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Theoretical study of the polarization by resonance scattering of the Hα line in spicules

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Abstract. One currently observes spicules in the Hα line, the underlying photospheric radiation being scattered by the matter of the spicules. In this paper we solve the equations of statistical equilibrium for a hydrogen atom having an ensemble velocity and being illuminated by the photospheric radiation field. This leads us to the determination of the Stokes parameters I, Q, U of the radiation diffused by the spicule towards the observer. The calculations include fine structure effects, limb darkening and depolarization by isotropic collisions.

One obtains the linear polarization degree as well as the rotation of the polarization plane as a function of the ascending velocity of matter in spicules.

Key words: line: formation – polarization – Sun: photosphere – Sun: chromosphere

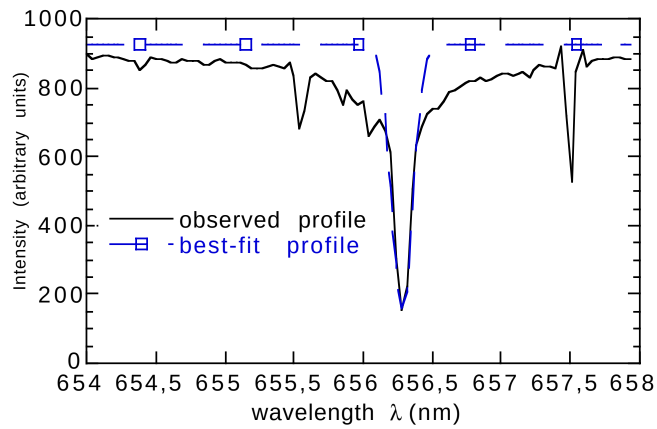

Fig. 1. Plot of the observed solar spectrum between 6540 Å and 6580 Å (full line). The continuum around Hα is arbitrarily set to 1000. Dotted line: fit to the observed spectrum by means of a gaussian profile (see text).

1. Introduction

One can describe spicules as cylindrical structures appearing at the boundaries of the supergranules of the solar atmosphere. Their diameters are about 700 km, and the spicules mount in the corona conducted by the magnetic field. One can observe matter up to a height of $15,000$ km with a mean inclination of $30°$ relative to the normal to the solar surface. This paper deals with the theory of the density matrix of photons resonantly scattered by atoms having an ensemble velocity. The theory of the multilevel atom scattering the incident photospheric radiation is outlined by a quantum formalism (Bommier 1977; Bommier & Sahal-Bréchot 1978). We apply the theory to neutral H atoms in spicules. We illustrate our method by calculating the Stokes parameters I, Q, U of Hα, one of the lines mostly used to observe spicules. We obtain a non negligible degree of polarization and rotation of the polarization direction due only to the effect of the velocity field, that is, neglecting the effect of any magnetic field on polarization.

The method of calculation described hereafter is derived from the theoretical method that has been developed for a two-level atom by Sahal-Bréchot et al. (1998), as outlined in Sahal-Bréchot et al. (1992) and in Sahal-Bréchot & Choucq-Bruston (1994). The present paper is concerned by the adaptation of the method to the particular multi-level case of the Hα line of hydrogen.

In Sects. 2 and 2.1 we define the photospheric profile of the Hα line used for our theoretical model, and also the density matrices of the incident and scattered radiation allowing for correlations due to the Doppler effect. In Sect. 2.1 and Appendix A:, we give the equations of statistical equilibrium in tensorial notation. We solve these equations and find an analytical approximation for the tensors (Sects. 3, 3.1 and 3.2). This enables us to calculate the density matrix of the scattered photons opening the way to the expression of the Stokes parameters I, Q, U of the line (Sect. 3.3). Sect. 3.4 is devoted to the integration over the velocity distribution accounting for the ensemble velocity of the matter in spicules. From the Stokes parameters one obtains the polarization degree and the rotation of the polarization plane. We summarize our results in Sects. 4, 4.1 and 4.2. Finally we consider the effect of collisions with protons and electrons on the atomic radiator (Sect. 5).

2. The Hα absorption profile

The starting point of our theory is to model the photospheric incident radiation around the Hα line by a continuum from which we substract a gaussian profile to fit the atlas data by Delbouille et al. (1973).

Such a profile is written as (see Fig. 1)

$$I(\nu) = I_0 \left(1 - \frac{I_i}{I_0} \exp\left[-\left(\frac{\Delta\nu}{\Delta\nu_{\mathrm{D}_i}}\right)^2\right]\right) \ , \quad (1)$$

where the constants are fixed by the data available in the atlas quoted (see Fig. 1).

The result of the fit brings to

$$\frac{I_i}{I_0} = 0.829$$

and to

$$\Delta\nu_{\mathrm{D}_i} = 5.90 \times 10^{10}\ \mathrm{s}^{-1} \ ,$$

corresponding to a line width at half maximum of 1.41 Å.

2.1. The density matrices of the problem

Taking into account the particular form of the profile, we use Eq. (1) of Sahal-Bréchot et al. (1992) to obtain the density matrix of the incident photons.

We recall that an atom having an individual velocity $\boldsymbol{v}$ absorbs, in its atomic frame, at the frequency ν such that

$$\nu = \nu_0 \left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{n}}{c}\right) \ , \quad (2)$$

where $\boldsymbol{n}$ is the unit vector of the ray of light whose direction is specified by the angles (α, β) (see Fig. 2.1), and ν_0 is the atomic frequency of the unperturbed transition corresponding to an upper level u and a lower level l which are assumed to be infinitly sharp (coherent redistribution in the atomic frame).

The density matrix now reads

$$\begin{aligned}\phi_Q^K(\boldsymbol{v}) &= \frac{1}{4\pi}\int_0^{\alpha_{\max}} \mathrm{d}\alpha\ f(\alpha)\sin\alpha \int_0^{2\pi} \mathrm{d}\beta \\ &\times \mathcal{M}_Q^K(\alpha,\beta)\left(1 - \frac{I_i}{I_0}\exp\left[-\left(\frac{\nu_0}{c}\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{\Delta\nu_{\mathrm{D}_i}}\right)^2\right]\right) \ , \end{aligned} \quad (3)$$

where $f(\alpha)$ is the limb darkening coefficient, and where the index K runs from 0 to 2. $\mathcal{M}_Q^K(\alpha,\beta)$ is a unitary matrix characterizing the angular behaviour of an unpolarized radiation beam propagating in the direction (α, β) (see Eq. (43) of Sahal-Bréchot et al. 1986).

Let R be the distance of the atom to the center of the Sun, and $R_\odot$ the solar radius; then the angle $\alpha_{\max}$ is defined by the equation (see Fig. 2.1)

$$\alpha_{\max} = \arcsin \frac{R_\odot}{R} \ . \quad (4)$$

The scalar product $\boldsymbol{v}\cdot\boldsymbol{n}$, depending on (α, β), leads to the appearance of coherences $(K = 2, Q = \pm1, \pm2)$ in $\phi_Q^K(\boldsymbol{v})$.

As we shall see later on, the Stokes parameters I, Q, U of the Hα line are obtained once the density matrix $\Phi_Q^K(\boldsymbol{v})$ of the reemitted photons is specified in the observer's frame $AXYZ$ of Fig. 2.1.

