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Relation of Cartesian and spherical multipole moments in general relativity

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Abstract. The Earth's gravitational field is represented by its multipole moments. Multipole moments have two kinds of equivalent forms, that is, the Cartesian symmetric and trace-free tensors and the spherical harmonic coefficients. The relation between these two forms is interesting and useful for some practical problems. Under Newtonian approximation, there exists a simple relation between the aforesaid two kinds of multipole moments (see Hartmann et al., 1994, for details). But in the 1PN approximation of general relativity, the relation mentioned above becomes complicated. This paper discusses how to turn the expansion of the 1PN Earth's gravitational potential, which consists of a scalar potential and a vector potential, in terms of BD moments into that in terms of a set of time-slowly-changing, observable multipole moments. Under a specific standard PN gauge, we derive the corresponding expansion of the potential in terms of spherical harmonics, obtain the relation between the 1PN spherical harmonic coefficients and the Cartesian multipole moments, and compute the expressions of the lowest order spherical harmonic coefficients including the relation between the 1PN Earth dynamical form-factor J_2 and the BD mass quadrupole moment of the Earth. As for the 1PN vector potential, we also discuss its expansion in terms of Cartesian multipole moments under the rigidity approximation. In this paper, we emphasize the choice of the coordinate gauge. Under our $ad\ hoc$ standard PN gauge, the results have simpler form and clearer physical meaning.

Key words: relativity – gravitation – Earth – celestial mechanics, stellar dynamics – reference systems

1. Introduction

The Earth's gravitational potential can be expanded in two forms: (1) spherical harmonics expansion (McCarthy, 1992), which astronomers and geophysicists are familiar with; (2) expansion in terms of rectangular coordinate multipole moments that are Cartesian symmetric and trace-free (STF) tensors, which is in favour with physicists. Two kinds of expressions of the gravitational potential are essentially equivalent, as both are the equivalent irreducible representation of the rotation group $SO(3)$. Various kinds of practical Earth's gravitational field models (e.g. Anderle, 1979, Lerch, 1985, Reigber, 1985) are built in spherical harmonics expansion. But many theoretical physicists trend to use Cartesian STF tensor expansion, which would make theories more elegant, simple and compact in form. A recent investigation (Hartmann et al., 1994) demonstrated that in numerical computation the STF tensors are more efficient than the spherical harmonic coefficients. Thorne (1980) once systematically summarized various kinds of spherical harmonics and STF tensors in gravitational physics. From then on, the STF tensors as a powerful mathematical tool have been extensively used in references on gravitation and general relativistic celestial mechanics, e.g. Kopejkin (1988), Blanchet and Damour (1989), Damour and Iyer (1991a, b), Damour et al. (1991, 1992, 1993, 1994) and Xu et al. (1997). Hartmann et al. (1994) presented an extensive discussion on the application of the STF tensors in classical celestial mechanics.

In the recent two decades, much progress has been made in relativistic celestial mechanics. Especially, a new 1PN celestial mechanics theory (hereafter cited as DSX theory) presented by Damour, Soffel and Xu (1991, 1992, 1993, 1994), is very attractive. The theory has extensive and bright prospects for its application in astronomy and other fields concerned. But at present, efforts have to be made to put DSX theory into its practical use. One difficulty lies in the basic physical quantities in DSX theory, i.e. relativistic mass and spin multipole moments (BD moments, Blanchet, Damour, 1989). Damour et al. (1991) considered BD moments as observable physical quantities. But BD moments are defined in the body-centered coordinate system whose axes could be slowly rotating with an 1PN angular velocity (Damour et al., 1991) and therefore they are generally fast-changing with time due to the irregular figure and the rotation of the body. It would be necessary to introduce the multipole moments that are defined in a co-rotating coordinate system with the body and therefore are slowly-changing with time. Furthermore, the evolution equations of BD moments expressed in terms of themself have not been obtained to our knowledge so far.

Hartmann et al. (1994) has derived a general relation between the spherical harmonic coefficients and the Cartesian STF multipole moments in Newtonian celestial mechanics. It would be also necessary to extend their results to the 1PN celestial

Send offprint requests to: T.-Y. Huang mechanics. In the latter case the 1PN gravitational field is described by a scalar potential and a 3-dimensional vector potential, which would change with a gauge transformation in DSX theory. A proper choice of the gauge condition might simplify the relation we are looking for.

This paper is arranged as follows: Sect. 2 briefly states the BD moments expansion of the 1PN gravitational potential, the existing problems in it and the gauge choice. Sect. 3 introduces a co-rotating coordinate system, then projects the above expansion into it, and obtains a time-slowly-changing multipole moments expansion of the 1PN potential. In Sect. 4, we write out the spherical harmonics expansion of the 1PN potential, and derive the relation between the lowest order spherical harmonic coefficients and the relevant Cartesian multipole moments. In Sect. 5, we discuss the Cartesian multipole moments expansion of the 1PN vector potential under the rigidity approximation. In Sect. 6, some main conclusions are summarized.

2. BD moments expansion of the 1PN Earth's gravitational potential

In this section, we first introduce some relevant results in Damour et al. (1991).

