta]} k^{\alpha;\beta}]^{1/2}, \quad (4)$$

of a congruence of null rays with tangent vector field k^μ . In Einstein gravity, the study of the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\theta^2 - |\sigma|^2 + \omega^2 - \frac{1}{2} R_{\mu\nu} k^\mu k^\nu \quad (5)$$

(where λ is an affine parameter along the null geodesics) in the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (6)$$

(the perturbations $h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, describe gravitational waves) leads, to first order in the waves amplitude h , to a vanishing Ricci tensor, and to the solution $\theta_{GR} = \mathcal{O}(h^2)$ for the expansion θ of the congruence (Bertotti 1971). This quantity describes the amplification of the beam in the geometric optics approximation, since the photon number is conserved (Schneider, Ehlers & Falco 1992). In scalar-tensor theories formulated in the Jordan conformal frame, one also has a gravitational scalar field $\phi = \phi_0 + \varphi$, where ϕ_0 is constant and $\mathcal{O}(\varphi/\phi_0) = \mathcal{O}(h)$. This leads to a nonvanishing term $R_{\alpha\beta} k^\alpha k^\beta$ on the right hand side of Eq. (5), which corresponds to a form of matter (scalar waves) in the beam; the first order amplification $\theta_{JF} = \mathcal{O}(h)$ arises as a consequence (Faraoni 1996). However, the expression $-R_{\mu\nu} k^\mu k^\nu$ oscillates with the frequency of φ , and this is a disturbing signal of the violation of the weak energy condition. The fact that $-R_{\mu\nu} k^\mu k^\nu$ is always negative whenever the energy conditions are satisfied, is essential in the proof of the singularity theorems (Wald 1984), hence the old adagio “matter always focuses”. The anomaly is in fact due to the violation of the weak energy condition in the Jordan frame, as will be explained in Sect. 2. Focusing of null geodesics is caused by the energy of the waves, and the anomalous dependence of the energy (linear in the second derivatives of the field, instead of quadratic in its first derivatives) in the Jordan frame version of scalar-tensor theories is reflected in the lensing effect.

It could be objected that a time-average gets rid of the offending first order expression; however, the problem is not solved. One can consider gravitational waves of astrophysical interest with relatively long periods (e.g. waves from μ -Sco, with period $3 \cdot 10^5$ s), for which the weak energy condition is violated on physically significant time scales.

There have been many debates in the literature on the issue of the conformal frame, which is still the subject of controversy. We do not repeat here these discussions but, rather, we refer the reader to (Magnano & Sokolowski 1994, and references therein). For our purposes, it is sufficient to remember that, in the Jordan frame formulation of scalar-tensor theories, the kinetic energy term for the scalar field in the Jordan action has indefinite sign, the system decays into a lower energy state *ad infinitum*, and it is unstable against small fluctuations. On the contrary, the reformulation of the scalar-tensor theory in the Einstein conformal frame exhibits a positive definite, canonical kinetic term for the Brans-Dicke-like scalar, and the theory has the desired stability property (Magnano & Sokolowski 1994,

and references therein). The present paper rephrases these arguments in terms of the weak energy condition – the reader should be warned that many current papers and most textbooks still present only the Jordan frame version of scalar-tensor theories.

The metric $\tilde{g}_{\mu\nu}$ in the Einstein frame is related to the Jordan frame metric $g_{\mu\nu}$ by the conformal transformation

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega = \sqrt{\tilde{\phi}}, \quad (7)$$

and the scalar fields in the two frames are related by the redefinition

$$\phi \longrightarrow \tilde{\phi} = \int \frac{(2\omega + 3)^{1/2}}{\phi} d\phi, \quad (8)$$

where $\omega > -3/2$. The necessity of the conformal transformation and arguments supporting the Einstein frame formulation were first advocated in Kaluza-Klein and Brans-Dicke theories (Sokolowski & Carr 1986; Bombelli et al. 1987; Sokolowski & Golda 1987; Sokolowski 1989a,b; Cho 1990, 1994) and later generalized to scalar-tensor theories (Cho 1997) and nonlinear gravity theories with gravitational part of the Lagrangian $\mathcal{L} = f(\phi, R)$ (Magnano & Sokolowski 1994, and references therein). It is important to reanalyse the calculations of (Faraoni 1996) in the Einstein conformal frame and to compute the magnitude of the amplification effect. The new calculation is presented in Sect. 2; photons propagating through a cosmological background of scalar-tensor gravitational waves are considered in Sect. 3, while Sect. 4 contains the conclusions.