In order to obtain the frequency dependence of the total scattered radiation we take into account the contributions of all atoms contained in a unit of volume of radiating matter. Due to the Doppler effect, each frequency ν' of the scattered radiation corresponds to an atom having a velocity v_Z such that

$$\nu' = \nu_0\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{n}'}{c}\right) = \nu_0\left(1 + \frac{v_Z}{c}\right) \quad (5)$$

We need to perform the average of the density matrix $\Phi_Q^K(\boldsymbol{v})$ over the atomic distribution of velocities projected onto a plane perpendicular to the line of sight, *i.e.*, over v_Y and v_X , and to multiply by the density $\mathcal{N}$ of the H atoms (Sahal-Bréchot et al. 1998). The distribution of the velocities is (Sahal-Bréchot et al. 1992, Sahal-Bréchot et al. 1998)

$$\begin{aligned} F(\boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v} \ &= \ \left(\frac{m_{\mathrm{H}}}{2\pi kT}\right)^{3/2} \exp\left[\frac{m_{\mathrm{H}}}{2kT}\left(v_X - V_X\right)^2\right] \\ &\times \exp\left[\frac{m_{\mathrm{H}}}{2kT}\left(v_Y - V_Y\right)^2\right] \\ &\times \exp\left[\frac{m_{\mathrm{H}}}{2kT}\left(v_Z - V_Z\right)^2\right]\ \mathrm{d}v_X\mathrm{d}v_Y\mathrm{d}v_Z \ , \end{aligned} \quad (6)$$

where $\boldsymbol{V}\ (V_X, V_Y, V_Z)$ or $\boldsymbol{V}\ (V, \alpha_V, \beta_V)$ is the velocity vector of the stream of radiating H atoms, and where m_{H} and k are, respectively, the atomic mass of Hydrogen and the Boltzmann constant.

We write the velocity distribution $F(v_Z)\mathrm{d}v_Z$ in units of frequency using the following transformations

$$\begin{aligned} v_Z &= \frac{c}{\nu_0}(\nu' - \nu_0) \ , \\ \mathrm{d}v_Z &= \frac{c}{\nu_0}\ \mathrm{d}\nu' \ , \\ \Delta\nu_{\mathrm{D}_d} &= \frac{\nu_0}{c}\sqrt{\frac{2kT_{\mathrm{D}_d}}{m_{\mathrm{H}}}} \ ; \end{aligned} \quad (7)$$

also

$$\Delta\nu_Z = \frac{\nu_0}{c}V_Z \ ,$$

and thus

$$\begin{aligned} F(v_Z)\ \mathrm{d}v_Z \ &= \ \mathrm{d}\nu'\ \frac{1}{\sqrt{\pi}}\frac{1}{\Delta\nu_{\mathrm{D}_d}} \\ &\times \exp\left[-\left(\frac{\nu' - \nu_0 - \Delta\nu_Z}{\Delta\nu_{\mathrm{D}_d}}\right)^2\right] \ . \end{aligned}$$

A temperature $T_{\mathrm{D}_d} \approx 10^4$ K (corresponding to the value $\Delta\nu_{\mathrm{D}_d} = 1.97 \times 10^{10}\ \mathrm{s}^{-1}$) is a reasonable assumption for the atomic radiators in the spicule.

We define, as usual, the polarization degree τ and the angle of rotation γ of the polarization direction by the formulae

$$\begin{aligned} \tau &= \sqrt{\frac{Q^2 + U^2}{I^2}} \ , \\ \tan 2\gamma &= \frac{U}{Q} \ , \end{aligned} \quad (8)$$

where the positive Q direction is defined as the tangent to the solar limb.

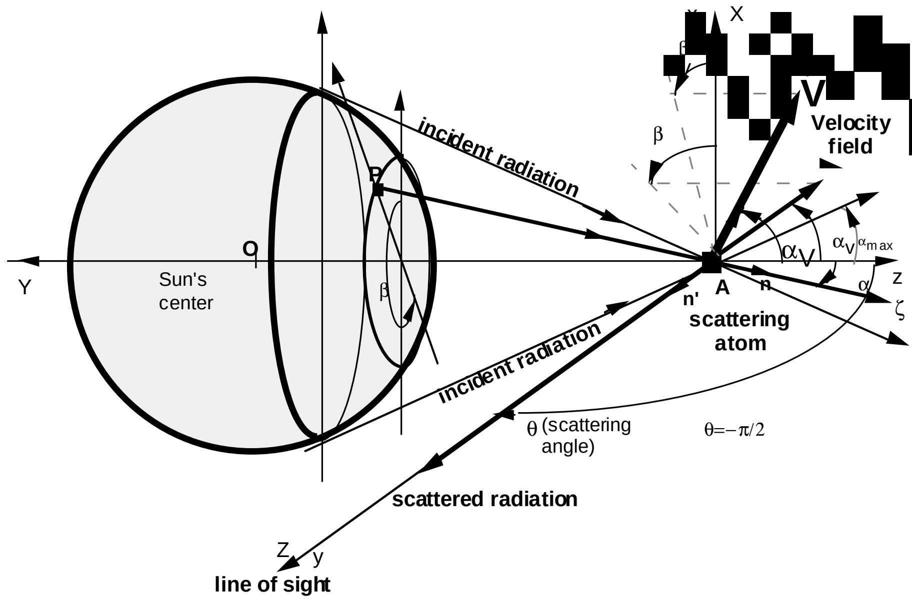

Fig. 2. Coordinates of the scattering atom A located at the height h; Oz is the preferred direction of incident radiation; AZ is the line of sight; Ax and AX are identical and perpendicular to the scattering plane Z[redacted] $\theta = (Az, AZ)$ is the scattering angle [redacted] to $-90°$ if (case of figure) A is in the[redacted]f the sky, then $AY = -Az$ and A[redacted]$= Ay$. The incident light comes from th[redacted]spherical cap limited by the angle $\alpha_{\max}$; [redacted]ch point P of the cap emits a ray $PA\zeta$ whose coordinates are the angles (α, β) in the frame $Axyz$. The atom A has the velocity $\boldsymbol{v} = (v, \alpha_v, \beta_v)$ and the mean velocity of the ensemble is $\boldsymbol{V} = (V, \alpha_V, \beta_V)$; $\boldsymbol{n}$ and $\boldsymbol{n}'$ are the unitary vectors of the directions $A\zeta$ and AZ. The angle χ is the angle between the normal to the surface of the Sun and the incident light direction at point P'. The direction of the Stokes parameter $Q > 0$ and $U = 0$ is that of a totally polarized radiation along Ax or AX. The direction of polarization is given by the angle γ with AX or Ax. The angle γ lies in the plane of the sky (xAz).

2.2. *The equations of statistical equilibrium*

We use the theory of atom-radiation interaction in the density matrix formalism given by Bommier (1977), and in particular Eq. (III-39) of that paper (see also Eq. (36) of Bommier & Sahal-Bréchot 1978).