The Earth's local coordinates are taken as DSX coordinates (T, X^a), $a = 1, 2, 3$. The corresponding metric tensor is $G_{\alpha\beta}$. The coordinates satisfy an algebraic coordinate condition: $G_{00}G_{ab} = -\delta_{ab} + O(4)$. Harmonic coordinates and standard PN coordinates are the special cases of DSX coordinates. Let $W_\alpha \equiv (W, W_a)$ denote the 1PN Earth's gravitational potential, $\Sigma^\alpha \equiv (\Sigma, \Sigma_a)$ the Earth's mass density and mass current density, here $\Sigma \equiv (T^{00} + T^{aa})/c^2$, $\Sigma^a \equiv T^{0a}/c$, and $T^{\alpha\beta}$ is the stress-energy tensor of the gravitational source. Σ^α satisfies the continuity equation: $\partial_T\Sigma + \partial_a\Sigma^a = O(2)$. According to DSX theory, W_α satisfies the linearized gravitational field equations

$$\begin{aligned}\Box W + 4c^{-2}\partial_T(\partial_T W + \partial_b W_b) &= -4\pi G\Sigma + O(4)\\ \Box W_a - \partial_a(\partial_T W + \partial_b W_b) &= -4\pi G\Sigma^a + O(2)\end{aligned} \tag{1}$$

where d'Alembertian $\Box \equiv -c^{-2}\partial_T^2 + \partial_a\partial_a$. Eq. (1) is gauge invariant under a gauge transformation: $W_\alpha \to {W'}_\alpha$

$$W' = W - c^{-2}\partial_T\lambda, \quad {W'}_a = W_a + \frac{1}{4}\partial_a\lambda \tag{2}$$

here $\lambda = \lambda(T, X^a)$ is an arbitrary function. Eq. (2) is equivalent to a coordinate transformation $(T, X^a) \to (T', {X'}^a)$

$$T' = T - \lambda/c^4, \quad {X'}^a = X^a \tag{3}$$

To further fix the time coordinate T, it is necessary to add a coordinate gauge condition to the field equations (1). Usually one would choose the harmonic gauge $\partial_T W + \partial_a W_a = O(2)$ or the standard PN gauge $3\partial_T W + 4\partial_a W_a = O(2)$. If we take

$$\begin{aligned}&\lambda = \tfrac{1}{2}\partial_T Z(T, \mathbf{X}),\\ &Z(T, \mathbf{X}) \equiv G\int_E d^3X'\Sigma(T, \mathbf{X}')|\mathbf{X} - \mathbf{X}'|\end{aligned} \tag{4}$$

Eq. (3) will turn the harmonic coordinates into the standard PN coordinates. If we take the harmonic gauge, the field equations (1) becomes very simple

$$\Box W_\alpha = -4\pi G\Sigma^\alpha + O(4, 2) \tag{5}$$

Its time-symmetric solution is

$$\begin{aligned}W_\alpha &= \Box^{-1}_{sym}(-4\pi G\Sigma^\alpha) + O(4, 2)\\ &= G\int_E d^3X'\Sigma^\alpha(T \pm |\mathbf{X} - \mathbf{X}'|/c, \mathbf{X}')/|\mathbf{X} - \mathbf{X}'|\\ &\quad +O(4, 2)\end{aligned} \tag{6}$$

In Eqs. (4) and (6), the subscript E means the Earth. Solution (6) and the gauge term in Eq. (2) construct the general solution of the field equations (1). It can be expanded in terms of the 1PN mass multipole moments M_L and the spin multipole moments S_L as follows

$$\begin{aligned}W(T, \mathbf{X}) &= G\sum_{l\geq 0}\tfrac{(-)^l}{l!}\partial_L[R^{-1}M_L(T \pm R/c)]\\ &\quad +c^{-2}\partial_T(\Lambda - \lambda) + O(4)\\ W_a(T, \mathbf{X}) &= G\sum_{l\geq 0}\tfrac{(-)^l}{(l+1)!}(\dot{M}_{aL} + l\epsilon_{ab<a_l}S_{L-1>b})\partial_L R^{-1}\\ &\quad -\tfrac{1}{4}\partial_a(\Lambda - \lambda) + O(2)\end{aligned} \tag{7}$$

where $R \equiv |\mathbf{X}|$, $\dot{M}_L \equiv dM_L/dT$, and

$$\begin{aligned}M_L(T) &\equiv \int_E d^3X\hat{X}^L\Sigma + \ddot{N}_L/2(2l + 3)c^2\\ &\quad -\dot{P}_L[4(2l + 1)/(l + 1)(2l + 3)c^2] + O(4)\\ S_L(T) &\equiv \int_E d^3X\epsilon^{ab<a_l}\hat{X}^{L-1>a}\Sigma^b + O(2), \quad l \geq 1\end{aligned} \tag{8}$$

where P_L and N_L are multipole-like moments, i.e. so-called "bad moments". They and Λ have the following expressions

$$\begin{aligned}&P_L(T) \equiv \int_E d^3X\hat{X}^{bL}\Sigma^b, \quad N_L \equiv \int_E d^3X\mathbf{X}^2\hat{X}^L\Sigma,\\ &\Lambda(T, \mathbf{X}) \equiv 4G\sum_{l\geq 0}\tfrac{(-)^l}{(l+1)!}\tfrac{2l+1}{2l+3}P_L\partial_L R^{-1} + O(2)\end{aligned} \tag{9}$$

The 1PN mass multipole moments M_L and the spin multipole moments S_L defined by Eq. (8) were first introduced by Blanchet and Damour (1989), so they are also called BD moments.