2. Einstein frame vs Jordan frame

Since the Maxwell equations in four dimensions are conformally invariant, photons follow null geodesics also in the Einstein frame, as expected in the geometric optics approximation. We begin by decomposing the Einstein frame metric and scalar field as follows:

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad (9)$$

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{\varphi}, \quad (10)$$

where $\tilde{\phi}_0 = \text{constant}$ and $\tilde{h}_{\mu\nu}, \tilde{\varphi}$ describe, respectively, tensor and scalar gravitational waves, with $|\tilde{h}_{\mu\nu}|, |\tilde{\varphi}/\tilde{\phi}_0| \ll 1$. The linearized field equations are

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{h}_{\mu\nu} = 0, \quad (11)$$

$$\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\varphi} = 0, \quad (12)$$

where $\tilde{\tilde{h}}_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}_\alpha^\alpha / 2$. The solutions of Eq. (12) are expressed as Fourier integrals of plane waves,

$$\tilde{\varphi} = \tilde{\varphi}_0 \cos(p_\alpha x^\alpha), \quad (13)$$

where $\tilde{\varphi}_0$ is a constant and $\eta_{\mu\nu} p^\mu p^\nu = 0$. The stress-energy tensor of the scalar field assumes the canonical form

$$\tilde{T}_{\mu\nu}[\tilde{\varphi}] = \partial_\mu \tilde{\varphi} \partial_\nu \tilde{\varphi} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \tilde{\varphi} \partial_\beta \tilde{\varphi}. \quad (14)$$

The term $\tilde{R}_{\mu\nu}k^\mu k^\nu$ on the right hand side of the Raychaudhuri equation (5), which is responsible for the first order amplification effect in the Jordan frame, is nonzero also in the Einstein frame, but it is now of second order. Eqs. (14) and (13) yield³

$$\begin{aligned} \tilde{R}_{\mu\nu}k^\mu k^\nu &= 4\pi [p_\alpha k^\alpha \tilde{\varphi}_0 \sin(p_\alpha x^\alpha)]^2 \\ &+ \tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}]k^\mu k^\nu, \end{aligned} \quad (15)$$

which is of second order and positive definite; $\tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}]$ is the Isaacson effective stress-energy tensor of the tensor modes $\tilde{h}_{\alpha\beta}$. Following the reasoning of (Bertotti 1971; Faraoni 1996), which we do not repeat here, it is straightforward to conclude that the amplification of the beam in the Einstein frame is of second order, $\theta_{EF} = \mathcal{O}(\tilde{h}^2)$, contrarily to the case of the Jordan frame. Since $|h_{\mu\nu}| \ll 1$, this changes the amplification by many orders of magnitude.

Why is there such a difference between the Jordan and the Einstein frame? This is due to the different orders of magnitude of the term $R_{\mu\nu}k^\mu k^\nu$ in the Raychaudhuri equation. The Ricci tensor changes under the conformal transformation (7) according to (Wald 1984)

$$\begin{aligned} \tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - 2\nabla_\alpha \nabla_\beta (\ln \Omega) - g_{\alpha\beta} g^{\rho\sigma} \nabla_\rho \nabla_\sigma (\ln \Omega) \\ &+ 2\nabla_\alpha (\ln \Omega) \nabla_\beta (\ln \Omega) \\ &- 2g_{\alpha\beta} g^{\rho\sigma} \nabla_\rho (\ln \Omega) \nabla_\sigma (\ln \Omega); \end{aligned} \quad (16)$$

the harmonic expansion of the Jordan frame scalar

$$\phi = \phi_0 + \varphi_0 \cos(l_\alpha x^\alpha) \quad (17)$$

yields the first order term $R_{\mu\nu} = \partial_\mu \partial_\nu \varphi / \phi_0$ (Eq. (6) of (Faraoni 1996)). This expression, which is the first term on the right hand side of Eq. (16), is exactly cancelled by the first order contribution to the next term $-2\nabla_\alpha \nabla_\beta (\ln \Omega) = -\partial_\alpha \partial_\beta \varphi / \phi_0 + \mathcal{O}(h^2)$; what is left on the right hand side of Eq. (16) is only of second order.

The expansion (17) is, of course, consistent with Eqs. (10), (13); in fact, from Eqs. (17), (8) it follows that

$$\tilde{\phi} = \frac{(2\omega_0 + 3)^{1/2}}{\phi_0} \varphi + C + \mathcal{O}(h^2), \quad (18)$$

where $\omega_0 = \omega(\phi_0)$ and C is a integration constant. To first order, Eq. (18) is nothing but Eq. (13) where

$$\tilde{\phi}_0 = C, \quad (19)$$

$$\tilde{\varphi}_0 = \frac{(2\omega_0 + 3)^{1/2}}{\phi_0} \varphi_0, \quad (20)$$

$$p^\alpha = l^\alpha. \quad (21)$$

The origin of the problem in the Jordan frame is the non-canonical form of the stress-energy tensor of the scalar field; the $T_{\mu\nu}[\phi]$ of the Brans-Dicke scalar in the Jordan frame violates

the weak energy condition, and its structure is also responsible for the first order amplification. For simplicity, we restrict our treatment to Brans-Dicke theory, in which ω is constant, with vanishing cosmological constant. Gravitational waves in the Jordan frame are then described by the metric and scalar field perturbations $h_{\mu\nu}$ and φ in Eqs. (6), (17). The field equations yield the linearized equations (Will 1993)