Neglecting coherences between level αJ and $\alpha J'$ (with $J \neq J'$) and neglecting the possible presence of a magnetic field, the equations assume the form given in Appendix A:, where $A(\alpha_1 J_1 \to \alpha_2 J_2)$ is the Einstein coefficient for spontaneous de-excitation between the levels $(\alpha_1 J_1 \to \alpha_2 J_2)$ and where ϕ_Q^K is given by Eq. (3).

We set

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\alpha_1 J_1}\rho_{q_1}^{k_1} = 0$$

looking for solutions of a stationary system. In these equations we take into account that the radiation field in the Lyα and Lyβ is isotropic, because these lines are opticaly thick in the solar chromosphere, so that

$$\phi^0_{0\mathrm{Ly}\alpha} = \phi^0_{0\mathrm{Ly}\beta} = \frac{1}{2\sqrt{3}}\,, \qquad \phi^2_{0\mathrm{Ly}\alpha} = \phi^2_{0\mathrm{Ly}\beta} = 0\,. \tag{9}$$

Our further assumption is to neglect any velocity effect on the profile of the Lyα and Lyβ lines for the radiating atom. If, for instance, we consider a velocity of 60 km/s for the atom, the shift in wavelength is of the order of 0.2 Å for both lines. At this distance from line center the profiles are nearly constant (Gouttebroze et al. 1978).

We also neglect the central dip of the profiles that would affect the radiator for smaller velocities.

Table 1. Values of the Einstein A coefficients.

$\alpha_1 J_1 \to \alpha_2 J_2$	$A(\alpha_1 J_1 \to \alpha_2 J_2)$ in units of $10^8\mathrm{s}^{-1}$
$3d_{5/2} \to 2p_{3/2}$	0.646
$3d_{3/2} \to 2p_{3/2}$	0.107
$3d_{3/2} \to 2p_{1/2}$	0.539
$3p_{3/2} \to 2s_{1/2}$	0.224
$3p_{1/2} \to 2s_{1/2}$	0.224
$3s_{1/2} \to 2p_{1/2}$	0.021
$3s_{1/2} \to 2p_{3/2}$	0.042
$3p_{1/2} \to 2s_{1/2}$	1.67
$3p_{3/2} \to 2s_{1/2}$	1.67
$2p_{3/2} \to 1s_{1/2}$	3.11
$2p_{1/2} \to 1s_{1/2}$	3.11

Table 2. Values for the f quantities.

Transition	λ Å	I_ν erg/cm^2/s/sr/Hz	f
Hα	6562.80	4.049×10^{-5}	0.288×10^{-1}
Lyα	1215.67	0.179×10^{-6}	0.179×10^{-6}
Lyβ	1025.72	0.119×10^{-8}	0.118×10^{-8}

3. The general solution

The system described by Eq. (A1) is a linear, homogeneous system of equations. A normalization condition on the populations

$$\sum_{\alpha_1 J_1} \sqrt{2J_1+1}\;{}^{\alpha_1 J_1}\rho_0^0 = \mathrm{Tr}\,\rho = 1 \tag{10}$$

Table 3. Analytical expressions and numerical values of the populations of the levels involved in all transitions.

Population	Analytical approximation	Analytical result	Exact numerical result
${}^{3d_{5/2}}\rho_0^0$	$\frac{3\sqrt{2}}{4}\, f_{\mathrm{Ly}\alpha} f_{\mathrm{H}\alpha}\, \phi_0^0$	0.138×10^{-8}	0.139×10^{-8}
${}^{3d_{3/2}}\rho_0^0$	$\frac{\sqrt{3}}{2}\, f_{\mathrm{Ly}\alpha} f_{\mathrm{H}\alpha}\, \phi_0^0$	0.113×10^{-8}	0.113×10^{-8}
${}^{3p_{3/2}}\rho_0^0$	$\frac{1}{2}\, f_{\mathrm{Ly}\beta}$	0.595×10^{-9}	0.596×10^{-9}
${}^{3p_{1/2}}\rho_0^0$	$\frac{\sqrt{2}}{4}\, f_{\mathrm{Ly}\beta}$	0.420×10^{-9}	0.421×10^{-9}
${}^{2p_{3/2}}\rho_0^0$	$\frac{1}{2}\, f_{\mathrm{Ly}\alpha}$	0.895×10^{-7}	0.896×10^{-7}
${}^{2p_{1/2}}\rho_0^0$	$\frac{\sqrt{2}}{4}\, f_{\mathrm{Ly}\alpha}$	0.632×10^{-7}	0.634×10^{-7}
${}^{3s_{1/2}}\rho_0^0$	$\frac{\sqrt{6}}{4}\, f_{\mathrm{Ly}\alpha} f_{\mathrm{H}\alpha}\, \phi_0^0$	0.801×10^{-9}	0.811×10^{-9}
${}^{2s_{1/2}}\rho_0^0$	$\frac{\sqrt{2}}{4\sqrt{3}}\, \frac{f_{\mathrm{Ly}\beta}}{f_{\mathrm{H}\alpha}}\, \frac{1}{\phi_0^0}$	0.332×10^{-7}	0.336×10^{-7}
${}^{1s_{1/2}}\rho_0^0$	$\frac{1}{\sqrt{2}}$	0.707	0.707

Table 4. Analytical expressions and numerical values of the higher tensorial components.

Tensorial component	Analytical approximation	Analytical result	Exact numerical result
${}^{3d_{5/2}}\rho_0^2$	$\frac{3\sqrt{7}}{10\sqrt{2}}\, f_{\mathrm{Ly}\alpha} f_{\mathrm{H}\alpha}\, \phi_0^2$	0.347×10^{-10}	0.348×10^{-10}
${}^{3d_{3/2}}\rho_0^2$	$\frac{7\sqrt{6}}{40}\, f_{\mathrm{Ly}\alpha} f_{\mathrm{H}\alpha}\, \phi_0^2$	0.265×10^{-10}	0.265×10^{-10}
${}^{3p_{3/2}}\rho_0^2$	$\alpha_1 \frac{\sqrt{2}}{4}\, f_{\mathrm{Ly}\beta}\, \frac{\phi_0^2}{\phi_0^0}$	0.236×10^{-11}	0.237×10^{-11}
${}^{2p_{3/2}}\rho_0^2$	0	0	0.343×10^{-11}

determines the solution.

We now specify the model atom used for our calculations. This is composed by the levels having $n = 1, 2$, and 3. The allowed transitions and Einstein coefficients for this model atom are given in Table 1.

As the tensor ϕ_Q^K contains only the multipolar components $K = 0$ and $K = 2$, only ρ_Q^K's with K even have to be considered. We have to solve a linear system of 38 equations plus an extra equation (the normalization condition).

Solving numerically the system for the populations, one finds that the population of the ground level ${}^{1s_{1/2}}\rho_0^0$ is nearly equal to $\sqrt{2}/2$. This is the value obtained in Eq. (10) setting all populations to zero except for the population of the ground level.

Obviously, there is a numerical solution of the system for each value of the atomic velocity, a parameter on which an average has to be performed. To circumvent this problem we prefer to obtain an approximate analytical solution for the ρ_Q^K.