The BD moments expansion of the 1PN gravitational potential, as shown by Eq. (7), depends on the choice of the gauge function $\lambda(T, \mathbf{X})$. Here we only consider two specific gauges. They respectively belong to the harmonic gauge and the standard PN one.

(i) $\lambda = \Lambda$, this belongs to the harmonic gauge. Damour et al. (1991) called it the skeletonized-body harmonic gauge. Under the gauge, the expansion of the 1PN gravitational potential is fully expressed in terms of BD moments, no "bad moments" P_L and N_L in it. Besides, the expansion of the vector potential is the simplest one in form. Eq. (7) can be written as

$$\begin{aligned}W(T, \mathbf{X}) &= G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[M_L - \ddot{M}_L R^2/2(2l - 1)c^2]\partial_L R^{-1}\\ &\quad +O(4)\end{aligned} \tag{10}$$

and

$$\begin{aligned}W_a(T, \mathbf{X}) &= G\sum_{l\geq 0}\tfrac{(-)^l}{(l+1)!}[\dot{M}_{aL} + l\epsilon_{ab<a_l}S_{L-1>b}]\partial_L R^{-1}\\ &\quad +O(2)\end{aligned} \tag{11}$$

Eq. (10) shows that $\ddot{M}_L \equiv d^2M_L/dT^2$ appearing in the expression of W breaks the simplicity of the expansion of W in classical mechanics.

(ii) If we take

$$\begin{aligned}\lambda &= \Lambda + \tfrac{1}{2}G\sum_{l\geq 0}\tfrac{(-)^l}{l!}\dot{M}_L\partial_L R \\ &= \tfrac{1}{2}\partial_T Z + G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[\tfrac{4(2l+1)}{(l+1)(2l+3)}P_L \\ &-\dot{N}_L/2(2l+3)]\partial_L R^{-1} + O(2)\end{aligned} \tag{12}$$

then it is easy to find that Eq. (12) belongs to the standard PN gauge. The primary advantage of this gauge lies in that the expansion of the 1PN scalar potential is formally identical with that of the Newtonian gravitational potential. This 1PN scalar potential can be viewed as a natural generalization of the Newtonian gravitational potential in the 1PN case. But the expansion of the vector potential is more complicated under this gauge than the one shown by Eq. (11). The reason for that is the gauge transformation will result in some additional terms. After some computation, we can write out the expansion of W_α under the gauge (12)

$$W(T,\mathbf{X}) = G\sum_{l\geq 0}\frac{(-)^l}{l!}M_L\partial_L R^{-1} + O(4) \tag{13}$$

and

$$\begin{aligned}W_a(T,\mathbf{X}) &= G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[\tfrac{1}{l+1}(\tfrac{7l+11}{4(2l+3)}\dot{M}_{aL} \\ &+l\epsilon_{ab<a_l}S_{L-1>b}) \\ &+\tfrac{l}{8(2l-1)}\delta_{a<a_l}\dot{M}_{L-1>}R^2]\partial_L R^{-1} + O(2)\end{aligned} \tag{14}$$

From Eqs. (13) and (14), we can find that these expansions include BD moments and their time-derivatives, and are not involved with "bad moments" P_L and N_L.

The 1PN gravitational effects are gauge-independent, but the gravitational potential W_α is gauge-dependent. The choice of gauge should be decided according to the specific problem under consideration. As for the expansion of the 1PN potential W_α, one of the principles for choosing gauge is to avoid the "bad moments" P_L and N_L appearing in the expansion, which is in accordance with that P_L and N_L have been absorbed into the definition of M_L. Gauge (i) and (ii) both follow this principle. Another is that the choice of gauge should make the expansion of the scalar potential as simple as possible for the reason that the vector potential only produces the 1PN gravitational effect and is negligible in many practical cases. In this paper, we will adopt the standard PN gauge shown by Eq. (12) (hereafter the standard PN gauge always means this gauge unless otherwise stated). This gauge complies with two principles mentioned above. Especially, under this gauge the expansion of the scalar potential is formally identical with that of the Newtonian gravitational potential. This means that the 1PN mass multipole moments generate the 1PN scalar potential in a similar way to that in classical mechanics.

3. Projections of BD moments in the co-rotating coordinate system

The BD moments are the basic physical quantities in DSX theory. They generate the gravitational field in place of the stress-energy tensor of the matter. They are defined in the local DSX coordinate system (e.g. dynamical non-rotating coordinate system) of a body. Due to the rotation of celestial bodies, BD moments generally change with time T. Only physical quantities that are time-independent or slowly-changing with time can be considered as astronomical or physical constants. Therefore, a set of new multipole moments that are time-independent or slowly-changing with time should be defined.

According to Newtonian mechanics, the mass multipole moments of a rigid body in its mass-centered and co-rotating reference system are time-independent. This prompts that the projections of the 1PN mass multipole moments of a body in its co-rotating system would become time-slowly-changing. Moreover, it is easy to infer that the projections of the time-derivatives of M_L ($\dot{M}_L$ and $\ddot{M}_L$) in the co-rotating system are also time-slowly-changing.