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi = 0, \quad (22)$$

$$R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = \frac{1}{\phi_0} \partial_\mu \partial_\nu \varphi. \quad (23)$$

By using the decomposition of φ in plane waves one has that, for each plane monochromatic wave,

$$T_{\mu\nu}[\varphi] \xi^\mu \xi^\nu = - (k_\mu \xi^\mu)^2 \frac{\varphi}{\phi_0} \quad (24)$$

for any timelike vector ξ^μ . Since φ is an oscillating quantity, the sign of the energy density measured by an observer with four-velocity ξ^μ changes with the frequency of φ , violating the weak energy condition. By contrast, the Einstein frame stress-energy tensor is the sum of the canonical tensor for a scalar field, plus the effective Isaacson tensor for spin 2 waves:

$$\begin{aligned} \tilde{T}_{\mu\nu}[\tilde{\varphi}] &= \tilde{\nabla}_\mu \tilde{\varphi} \tilde{\nabla}_\nu \tilde{\varphi} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}^\alpha \tilde{\varphi} \tilde{\nabla}_\alpha \tilde{\varphi} + \tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}] \\ &= \mathcal{O}(h^2). \end{aligned} \quad (25)$$

One obtains, to the lowest order in the Einstein frame,

$$\begin{aligned} \tilde{T}_{\mu\nu} \xi^\mu \xi^\nu &= [\xi^\alpha p_\alpha \varphi_0 \sin(p_\beta x^\beta)]^2 \\ &+ \tilde{T}_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}] \xi^\mu \xi^\nu \geq 0. \end{aligned} \quad (26)$$

Besides violating the weak energy condition, the non-canonical form of $T_{\mu\nu}[\varphi]$ in the Jordan frame is also responsible for the order of magnitude of the term $R_{\mu\nu}k^\mu k^\nu$ in the Raychaudhuri equation; $T_{\mu\nu}[\varphi]$ is not a quadratic form in the scalar field derivatives but contains a term that depends linearly from the second derivatives of φ – this is the only term that survives for linearized waves. By contrast, the Einstein frame $\tilde{T}_{\mu\nu}[\tilde{\varphi}]$ complex (associated to the usual energy functional) is quadratic in the scalar field derivatives, and hence it is positive definite.

3. The gravitational wave background

The propagation of light through the cosmological gravitational wave background (Matzner 1968) has been studied in Einstein gravity in order to ascertain whether the deflection and frequency shift of the photons (which are of first order in the gravitational wave amplitudes, and therefore small) cumulate with the travelled distance D . Since D can be a cosmological distance, this secular or “ D -effect”, if present, would significantly enhance the deflections and frequency shifts, and it was considered also in relation with redshift anomalies and periodicities in galaxy groups and clusters (Rees 1971; Dautcourt 1974), and with proper motions of quasars (Gwinn et al. 1997). Naively, one would expect that, if a photon undergoes N scatterings in a

³ The order of magnitude of the perturbations is the same in both conformal frames (see the Appendix).

background of random gravitational waves, the deflections add stochastically, resulting in a D -effect proportional to \sqrt{N} (Winterberg 1968; Marleau & Starkman 1996). This is not the case, due to the equality between the speed of the propagating signals and that of the random inhomogeneities of the medium (Zipoy 1966; Zipoy & Bertotti 1968; Dautcourt 1974; Bertotti & Cattacci 1975; Linder 1986, 1988; Braginsky et al. 1990; Kaiser & Jaffe 1997). Is the D -effect present in a stochastic background of scalar-tensor gravitational waves? This question is non-trivial because random inhomogeneities due to fields of different spin produce different results for the rms deflection, and spin 0 waves go hand in hand with spin 2 modes in scalar-tensor gravity.