3.1. The analytical solution for ρ_Q^K

We introduce the following notations

$$f_{\mathrm{H}\alpha} = \frac{c^3 u_{\nu_{32}}}{8\pi h \nu_{32}^3} \;,\qquad f_{\mathrm{Ly}\alpha} = \frac{c^3 u_{\nu_{21}}}{8\pi h \nu_{21}^3} \;,\qquad f_{\mathrm{Ly}\beta} = \frac{c^3 u_{\nu_{31}}}{8\pi h \nu_{31}^3} \;, \tag{11}$$

where $\nu_{32}, \nu_{21}, \nu_{31}$ stand for the frequencies of Hα , Lyα, Lyβ, respectively. The density of radiation is

$$u_\nu = \frac{4\pi I_\nu}{c} \;.$$

Using the data available for the Sun, Delbouille et al. (1973) and Allen (1973) for the Hα line and the results of Gouttebroze et al. (1978) for the Lyα and Lyβ lines, we summarize the results in Table 2. Note that $f_{\mathrm{H}\alpha}$, $f_{\mathrm{Ly}\alpha}$, $f_{\mathrm{Ly}\beta}$ are dimensionless quantities proportional to the number photons at frequency ν. The values of the intensites I_ν of the Lyα and Lyβ lines are taken at the center of the lines and at disk center, while the intensity I_ν of the Hα line is taken in the continuum surrounding the line at disk center.

We will not write down explicitly the Kramer system $\mathcal{A}X = b$ where $\mathcal{A}$ is the $(39, 39)$ matrix of coefficients. But when all the coefficients are calculated (which implies a heavy use of Racah algebra), it is readily seen that one can neglect some couplings between the elements and thus reduce the whole system to a set of subsystems of maximum dimension 3. Another simplification is made by writing

$$^{1s_{1/2}}\rho_0^0 = \frac{\sqrt{2}}{2}$$

for the population of the ground level.

For the populations we get the results contained in Table 3, where the last column is obtained by solving numerically the Kramer system, for a zero velocity.

The results refer to the value

$$\phi_0^0 = \frac{1}{2\sqrt{3}}\left(1-\cos\alpha_{\max}\right) = 0.254\ ,$$

valid for an atom at a height $h = 5\times 10^3$ km. The results for the higher tensorial components are given in Table 4.

The agreement between the analytical approximation and the exact numerical results is very good.

The quantity α_1 appearing in Table 4 is defined by

$$\alpha_1 = \frac{A\left(3p_{3/2}\to 2s_{1/2}\right)}{A\left(3p_{3/2}\to 2s_{1/2}\right)+A\left(3p_{3/2}\to 1s_{1/2}\right)}\ .$$

Moreover,

$$\begin{aligned}\phi_0^2 &= \frac{1}{4\sqrt{6}}\cos\alpha_{\max}\sin^2\alpha_{\max}\\ &= 1.20\times 10^{-2}\ .\end{aligned}$$

We can observe that all the tensorial components of the d states $^{3d_{3/2}}\rho_Q^K$ and $^{3d_{5/2}}\rho_Q^K$ are proportional to the product $f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}$. This is indeed a process of two step excitation, $1s\to 2p\to 3d$ from the ground level.

3.2. The density matrix of the reemitted photons

The density matrix of the reemitted photons in Hα is given by the expression (Bommier 1977, Eq. (III-45); see also Bommier & Sahal-Bréchot 1978, Eq. (38))

$$\begin{aligned}\Phi_Q^K &= \sum_{\substack{\alpha_i J_i\\ \alpha_f J_f}}\left(2J_i+1\right)A\left(\alpha_i J_i\to\alpha_f J_f\right)\\ &\quad\times(-1)^{J_i+J_f+K+1}\begin{Bmatrix}K & J_i & J_i\\ J_f & 1 & 1\end{Bmatrix}\ ^{\alpha_i J_i}\rho_Q^K\end{aligned}\qquad(12)$$

where the sum has to be performed over all initial states with $n=3$ and all final states with $n=2$.

Using the useful relations between Einstein A coefficients

$$\begin{aligned}&A\left(3d_{5/2}\to 2p_{3/2}\right) = 6A\left(3d_{3/2}\to 2p_{3/2}\right)\ ,\\ &A\left(3d_{3/2}\to 2p_{1/2}\right) = 5A\left(3d_{3/2}\to 2p_{3/2}\right)\ ,\\ &A\left(3p_{3/2}\to 2s_{1/2}\right) = A\left(3p_{1/2}\to 2s_{1/2}\right)\ ,\\ &A\left(3s_{1/2}\to 2p_{3/2}\right) = 2A\left(3s_{1/2}\to 2p_{1/2}\right)\ ,\end{aligned}\qquad(13)$$

we find after some calculations

$$\begin{aligned}\Phi_0^0 &= 2A\left(3d_{3/2}\to 2p_{3/2}\right)\left[3\sqrt{2}\ ^{3d_{5/2}}\rho_0^0 + 2\sqrt{3}\ ^{3d_{3/2}}\rho_0^0\right]\\ &+\frac{\sqrt{3}}{3}A\left(3p_{1/2}\to 2s_{1/2}\right)\left[2\ ^{3p_{3/2}}\rho_0^0+\sqrt{2}\ ^{3p_{1/2}}\rho_0^0\right]\\ &+\sqrt{6}A\left(3s_{1/2}\to 2p_{1/2}\right)\ ^{3s_{1/2}}\rho_0^0\ ,\end{aligned}$$

$$\begin{aligned}\Phi_0^2 &= 6A\left(3d_{5/2}\to 2p_{3/2}\right)\left[\frac{1}{15}\sqrt{\frac{7}{2}}\ ^{3d_{5/2}}\rho_0^2\right]\\ &-4A\left(3d_{3/2}\to 2p_{3/2}\right)\left[\frac{1}{5}\sqrt{\frac{2}{3}}\ ^{3d_{3/2}}\rho_0^2\right]\\ &+4A\left(3d_{3/2}\to 2p_{1/2}\right)\left[\frac{1}{2\sqrt{6}}\ ^{3d_{3/2}}\rho_0^2\right]\\ &+4A\left(3p_{3/2}\to 2s_{1/2}\right)\left[\frac{1}{2\sqrt{6}}\ ^{3p_{3/2}}\rho_0^2\right]\ ,\end{aligned}\qquad(14)$$

and, replacing the ρ_Q^K with their analytical approximation,

$$\begin{aligned}\Phi_0^0 &= 3f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}\phi_0^0(\boldsymbol{v})\\ &\quad\times\left[5A\left(3d_{3/2}\to 2p_{3/2}\right)+\frac{1}{2}A\left(3s_{1/2}\to 2p_{1/2}\right)\right]\\ &+\frac{\sqrt{3}}{2}f_{\mathrm{Ly}\beta}A\left(3p_{1/2}\to 2s_{1/2}\right)\ ,\\ \Phi_Q^2 &= \frac{399}{100}A\left(3d_{3/2}\to 2p_{3/2}\right)f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}\phi_Q^2(\boldsymbol{v})\\ &+\alpha_1 f_{\mathrm{Ly}\beta}\frac{\sqrt{3}}{6}\frac{\phi_Q^2(\boldsymbol{v})}{\phi_0^0(\boldsymbol{v})}A\left(3p_{1/2}\to 2s_{1/2}\right)\ ,\end{aligned}\qquad(15)$$

where the $\phi_Q^K(\boldsymbol{v})$ is defined in Eq. (3).