Two facts make things more complicated. They are: (a) There is no " rigid body" in general relativity; (b) The real celestial body such as the Earth is not rigid under tidal influence and geophysical mass redistribution even in Newtonian mechanics. So there is no unique co-rotating system. We must define the co-rotating system of a body to assure that the spin of the body with respect to this system is as small as possible.

Let (T, x^i) be the co-rotating coordinate system of the Earth, which is connected with the local DSX coordinate system of the Earth by an orthogonal matrix $R^a_i(T)$. The relation between the two coordinate systems is $X^a = R^a_i(T)x^i$, or $x^i = R^a_i(T)X^a$. The matrix $R^a_i(T)$ satisfies

$$R^a_i(T)R^a_j(T) = \delta_{ij}, \quad R^a_i(T)R^b_i(T) = \delta_{ab} \tag{15}$$

Apparently, $r \equiv |\mathbf{x}| = |\mathbf{X}| \equiv R$. The rotational angular velocity $\mathbf{\Omega}$ reads

$$\Omega^a = \frac{1}{2}\epsilon_{abc}\Omega_{bc}, \quad \Omega_{bc} \equiv R^b_i\dot{R}^c_i, \quad \Omega^a = R^a_i\omega^i \tag{16}$$

where ω^i is the projection of Ω^a on the spatial axes of the co-rotating system, $\omega^i{=}\frac{1}{2}\epsilon_{ijk}\omega_{jk}$, $\omega_{jk}{=}\dot{R}^a_j R^a_k$. Making use of $R^a_i(T)$ and Eq. (15), W and W_a can be projected into the co-rotating system (T, x^i), and obtain

$$\begin{aligned}&W(T,X^a) = W(T,R^a_i x^i) \equiv w(t,x^i), \\ &W_a(T,X^b) = W_a(T,R^b_j x^j) \equiv R^a_i w_i(t,x^j)\end{aligned} \tag{17}$$

where w and w_i have expansions formally similar to those of W and W_a. If the harmonic gauge is adopted: $\lambda = \Lambda$, corresponding to Eqs. (10) and (11), w and w_i can be written as

$$\begin{aligned}&w(T,\mathbf{x}) = \\ &G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[m_{L'} - m''_{L'}r^2/2(2l-1)c^2]\partial_{L'}r^{-1} + O(4)\end{aligned} \tag{18}$$

and

$$\begin{aligned}&w_i(T,\mathbf{x}) = G\sum_{l\geq 0}\tfrac{(-)^l}{(l+1)!}[m'_{iL'} + l\epsilon_{ij<i_l}s_{L'-1>j}]\partial_{L'}r^{-1} \\ &+O(2)\end{aligned} \tag{19}$$

respectively, where

$$m_{L'} \equiv R^L_{L'} M_L, \quad m'_{L'} \equiv R^L_{L'} \dot{M}_L, \quad s_{L'} \equiv R^L_{L'} S_L,$$
$$m''_{L'} \equiv R^L_{L'} \ddot{M}_L \quad \partial_{L'} \equiv R^L_{L'} \partial_L, \quad L \equiv < a_1 a_2 \cdots a_l >, \tag{20}$$
$$L' \equiv < i_1 i_2 \cdots i_l >, \quad R^L_{L'} \equiv R^{<a_1}_{<i_1} R^{a_2}_{i_2} \cdots R^{a_l>}_{i_l>}$$

If the standard PN gauge is adopted, corresponding to Eqs. (13) and (14), w and w_i read

$$w(T, \mathbf{x}) = G \sum_{l \geq 0} \frac{(-)^l}{l!} m_{L'} \partial_{L'} r^{-1} + O(4) \tag{21}$$

and

$$w_i(T, \mathbf{x}) = G \sum_{l \geq 0} \frac{(-)^l}{l!} [\frac{1}{l+1} (\frac{7l+11}{4(2l+3)} m'_{iL'} + l \epsilon_{ij<i_l} s_{L'-1>j}) + \frac{l}{8(2l-1)} r^2 \delta_{i<i_l} m'_{L'-1>}] \partial_{L'} r^{-1} + O(2) \tag{22}$$

respectively. In Eqs. (18)-(22), $m_{L'}$, $m'_{L'}$, $m''_{L'}$ and $s_{L'}$ are the projections of M_L, $\dot{M}_L$, $\ddot{M}_L$ and S_L, respectively. They are all time-slowly-changing physical quantities. It should be noted that $m'_{L'}$ and $m''_{L'}$ are in general not small quantities, though $\dot{m}_{L'}$ and $\ddot{m}_{L'}$ are small indeed.

One can find that the right-hand side of Eq. (21) is simpler than that of Eq. (18), which is just the reason to choose the standard PN gauge (Eq. (12)) other than the harmonic gauge $\lambda = \Lambda$. The merits of choosing the gauge (Eq. (12)) have been stated in Sect. 2. One more argument is : if we take up the harmonic gauge $\lambda = \Lambda$, w is expressed by Eq. (18), and it can be formally expanded in terms of spherical harmonics, but, the spherical harmonic coefficients C_{lm} and S_{lm} must be related to r. This would cause that C_{lm} and S_{lm} are not constants even in the rigid case.