The solution to this problem is contained in Linder's (1986) paper; although he did not explicitly consider alternative theories of gravity, he studied light propagation through a random medium with inhomogeneities due to fields of spin 0, 1 or 2, which are allowed to propagate at any speed less than, or equal to, the speed of light. Adapting Linder's (1986) result to the case of scalar modes propagating at the speed of light, one obtains that a photon whose unperturbed path is parallel to the z -axis, experiences the rms deflection

$$(\delta k^x)_{rms} = (\delta k^y)_{rms} = \left(\frac{\tilde{\varphi}}{\tilde{\phi}_0} \right)_{rms} \left[\ln \left(\frac{2\pi D}{\lambda_{gw}} \right) \right]^{1/2} \quad (27)$$

$$(\delta k^z)_{rms} = 0. \quad (28)$$

The same dependence was obtained in a recent paper by Kaiser & Jaffe (1997). Albeit qualitatively different from Einstein gravity, the dependence of $(\delta k^\mu)_{rms}$ from the distance D is hardly significant: to give an idea of the orders of magnitude involved, we consider a gravitational wavelength $\lambda_{gw} = 5$ cm (approximately corresponding to the 1 K cosmic gravitational wave background (Matzner 1968)) and a cosmological distance $D = 500$ Mpc, which give

$$(\delta k)_{rms} \simeq 7.92 \left(\frac{\tilde{\varphi}}{\tilde{\phi}_0} \right)_{rms}, \quad (29)$$

an enhancement of less than one order of magnitude with respect to general relativity. Again, lensing by gravitational waves in scalar-tensor gravity is not much more efficient than in Einstein gravity.

4. Conclusions

From the theoretical point of view, our analysis is relevant to the issue of the conformal frame in scalar-tensor theories of gravity. The violation of the weak energy condition in the Jordan frame was shown in Sect. 2. From the point of view of the applications of the theory, we have given a negative answer to the problem of whether, in scalar-tensor theories, the gravity wave-induced amplification of a light beam is enhanced by many orders of magnitude in comparison to Einstein gravity. If this was true, a door would be open for discriminating between general relativity and scalar-tensor theories using astronomical observations and present technology. Our study is relevant for the ongoing

VLBI observations of the radio source 2022+171 (Pogrebenko et al. 1994a,b, 1996) aimed at detecting gravity wave-induced scintillation effects, and in view of the observations proposed by Labeyrie (1993) (see also Bracco 1997). Unfortunately, when the amplification of a light beam due to gravitational waves is computed in the Einstein conformal frame, to which the observations must be referred, the effect is not much larger in scalar-tensor gravity than it is in general relativity. This leaves little hope for an easy detection of the scintillation effect, and for the determination of the correct theory of gravity using astronomy. This rather pessimistic conclusion is based on the assumption that scalar and tensor modes have the same amplitude, $O(\tilde{\varphi}/\tilde{\phi}_0) = O(\tilde{h}_{\mu\nu})$; perhaps this assumption is relaxed to a certain extent if scalar modes are emitted at a significantly larger rate than tensor modes in processes of astrophysical relevance. For example, gravitational collapse with spherical symmetry produces spin 0, but not spin 2, waves.

Taking a broader point of view, it would be premature to conclude that the amplification induced by gravitational waves (in general relativity or in its scalar-tensor competitors) is impossible to detect with present technology. In fact, the optical scalars formalism used in our calculation breaks down when the gravitational lens exhibits caustics and critical lines, which separate regions corresponding to different numbers of images of the light source. In this situation, high amplification events occur if the light source crosses a caustic (Schneider, Ehlers & Falco 1992). In (Faraoni 1997), it was shown that the optical scalars formalism does not tell the whole story: the actual amplification is of order

$$A \approx \left(\frac{hD}{\lambda_{gw}} \right)^2, \quad (30)$$

where λ_{gw} is the gravitational wavelength, and D is the distance between the observer and the light source. Large values of the ratio D/λ_{gw} can balance small values of h and a non-negligible amplification is still possible. A detailed study of realistic gravitational waveforms within the formalism developed in (Faraoni 1992a,b, 1997) and a feasibility study of the *VLBI* detection of scintillation effects induced by gravitational waves will be the subject of a future publication. The conclusion of the present paper is that scalar-tensor gravity does not fare much better than general relativity in inducing this kind of effects.

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Appendix A

Insight into the nature of scalar-tensor waves in the Einstein frame is obtained by combining Eq. (7) and the metric decompositions (6), (9) to obtain

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{\varphi}{\phi_0} \eta_{\mu\nu} + O(h^2). \quad (A.1)$$

According to Eq. (A.1), the Einstein frame gravitational waves are a mixture of spin 2 ($h_{\mu\nu}$) and spin 0 ($\eta_{\mu\nu}\varphi/\phi_0$) modes in the language of the Jordan frame. Moreover, the metric perturbations have the same order of magnitude in the two conformal frames:

$$O(\tilde{h}_{\mu\nu}) = O(h_{\mu\nu}), \quad (\text{A.2})$$

$$O\left(\frac{\tilde{\varphi}}{\tilde{\phi}_0}\right) = O\left(\frac{\varphi}{\phi_0}\right) \quad (\text{A.3})$$

(where the last equality follows from Eq. (18)). It is this property that allows a meaningful comparison of the amplification effect in the two conformal frames.

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