3.3. The Stokes parameters of the Hα line

We now have to transform the matrix Φ_Q^K to the frame $AXYZ$ of the observer (see Fig. 2.1), which is obtained after a rotation of the frame $Axyz$ by the Euler angles (Sahal-Bréchot et al. 1998)

$$\left(-\frac{\pi}{2},\theta,\frac{\pi}{2}\right)\ .$$

The general transformation of the tensors, under a rotation characterized by the Euler angles (α,β,γ), is

$$\Phi'^K_Q = \sum_{Q'} e^{\mathrm{i}\gamma Q}\ r^K_{QQ'}(-\beta)\ e^{\mathrm{i}\alpha Q'}\ \Phi^K_{Q'}\ ,\qquad(16)$$

where $r^K_{QQ'}$ are the rotation operators as defined in Messiah (1959). For our particular rotation, we have

$$\Phi'^K_Q = \sum_{Q'} e^{\mathrm{i}\frac{\pi}{2}Q}\ r^K_{QQ'}\left(-\theta\right)\ e^{-\mathrm{i}\frac{\pi}{2}Q'}\ \Phi^K_{Q'}\ . \quad (17)$$

For the matrix Φ'^K_Q expressed in the observer's frame we thus obtain (Sahal-Bréchot et al. 1998)

$$\Phi'^0_0 = \Phi^0_0\ ,$$

$$\begin{aligned}\Phi'^2_0 &= \frac{1}{2}\left(3\cos^2\theta - 1\right)\Phi^2_0 \\ &-\sqrt{\frac{3}{2}}\sin 2\theta\ \ \mathrm{Im}\,\Phi^2_1 - \sqrt{\frac{3}{2}}\sin^2\theta\ \ \mathrm{Re}\,\Phi^2_2\ ,\end{aligned} \quad (18)$$

$$\begin{aligned}\Phi'^2_2 &= -\sqrt{\frac{3}{8}}\sin^2\theta\ \Phi^2_0 \\ &-\frac{1}{2}\sin 2\theta\ \ \mathrm{Im}\,\Phi^2_1 + \mathrm{i}\sin\theta\ \ \mathrm{Re}\,\Phi^2_1 \\ &+\frac{1}{2}\left(1+\cos^2\theta\right)\ \mathrm{Re}\,\Phi^2_2 + \mathrm{i}\cos\theta\ \ \mathrm{Im}\,\Phi^2_2\ .\end{aligned}$$

The Stokes parameters of the Hα line are then given by (Sahal-Bréchot et al. 1998)

$$\begin{aligned}I(\nu)\mathrm{d}\nu &= F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\ \mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{4\pi}\left(\frac{1}{\sqrt{3}}\Phi'^0_0 + \frac{1}{\sqrt{6}}\Phi'^2_0\right)\ , \\ Q(\nu)\mathrm{d}\nu &= -F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{8\pi}2\,\mathrm{Re}\,\Phi'^2_2\ , \\ U(\nu)\mathrm{d}\nu &= F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\ \mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{8\pi}2\,\mathrm{Im}\,\Phi'^2_2\ .\end{aligned} \quad (19)$$

The distribution of velocities used here is the shifted Maxwellian of Eq. (6), where $\boldsymbol{V}\ (V_X, V_Y, V_Z)$ or $\boldsymbol{V}\ (V, \alpha_V, \beta_V)$ is the velocity of the stream of particles in the spicule.

We simplify the final expression by setting $\theta = -\pi/2$, that is by supposing the atom is in the plane of the sky. We thus obtain

$$\begin{aligned}I(\nu)\mathrm{d}\nu &= F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\ \mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{4\pi}\left(\frac{1}{\sqrt{3}}\Phi^0_0 - \frac{1}{2\sqrt{6}}\Phi^2_0 - \frac{1}{2}\,\mathrm{Re}\,\Phi^2_2\right)\ , \\ Q(\nu)\mathrm{d}\nu &= -F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{4\pi}\left(-\sqrt{\frac{3}{8}}\Phi^2_0 + \frac{1}{2}\,\mathrm{Re}\,\Phi^2_2\right)\ , \\ U(\nu)\mathrm{d}\nu &= -F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\ \mathrm{d}v_X\mathrm{d}v_Y \\ &\times\frac{3}{4\pi}\left(\mathrm{Re}\,\Phi^2_1\right)\ .\end{aligned} \quad (20)$$

We now substitute Eqs. (15), and, in order to simplify the resulting formulae, we intoduce some notations. Putting

$$P(\nu) = \frac{1}{4\pi}\left\{1 - \frac{I_i}{I_0}\exp\left[-\left(\frac{\nu_0}{\Delta\nu_{\mathrm{D}_i}}\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{c}\right)^2\right]\right\}\ , \quad (21)$$

the density matrix elements of the Hα incoming radiation can be written as (Sahal-Bréchot & Choucq-Bruston 1994, Sahal-Bréchot et al. 1998)

$$\begin{aligned}\phi^0_0 &= \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta\ P(\nu)\frac{1}{\sqrt{3}}\ , \\ \phi^2_0 &= \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta \\ &\quad\times P(\nu)\frac{1}{2\sqrt{6}}\left(3\cos^2\alpha - 1\right)\ , \\ \mathrm{Re}\,\phi^2_1 &= -\int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta \\ &\quad\times P(\nu)\frac{1}{4}\sin 2\alpha\cos\beta\ , \\ \mathrm{Re}\,\phi^2_2 &= \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta \\ &\quad\times P(\nu)\frac{1}{4}\sin^2\alpha\cos 2\beta\ , \\ \mathrm{Im}\,\phi^2_1 &= \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta \\ &\quad\times P(\nu)\frac{1}{4}\sin 2\alpha\sin\beta\ , \\ \mathrm{Im}\,\phi^2_2 &= -\int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\ \mathrm{d}\alpha \int_0^{2\pi}\mathrm{d}\beta \\ &\quad\times P(\nu)\frac{1}{4}\sin^2\alpha\sin 2\beta\ .\end{aligned} \quad (22)$$

Introducing the quantities (depending on frequencies and direction)

$$\begin{aligned}I_{Z_1} &= f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}3\left(5A_{3d} + \frac{1}{2}A_{3s}\right)P\left(\nu\right)\ , \\ I_{Z_2} &= \frac{3.99}{8}A_{3d}f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha} \\ &\times\left[\left(3\cos^2\alpha - 1\right) + 3\sin^2\alpha\cos 2\beta\right]P\left(\nu\right)\ , \\ Q_{Z_3} &= \frac{11.97}{8}A_{3d}f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha} \\ &\times\left[\left(3\cos^2\alpha - 1\right) - \sin^2\alpha\cos 2\beta\right]P\left(\nu\right)\ ,\end{aligned} \quad (23)$$

$$U_{Z_4} = \frac{11.97}{4} A_{3d} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \sin 2\alpha \cos\beta \; P(\nu) \; ,$$

one obtains

$$\begin{aligned} I(\nu)\,\mathrm{d}\nu &= \frac{1}{4\pi} F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \\ &\times \left\{ \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta \left(I_{Z_1} + f_{\mathrm{Ly}\beta}\frac{3}{2}A_{3p} \right) \right. \\ &\left. - \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta\, I_{Z_2} \right\} \; , \\ Q(\nu)\,\mathrm{d}\nu &= \frac{1}{4\pi} F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \\ &\times \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta\, Q_{Z_3} \; , \\ U(\nu)\,\mathrm{d}\nu &= \frac{1}{4\pi} F(v_Z)\mathrm{d}v_Z \iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \\ &\times \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta\, U_{Z_4} \; . \end{aligned} \tag{24}$$