4. Spherical harmonic expansions of w and w_i

Adopting the standard PN gauge expressed by Eq. (12), Eqs. (17), (21) and (22) determine the expression of the 1PN potential in terms of time-slowly-changing multipole moments $m_{L'}, m'_{L'}, m''_{L'}$ and $s_{L'}$, which are the Cartesian STF multipole moments. The next step is to find their relation with spherical harmonic coefficients. In Eqs. (21) and (22), $\partial_{L'} r^{-1}$ is an explicit factor in the expressions of w and w_i. This means that w and w_i can certainly be expanded in terms of spherical harmonics. Comparing the expressions of w and w_i in terms of the spherical harmonics with those in terms of the Cartesian multipole moments, we can immediately get the corresponding relations between the Cartesian multipole moments and the spherical harmonic coefficients.

The spherical harmonics expansions corresponding to Eqs. (21) and (22) can be written as

$$\begin{aligned} w(T, r, \phi, \lambda) &= (GM/r) \sum_{l \geq 0} \sum_{m=0}^{l} (\tfrac{a_e}{r})^l P_{lm}(\sin\phi) \times \\ &\quad [C_{lm} \cos m\lambda + S_{lm} \sin m\lambda] + O(4) \\ &= (GM/r) \{1 + \sum_{l \geq 1} \sum_{m=0}^{l} (\tfrac{a_e}{r})^l P_{lm}(\sin\phi) \times \\ &\quad [C_{lm} \cos m\lambda + S_{lm} \sin m\lambda]\} + O(4) \end{aligned} \tag{23}$$

and

$$\begin{aligned} w_i(T, r, \phi, \lambda) &= (GMc/r) \sum_{l \geq 0} \sum_{m=0}^{l} (\tfrac{a_e}{r})^l P_{lm}(\sin\phi) \times \\ &\quad [C^i_{lm} \cos m\lambda + S^i_{lm} \sin m\lambda] + O(2) \end{aligned} \tag{24}$$

The introduction of a factor GMc/r in Eq. (24) is to make coefficients C^i_{lm} and S^i_{lm} dimensionless. In Eqs. (23) and (24), λ and ϕ are the longitude and latitude of a field point relating to $\mathbf{x}$ respectively, M is the BD mass of the Earth, a_e is the equatorial radius of the Earth. After taking the 1PN barycenter of the Earth as the origin of spatial coordinates, i.e. $M_a(T) = O(4)$, $a = 1, 2, 3$, all the terms of $l = 1$ in Eqs. (21) and (23), $l = 0$ in Eqs. (22) and (24) will vanish.

Comparing Eq. (21) with Eq. (23), we can determine C_{lm} and S_{lm} in terms of $m_{L'}$, the result is identical with that of Hartmann et al. (1994). e.g., $l = 2$, we have

$$-J_2 \equiv C_{20} = 3m_{zz}/2Ma_e^2,$$
$$C_{21} = m_{xz}/Ma_e^2 \quad C_{22} = (m_{xx} - m_{yy})/4Ma_e^2, \tag{25}$$
$$S_{21} = m_{yz}/Ma_e^2, \quad S_{22} = m_{xy}/2Ma_e^2$$

Now we can find that, under the standard PN gauge Eq. (12), the expansion of the 1PN scalar potential in terms of the Cartesian STF multipole moments or the spherical harmonics is formally similar to that of the Newtonian potential in classical mechanics. As an astronomical constant, the Earth dynamical form-factor J_2 under the standard PN gauge can be completely determined by the BD mass M, the equatorial radius a_e and the projection of mass quadrupole moment m_{zz} (which equals $R^a_z R^b_z M_{ab}$) of the Earth. The 1PN J_2 is generally non-unique but gauge-dependent. This conclusion is the same as that of Brumberg et al. (1996). So the adopted gauge connected with the 1PN J_2 should be explicitly indicated. The first equation in Eq. (25) provides the simplest definition of the 1PN J_2. That results from the choice of the standard PN gauge. Actually, it is necessary to choose the standard PN gauge in the spherical harmonics expansion of the 1PN scalar potential, otherwise the spherical harmonic coefficients C_{lm} and S_{lm} will depend on r because the 1PN scalar potential does not satisfy the Poisson equation. In other words, the 1PN scalar potential is exactly the solution of the Poisson equation under the standard PN gauge.

Eqs. (22) and (24) show that the expansion of the vector potential is much more complicated than that of the scalar potential. For the vector potential, three sets of expansion coefficients (C^i_{lm} and S^i_{lm}, $i = 1, 2, 3$) are needed. Comparing Eq. (22) with

Eq. (24), they are determined in terms of $m'_{iL'}$, $\epsilon_{ij<i_l}s_{L'-1>j}$ and $r^2\delta_{i<i_l}m'_{L'-1>}$. Therefore, C^i_{lm} and S^i_{lm} depend on the radius r. The reason is that the 1PN vector potential does not follow the Poisson equation under the standard PN gauge.

As a useful example, here we give the expressions of the coefficients C^i_{lm} and S^i_{lm} of the lowest order ($l=1$)

$$C^i_{10} = (\tfrac{9}{10}m'_{i3} + \epsilon_{3ij}s_j)/2Mca_e$$

$$C^i_{11} = (\tfrac{9}{10}m'_{i1} + \epsilon_{1ij}s_j)/2Mca_e, \qquad (i=1,2,3) \tag{26}$$

$$S^i_{11} = (\tfrac{9}{10}m'_{i2} + \epsilon_{2ij}s_j)/2Mca_e$$

In many practical problems, it is usually accurate enough to consider only the largest one of the 1PN terms in the gravitational field of the Earth, i.e. the 1PN term generated by the mass of the Earth. In this case the 1PN vector potential is completely negligible. When a higher accuracy is needed, it would be enough to consider the 1PN terms generated by the mass, mass quadrupole moment and spin of the Earth. In this case, the right-hand side of Eq. (24) only contains the terms of $l=1$ and $l=3$. Generally speaking, if we set up the model of the 1PN gravitational field of the Earth, the vector potential should certainly be taken into account. Only a few terms in Eq. (24) need to be retained, the terms with large values of l are actually negligible. In other words, for a practical model of the gravitational field, the expressions of the scalar and vector potentials must be truncated at some value of l, which is determined according to the practical accuracy of observation.

5. Expansion of w_i under the rigidity approximation

Eq. (14) has shown that, under the standard PN gauge the 1PN vector potential W_a can be expanded in terms of $\dot{M}_L$ and S_L. We find that the expansion of W_a is not complicated in form. But $\dot{M}_L$, the time-derivative of M_L, is a set of physical quantities that are algebraically independent of M_L. So it is safer to say that three sets of Cartesian multipole moments M_L, $\dot{M}_L$ and S_L generate together the 1PN potential W_α. Up to now, there is no explicit expressions of $\dot{M}_L$ $(l\geq 2)$, i.e. the evolution equations of M_L $(l\geq 2)$. This makes it difficult to compute W_a. But in fact, we can use some approximate methods or models for computing W_a.

Because the vector potential W_a only generate the 1PN effects, all the physical quantities in the expression of W_a only need to reach the accuracy of Newtonian order. Thus we can adopt the rigidity approximation, i.e. assume that the Earth is a rigid body rotating at an angular velocity $\mathbf{\Omega}$, neglecting the non-rigidity correction to the real Earth. Under this assumption, the vector potential can be expressed in terms of M_L and N_L. It is preferable that, we think, N_L in place of $\dot{M}_L$ would make the expression of W_a independent of time-derivatives of Cartesian multipole moments. Moreover, the multipole moments in W_α are also reduced to two sets (M_L and N_L replace M_L, $\dot{M}_L$ and S_L).

According to the rigidity approximation, we have $\Sigma^a{=}\epsilon_{abc}\Omega^b X^c\Sigma$. Inserting it into Eq. (14), and after a tedious calculation (see Appendix), the approximate expression of the vector potential W_a can be obtained as

$$\begin{aligned} W_a(T,\mathbf{X}) =\ & G\Omega^b \sum_{l\geq 0} \tfrac{(-)^l}{l!}\{\tfrac{8l+11}{4(2l+3)}\epsilon_{abc}M_{cL} \\ &+ \tfrac{l}{2l+1}\epsilon_{ab<a_l}N_{L-1>} \\ &+ \delta_{aa_l}\epsilon_{bca_{l-1}}[\tfrac{l(l-1)}{8(2l-1)}R^2 M_{cL-2} + \tfrac{l-1}{2l+1}N_{cL-2}] \\ &- \tfrac{l}{4(2l+3)}\epsilon_{bc<a_l}M_{L-1>ca}\}\partial_L R^{-1} + O(2) \end{aligned} \tag{27}$$

Inserting Eq. (27) into Eq. (17), we have

$$w_i(T,x^j) = R^a_i W_a(T,X^b)$$

w_i can be expanded in terms of $m_{L'}$ and $n_{L'}{\equiv}R^L_{L'}N_L$, which are the projections of M_L and N_L in the co-rotating system. The expression of w_i is similar to Eq. (27), it reads

$$\begin{aligned} w_i(T,\mathbf{x}) =\ & G\omega^j \sum_{l\geq 0} \tfrac{(-)^l}{l!}\{\tfrac{8l+11}{4(2l+3)}\epsilon_{ijk}m_{kL'} \\ &+ \tfrac{l}{2l+1}\epsilon_{ij<i_l}n_{L'-1>} \\ &+ \delta_{ii_l}\epsilon_{jki_{l-1}}[\tfrac{l(l-1)}{8(2l-1)}r^2 m_{kL'-2} + \tfrac{l-1}{2l+1}n_{kL'-2}] \\ &- \tfrac{l}{4(2l+3)}\epsilon_{jk<i_l}m_{L'-1>ki}\}\partial_{L'} r^{-1} + O(2) \end{aligned} \tag{28}$$

The terms containing $n_{kL'-2}$ in Eq. (28) and N_{cL-2} in Eq. (27) are zero as $l=0$. Comparing Eq. (28) with Eq. (24), we can obtain the relations between C^i_{lm}, S^i_{lm} and $m_{L'}$, $n_{L'}$. For example, as $l=1$ we have

$$C^i_{10} = \omega^j(\tfrac{19}{20}\epsilon_{ijk}m_{k3} - \tfrac{1}{20}\epsilon_{jk3}m_{ki} + \tfrac{1}{3}\epsilon_{ij3}N)/Mca_e$$

$$C^i_{11} = \omega^j(\tfrac{19}{20}\epsilon_{ijk}m_{k1} - \tfrac{1}{20}\epsilon_{jk1}m_{ki} + \tfrac{1}{3}\epsilon_{ij1}N)/Mca_e \tag{29}$$

$$S^i_{11} = \omega^j(\tfrac{19}{20}\epsilon_{ijk}m_{k2} - \tfrac{1}{20}\epsilon_{jk2}m_{ki} + \tfrac{1}{3}\epsilon_{ij2}N)/Mca_e$$

where $i=1,\ 2,\ 3$. Applying the rigidity approximation, i.e.

$$m'_{ij} = (\epsilon_{ikl}m_{lj} + \epsilon_{jkl}m_{il})\omega^k, \quad s_i = (\frac{2}{3}N\delta_{ij} - m_{ij})\omega^j \tag{30}$$

and using the following identity

$$\epsilon_{ijl}m_{lk} + \epsilon_{jkl}m_{li} + \epsilon_{kil}m_{lj} \equiv 0 \tag{31}$$

it is easy to show that Eq. (29) is consistent with Eq. (26).