Here A_{3d}, A_{3s}, A_{3p} stand, respectively, for $A\left(3d_{3/2} \to 2p_{3/2}\right)$, $A\left(3s_{1/2} \to 2p_{1/2}\right)$, $A\left(3p_{1/2} \to 2s_{1/2}\right)$. Note that we neglected the terms proportional to

$$\alpha_1 \frac{\phi^2_{0,\pm1,\pm2}(\boldsymbol{v})}{\phi^0_0(\boldsymbol{v})} \; .$$

These terms are indeed negligible due to the fact that

$$\alpha_1 = 0.119 \; ,$$

and that

$$\phi^2_Q \ll \phi^0_0 \; .$$

3.4. Integration over the velocity distribution

From Eq. (21) we see that we can split the calculation of the integral into two parts: part I coming from the first term (the continuum), and part II coming from the the second term (containing the profile of the Hα line). Part I is easily evaluated since the velocity distribution is normalized, so that

$$\iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y = 1 \; . \tag{25}$$

We obtain

$$\begin{aligned} I_C(\nu)\,\mathrm{d}\nu &= \frac{1}{8\pi} F(v_Z)\mathrm{d}v_Z \\ &\times \left\{ 3 f_{\mathrm{Ly}\beta} A_{3p} + f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \right. \\ &\left. \times \left[3\left(5A_{3d} + \frac{1}{2}A_{3s}\right) - \frac{3.99}{8} A_{3d}\left(3\cos^2\alpha - 1\right) \right] \right\} \; , \\ Q_C(\nu)\,\mathrm{d}\nu &= \frac{11.97}{64\pi} F(v_Z)\mathrm{d}v_Z \\ &\times f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} A_{3d} \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \; \left(3\cos^2\alpha - 1\right) \; , \\ U_C(\nu) &= 0 \; . \end{aligned} \tag{26}$$

For part II we shall not give here the details of the proof of the following formula, which will be published elsewhere for any θ value (Sahal-Bréchot et al. 1998). Introducing the quantities

$$C = \frac{1}{4\pi}\frac{1}{\pi}\frac{\sqrt{\pi}\Delta\nu_{\mathrm{D}_i}}{\Delta\nu_{\mathrm{D}_d}}\frac{I_i}{4\pi I_0} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \; ,$$

$$s(\alpha,\beta) = \frac{\nu_0}{c}\left[V_Z + \frac{\Delta\nu^2_{\mathrm{D}_d}}{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}} \sin\alpha\sin\beta \right. \\ \left. \times \left(V_x \sin\alpha\cos\beta + V_y\sin\alpha\sin\beta + V_z\cos\alpha\right) \right] \; ,$$

$$d(\alpha,\beta) = \frac{\nu_0^2}{c^2} \\ \times \frac{\left(V_x \sin\alpha\cos\beta + V_y\sin\alpha\sin\beta + V_z\cos\alpha\right)^2}{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}} \; ,$$

$$w(\alpha,\beta) = \Delta\nu_{\mathrm{D}_d} \\ \times \left[\frac{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}\left(1 - \sin^2\alpha\sin^2\beta\right)}{\Delta\nu^2_{\mathrm{D}_d} + \Delta\nu^2_{\mathrm{D}_i}} \right]^{1/2} \; ,$$

$$E(\alpha,\beta) = \frac{1}{\left[\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}\left(1 - \sin^2\alpha\sin^2\beta\right)\right]^{1/2}} \\ \times \exp\left[-d(\alpha,\beta)\right] \; \exp\left\{ - \left[\frac{\nu - \nu_0 - s(\alpha,\beta)}{w(\alpha,\beta)} \right]^2 \right\} \; ,$$

$$C_1 = \left[\left(3\cos^2\alpha - 1\right) + 3\sin^2\alpha\cos 2\beta\right] \; ,$$

and introducing the formal operator

$$\mathcal{D} = C \int_0^{\alpha_{\max}} f(\alpha)\sin\alpha\,\mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta \; E(\alpha,\beta) \; , \tag{27}$$

we can write the Stokes parameters as

$$\begin{aligned} I_L(\nu) &= -\mathcal{D}\left\{ 3\left(5A_{3d} + \frac{1}{2}A_{3s}\right) - \frac{3.99}{8}A_{3d}C_1 \right\} \\ Q_L(\nu) &= -\mathcal{D}\left\{ \frac{11.97}{8}A_{3d} \right. \\ &\left. \times \left[\left(3\cos^2\alpha - 1\right) - \sin^2\alpha\cos 2\beta\right] \right\} \\ U_L(\nu) &= -\mathcal{D}\left\{ \frac{11.97}{4}A_{3d}\sin 2\alpha\cos\beta \right\} \end{aligned} \tag{28}$$

Integrating $I_L(\nu)$, $Q_L(\nu)$, $U_L(\nu)$ over frequencies, we obtain the simpler formula

$$\begin{pmatrix} I_S \\ Q_S \\ U_S \end{pmatrix} = -\frac{1}{4\pi} \frac{\Delta\nu_{\mathrm{D}_i}}{\sqrt{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}}} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \frac{I_i}{4\pi I_0} \times \int_0^{\alpha_{\max}} f(\alpha) \sin\alpha \, \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\beta \, \exp\left[-d(\alpha,\beta)\right] \times \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \end{pmatrix} , \tag{29}$$

where

$$\begin{pmatrix} I_0 \\ Q_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} \left\{ 3\left(5A_{3d} + \frac{1}{2}A_{3s}\right) - \frac{3.99}{8} A_{3d} \times \left[(3\cos^2\alpha - 1) + 3\sin^2\alpha\cos 2\beta\right] \right\} \\ \frac{11.97}{8} A_{3d} \left[(3\cos^2\alpha - 1) - \sin^2\alpha\cos 2\beta\right] \\ \frac{11.97}{4} A_{3d} \sin 2\alpha \cos\beta \end{pmatrix} . \tag{30}$$

In particular, for $V = 0$, we have

$$\begin{pmatrix} I_S \\ Q_S \\ U_S \end{pmatrix} = -\frac{1}{8\pi} \frac{\Delta\nu_{\mathrm{D}_i}}{\sqrt{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}}} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \frac{I_i}{I_0} \times \int_0^{\alpha_{\max}} f(\alpha) \sin\alpha \, \mathrm{d}\alpha \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \end{pmatrix} , \tag{31}$$

and, for $V \to \infty$

$$\begin{pmatrix} I_S \\ Q_S \\ U_S \end{pmatrix} = 0 .$$

4. Limits of the polarization degree

Neglecting limb darkening (*i.e.*, setting $f(\alpha) = 1$), we can integrate I_S, Q_S, I_C, Q_C over the angle α. For the polarization degree we have, in the two limiting cases $V \to \infty$ and $V \to 0$

$$\tau_{V\to\infty} \approx \frac{3}{2}\frac{A}{B} , \tag{32}$$

where

$$A = A_{3d} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \sin^2\alpha_{\max} \cos\alpha_{\max} ,$$

$$B = 3 f_{\mathrm{Ly}\beta} A_{3p} + f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \left[3\left(5A_{3d} + \frac{1}{2}A_{3s}\right)(1 - \cos\alpha_{\max}) - \frac{1}{2} A_{3d} \sin^2\alpha_{\max}\cos\alpha_{\max} \right] .$$