6. Conclusions

Through the discussions of this paper, we obtain the following conclusions: a) The 1PN gravitational potential can be expanded in terms of the time-slowly-changing multipole moments such as $m_{L'}$, $m'_{L'}$, $s_{L'}$, which can be determined by observation. Its expansion is formally similar to that in terms of BD moments M_L, $\dot{M}_L$ and S_L. b) For the multipole moments expansion of the 1PN potential, we suggest to adopt the standard PN gauge shown by Eq. (12). Under this gauge, the expansion of the 1PN scalar potential has the simplest form, and it is formally identical with the expansion of the Newtonian gravitational potential. c) The scalar potential under the aforesaid standard PN gauge is the simplest generalization of the Newtonian gravitational potential in the 1PN case. Then the conventional relation between the dynamical form-factor J_2 and the mass quadrupole moment of the Earth would be kept in the 1PN case, if the latter be identified as the projection of BD moments of the Earth in its co-rotating system. d) The vector potential under the aforesaid gauge can be also expanded in terms of spherical harmonics. Its expansion coefficients are related with the Cartesian time-slowly-changing multipole moments, $m'_{L'}$, $s_{L'}$ and radius r. Although the expansion of the vector potential is not simple under the standard PN gauge, this does not cause too much trouble at all, since in astronomical practice only few terms in the expansion of the vector potential would be needed. e) Under the standard PN gauge and rigidity approximation, the vector spherical harmonic coefficients, C^i_{lm} and S^i_{lm}, are determined by $m_{L'}$ and $n_{L'}$, the projections of M_L and N_L in the Earth's co-rotating reference system.

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Appendix: Expansion of the 1PN vector potential under the standard PN gauge and rigidity approximation

In this appendix, we shall briefly derive Eq. (27) in Sect. 5. The standard PN gauge means Eq. (12) in Sect. 2. The rigidity approximation is just the following expression

$$\Sigma^a(T,\mathbf{X}) = \epsilon_{abc}\Omega^b X^c \Sigma(T,\mathbf{X}) \qquad (32)$$

Under the gauge (12), the vector potential can be divided into three parts

$$W_a = W_a^{(1)} + W_a^{(2)} + W_a^{(3)} + O(2) \qquad (33)$$

$$W_a^{(1)} = G\int_E d^3X'\Sigma^a(T,\mathbf{X}')/|\mathbf{X}-\mathbf{X}'| \qquad (34)$$

$$W_a^{(2)} = \frac{G}{8}\partial_a\partial_T\int_E d^3X'\Sigma(T,\mathbf{X}')|\mathbf{X}-\mathbf{X}'| \qquad (35)$$

$$W_a^{(3)} = \frac{G}{8}\partial_a\sum_{l\geq 0}\frac{(-)^l}{l!}\left[\frac{8(2l+1)}{(l+1)(2l+3)}P_L - \frac{1}{2l+3}\dot{N}_L\right]\partial_L R^{-1} \qquad (36)$$

here $W_a^{(1)}$ comes from Eq. (6), $W_a^{(2)}$ and $W_a^{(3)}$ are gauge terms. We first calculate $W_a^{(3)}$. According to the formula $\dot{N}_L = 2P_L + \frac{l(2l+3)}{2l+1}Q_L$, where $Q_L \equiv \int_E d^3X\mathbf{X}^2\hat{X}^{<L-1}\Sigma^{a_l>}$, and applying (32) on P_L and Q_L, we have

$$P_L\partial_L R^{-1} = -\frac{l}{2l+1}\epsilon_{abc}\Omega^b N_{cL-1}\partial_{aL-1}R^{-1} + O(2) \qquad (37)$$

$$Q_L\partial_L R^{-1} = \epsilon_{abc}\Omega^b N_{cL-1}\partial_{aL-1}R^{-1} + O(2) \qquad (38)$$

$$\dot{N}_L\partial_L R^{-1} = l\epsilon_{abc}\Omega^b N_{cL-1}\partial_{aL-1}R^{-1} + O(2) \qquad (39)$$

Inserting (38) and (39) into (36), $W_a^{(3)}$ reads

$$W_a^{(3)} = \tfrac{G}{8}\Omega^b\sum_{l\geq 2}\tfrac{(-)^l}{l!}\tfrac{(l-1)(l+8)}{2l+1}\delta_{aa_l}\epsilon_{bca_{l-1}}N_{cL-2}\partial_L R^{-1} + O(2) \qquad (40)$$