In particular, for $\alpha_{\max} = \pi/2$ (the atom is at the limb), one gets

$$\tau_{V\to\infty} = 0 .$$

On the other hand,

$$\tau_{V\to 0} = \frac{11.97}{8}\frac{A'}{B'} , \tag{33}$$

where, setting

$$\Delta = \left(1 - \frac{\Delta\nu_{\mathrm{D}_i}}{\sqrt{\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}}} \frac{I_i}{I_0}\right) ,$$

$$A' = A_{3d} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \Delta \sin^2\alpha_{\max} \cos\alpha_{\max} ,$$

$$B' = 3 f_{\mathrm{Ly}\beta} A_{3p} + f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \Delta \left[3\left(5A_{3d} + \frac{1}{2}A_{3s}\right)(1 - \cos\alpha_{\max}) - \frac{3.99}{8} A_{3d} \sin^2\alpha_{\max}\cos\alpha_{\max} \right] .$$

Again, for $\alpha_{\max} = \pi/2$, one gets

$$\tau_{V\to 0} = 0 .$$

4.1. *Limiting values of τ with limb darkening*

It is well established that, as a first order approximation, limb darkening can be described by a linear equation of the form

$$\frac{I_\nu(\chi)}{I_\nu(0)} = f(\chi) = 1 - u + u\cos\chi , \tag{34}$$

where χ is defined in Fig. 2.1 and where u is a parameter given by $u_C \approx 0.5$ in the continuum surrounding Hα (see for instance Allen 1973).

From elementary geometry we find

$$\cos\chi = \sqrt{1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}} , \tag{35}$$

$$f(\alpha) = 1 - u + u\sqrt{1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}} = a_1 + a_2\sqrt{1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}} , \tag{36}$$

where

$$a_1 = 1 - u , \quad a_2 = u .$$

We therefore must evaluate integrals of the form

$$\int_0^{\alpha_{\max}} f(\alpha) \left(3\cos^2\alpha - 1\right) \sin\alpha \, \mathrm{d}\alpha ,$$

i.e., integrals such as

$$I = \int_0^{\alpha_{\max}} \sin\alpha \, \mathrm{d}\alpha = (1 - \cos\alpha_{\max}) \ ,$$

$$\begin{aligned} I_0 \ &= \int_0^{\alpha_{\max}} \left(3\cos^2\alpha - 1\right) \sin\alpha \, \mathrm{d}\alpha \\ &= \cos\alpha_{\max} \sin^2\alpha_{\max} \ , \end{aligned} \tag{37}$$

$$I_1 = \int_0^{\alpha_{\max}} \left(1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}\right)^{1/2} \sin\alpha \, \mathrm{d}\alpha \ ,$$

$$I_2 = \int_0^{\alpha_{\max}} \left(1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}\right)^{1/2} \cos^2\alpha \sin\alpha \ \ \mathrm{d}\alpha \ .$$

The integrals I_1 and I_2 can be found in Gradshteyn and Ryzhik (1965). One gets

$$I_1 = \frac{1}{2} + \frac{\cos^2\alpha_{\max}}{2\sin\alpha_{\max}} \log_e \left(\frac{\cos\alpha_{\max}}{1 + \sin\alpha_{\max}}\right) \ ,$$

$$\begin{aligned} I_2 \ &= \ \frac{1}{8}\Bigg[1 + \sin^2\alpha_{\max} \\ &\quad + \frac{\cos^4\alpha_{\max}}{\sin\alpha_{\max}} \log_e \left(\frac{\cos\alpha_{\max}}{1 + \sin\alpha_{\max}}\right)\Bigg] \end{aligned} \tag{38}$$

For the continuum, we use Eq. (26), and we split the integrals relative to a_1 and a_2 to obtain

$$\begin{aligned} J_1 &= a_1 \int_0^{\alpha_{\max}} \sin\alpha \, \mathrm{d}\alpha \\ &\times \left[15A_{3d} + \frac{3}{2}A_{3s} - \frac{3.99}{8}A_{3d}\left(3cos^2\alpha - 1\right)\right] \\ &\approx a_1 \left(15A_{3d} + \frac{3}{2}A_{3s}\right) I - a_1 \frac{1}{2} A_{3d} I_0 \ , \\ J_2 &\approx a_2 \int_0^{\alpha_{\max}} \left(1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}\right)^{1/2} \sin\alpha \, \mathrm{d}\alpha \\ &\quad \times \left(15A_{3d} + \frac{3}{2}A_{3s} + \frac{1}{2}A_{3d}\right) \\ &\quad - a_2 \frac{3}{2} A_{3d} \int_0^{\alpha_{\max}} \cos^2\alpha \sin\alpha \, \mathrm{d}\alpha \\ &\quad \times \left(1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}\right)^{1/2} \\ &\approx a_2 \left(15A_{3d} + \frac{3}{2}A_{3s} + \frac{1}{2}A_{3d}\right) I_1 - \frac{3}{2} a_2 A_{3d} I_2 \\ &= \frac{1}{2} a_2 \left[A_{3d}\left(31I_1 - 3I_2\right) + 3A_{3s}I_1\right] \ . \end{aligned} \tag{39}$$

Also

$$\begin{aligned} Q_1 &= a_1 A_{3d} \int_0^{\alpha_{max}} \left(3\cos^2\alpha - 1\right) \sin\alpha \, \mathrm{d}\alpha \\ &= a_1 A_{3d} I_0 \ , \end{aligned}$$

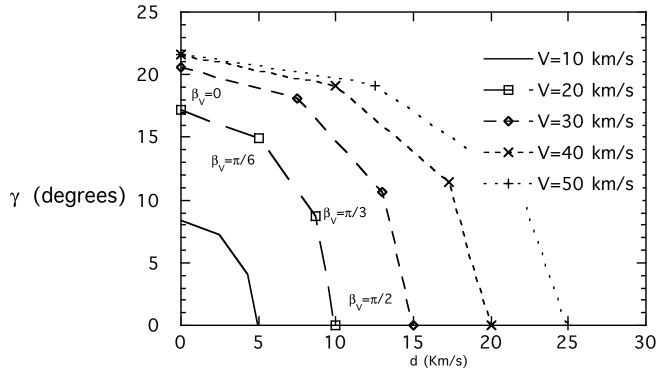

Fig. 3. Plot of the angle of rotation of polarization γ, as a function of the line shift d for $\alpha_V = 30^\circ$ and β_V varying from 0 to $\pi/2$. The plots are performed for 5 values of the modulus V. Other parameters are specified in the text. Note that the rotation angle goes to zero when the velocity V goes to zero.