Then, we calculate $W_a^{(1)}$. Using Eq. (32) and the formula $X^c\hat{X}^L = \hat{X}^{cL} + \frac{l}{2l+1}R^2\delta^{c<a_l}\hat{X}^{L-1>}$, we immediately have

$$W_a^{(1)} = G\Omega^b \times \sum_{l\geq 0}\tfrac{(-)^l}{l!}[\epsilon_{abc}M_{cL} + \tfrac{l}{2l+1}\epsilon_{ab<a_l}N_{L-1>}]\partial_L R^{-1} + O(2) \qquad (41)$$

At last, we calculate $W_a^{(2)}$. For convenience, a caret is attached to a STF tensor, e.g. $\hat{M}_L$, $\hat{N}_L$ and so on. Now M_L and N_L only represent symmetric tensors. From (35) it is easy to obtain

$$8W_a^{(2)} = G\epsilon_{bcd}\Omega^c\sum_{l\geq 0}\frac{(-)^{l+1}}{l!}M_{dL}\partial_{abL}R + O(2) \qquad (42)$$

Making use of

$$\partial_L R = [-R^2\partial_L R^{-1} + l(l-1)\delta_{(a_l a_{l-1}}\partial_{L-2)}R^{-1}]/(2l-1)$$

we have

$$M_{dL}\partial_L R = [-R^2 M_{d<L>}\partial_L R^{-1} + l(l-1)N_{d<L-2>}\partial_{L-2}R^{-1}]/(2l-1) \qquad (43)$$

Doing with

$$\partial_b R\partial_L R^{-1} = -(R/(2l+1))[\partial_{bL}R^{-1} + l(2l-1)R^{-2}\delta_{b<a_l}\partial_{L-1>}R^{-1}],$$

we obtain

$$M_{dL}\partial_{bL}R = [-R^2 M_{d<L>}\partial_{bL}R^{-1} + 2lM_{d<bL-1>}\partial_{L-1}R^{-1}]/(2l+1) + \tfrac{l(l-1)}{2l-1}N_{d<L-2>}\partial_{bL-2}R^{-1} \qquad (44)$$

and

$$M_{dL}\partial_{abL}R = [-R^2 M_{d<L>}\partial_{abL}R^{-1} + 2(l+1)M_{d<L>}\delta_{a<b}\partial_{L>}R^{-1}]/(2l+3) + \tfrac{2l}{2l+1}M_{d<bL-1>}\partial_{aL-1}R^{-1} + \tfrac{l(l-1)}{2l-1}N_{d<L-2>}\partial_{abL-2}R^{-1} \qquad (45)$$

Then employing the following formulae

$$(l+1)\delta_{a<b}\hat{\partial}_{L>} = \delta_{ab}\hat{\partial}_L + l\delta_{a<a_l}\hat{\partial}_{L-1>b} - \tfrac{2l}{2l+1}\delta_{b<a_l}\hat{\partial}_{L-1>a},$$

$$M_{d<L>} = \hat{M}_{dL} + \tfrac{l}{2l+1}\delta_{d<a_l}\hat{N}_{L-1>},$$

$$N_{d<L-2>} = \hat{N}_{dL-2} + \tfrac{l-2}{2l-3}\delta_{d<a_{l-2}}\hat{N}'_{L-3>},$$

$$\hat{N}'_L \equiv \int_E d^3X\mathbf{X}^4\hat{X}^L\Sigma(T,\mathbf{X}),$$

$$M_{d<aL-1>} = \hat{M}_{daL-1} + \tfrac{1}{2l+1}[\delta_{da}\hat{N}_{L-1} + (l-1)\delta_{d<a_{l-1}}\hat{N}_{L-2>a} - \tfrac{2(l-1)}{2l-1}\delta_{a<a_{l-1}}\hat{N}_{L-2>d}] \qquad (46)$$

After some calculation, we discover

$$\epsilon_{bcd} M_{dL} \partial_{abL} R = \epsilon_{bcd} [(-R^2 \hat{M}_{dL} \partial_{abL} R^{-1}$$
$$+2\hat{M}_{dL} \delta_{ab} \partial_L R^{-1})/(2l+3) \tag{47}$$
$$+\frac{2l}{2l+3} \hat{M}_{daL-1} \partial_{bL-1} R^{-1} + \frac{l(l-1)}{2l+1} \hat{N}_{dL-2} \partial_{abL-2} R^{-1}]$$

Inserting (47) into (42), and now letting M_L and N_L in place of $\hat{M}_L$ and $\hat{N}_L$ to represent the STF multipole moments, then $W_a^{(2)}$ reads as follows

$$W_a^{(2)} = (G/8)\Omega^b \sum_{l \geq 0} \frac{(-)^l}{l!} \{ -\frac{2}{2l+3} [\epsilon_{abc} M_{cL}$$
$$+l \epsilon_{bc<a_l} M_{L-1>ca}] + \frac{l(l-1)}{2l-1} \delta_{aa_l} \epsilon_{bca_{l-1}} \times \tag{48}$$
$$[R^2 M_{cL-2} - \frac{2l-1}{2l+1} N_{cL-2}]\} \partial_L R^{-1} + O(2)$$

Finally, inserting (40), (41) and (48) into (33), we obtain the expression of the 1PN vector potential, which is expanded in terms of Cartesian multipole moments M_L and N_L, under the standard PN gauge and the assumption (32), which is exactly Eq. (27) in Sect. 5.

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