$$\begin{aligned} Q_2 &= a_2 A_{3d} \int_0^{\alpha_{max}} \left(3\cos^2\alpha - 1\right) \sin\alpha \, \mathrm{d}\alpha \\ &\quad \times \left(1 - \frac{\sin^2\alpha}{\sin^2\alpha_{\max}}\right)^{1/2} \\ &= a_2 A_{3d} \left(3I_2 - I_1\right) \ . \end{aligned} \tag{40}$$

Setting $u = 0.5$ we obtain for τ

$$\begin{aligned} \tau \ &= \ \left|\frac{Q_C}{I_C}\right| \\ &= \ \frac{11.97}{8} \ \frac{f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \left(Q_1 + Q_2\right)}{3 f_{\mathrm{Ly}\beta} A_{3p} + f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \left(J_1 + J_2\right)} \end{aligned} \tag{41}$$

For $\alpha_{\max} = \pi/2$ (atom at the limb), we have

$$I = I_0 = 0 \ ,$$

$$I_1 = \frac{1}{2} \ ,$$

$$I_2 = \frac{1}{4} \ ,$$

so that

$$\begin{aligned} \tau \ &\approx \ \frac{3}{2} \ \frac{A_{3d} f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha}}{24 A_{3p} f_{\mathrm{Ly}\beta} + f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha} \left(\frac{59}{2} A_{3d} + 3A_{3s}\right)} \\ &= \ \frac{3}{59 + \frac{48 f_{\mathrm{Ly}\beta}}{f_{\mathrm{H}\alpha} f_{\mathrm{Ly}\alpha}} \ \frac{A_{3p}}{A_{3d}} + 6\frac{A_{3s}}{A_{3d}}} \\ &= \ 0.0413 \ . \end{aligned}$$

4.2. The rotation of the plane of polarization

To obtain the profiles of the Stokes parameters of the scattered radiation, one has to go back to calculate the integrals appearing in Eqs. (28).

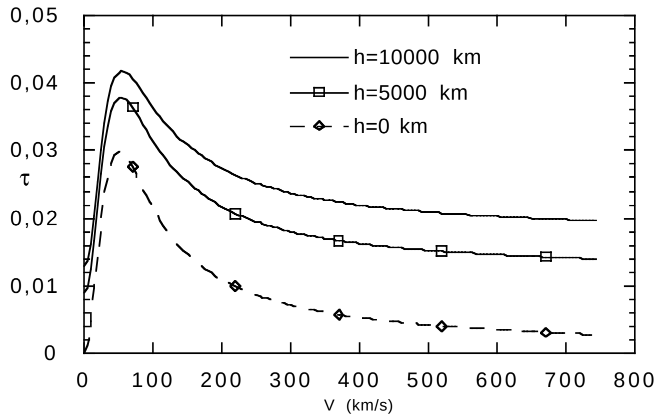

Fig. 4. The polarization degree is plotted as a function of V for three different altitudes. Collisional rates and limb darkening are neglected.

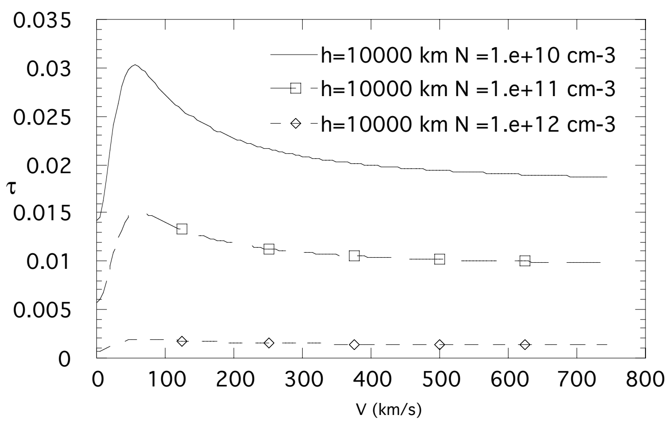

Fig. 5. The polarization degree is plotted as a function of V for three values of the electron density. The atom is at an altitude of $h = 10,000$ km above the limb. Limb darkening is taken into account.

By setting

$$K_0 = \frac{1}{16}\frac{1}{\pi^3}\frac{\sqrt{\pi}\Delta\nu_{\mathrm{D}_i}}{\Delta\nu_{\mathrm{D}_d}}\frac{11.97}{8}A_{3d}f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}\frac{I_i}{I_0} ,$$

$$D(\alpha,\beta) = \frac{\exp\left\{-d(\alpha,\beta)\right\}}{\left[\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}\left(1-\sin^2\alpha\sin^2\beta\right)\right]^{1/2}} ,$$

$$K_1 = \frac{\left(\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}\right)^{1/2}}{\Delta\nu_{\mathrm{D}_d}\left[\Delta\nu^2_{\mathrm{D}_i} + \Delta\nu^2_{\mathrm{D}_d}\left(1-\sin^2\alpha\sin^2\beta\right)\right]^{1/2}} ,$$

one obtains

$$\begin{aligned} Q(\nu) &= Q_C(\nu) \\ &-K_0\int_0^{\alpha_{\max}} f(\alpha)\ \sin\alpha\ \mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta\ D(\alpha,\beta) \\ &\times\exp\left\{-\left[K_1\left(\nu-\nu_0-s(\alpha,\beta)\right)\right]^2\right\} \\ &\times\left[\left(3\cos^2\alpha-1\right)-\sin^2\alpha\cos2\beta\right] , \end{aligned} \tag{42}$$

Table 5. Inelastic transition $3d \to 2p$. Values of the inelastic collisional coefficient $\alpha\,(3d \to 2p)$ for various temperatures and densities.

$\alpha\,(3d \to 2p)$ $\mathrm{cm^3 s^{-1}}$	T_e K	N_e $\mathrm{cm^{-3}}$
0.2327×10^{-7}	$9,300$	0.189×10^{-12}
0.6684×10^{-7}	$14,200$	0.9×10^{-11}
0.875×10^{-7}	$16,500$	0.3×10^{-9}

$$\begin{aligned} U(\nu) &= -2K_0\int_0^{\alpha_{\max}} f(\alpha)\ \sin\alpha\ \mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta\ D(\alpha,\beta) \\ &\times\exp\left\{-\left[K_1\left(\nu-\nu_0-s(\alpha,\beta)\right)\right]^2\right\} \\ &\times\sin2\alpha\cos\beta . \end{aligned}$$

The integrals over the two variables α, β can be performed by using a numerical Gauss-Legendre integration procedure with 32 points. The results of the integration depend on h, the height of the atom above the limb, the magnitude and direction of the velocity flow $\boldsymbol{V}(V,\alpha_V,\beta_V)$, the temperature T_{D_d} (and therefore $\Delta\nu_{\mathrm{D}_d}$) of the scattering region, and the law of limb darkening specified by the function $f(\alpha)$. Fig. 3, obtained for $T_{\mathrm{D}_d} \approx 10^4$ K , $h = 10,000$ km, $f(\alpha) = 1$, shows the rotation angle versus the line shift $\Delta\nu_Z$ for various directions and moduli of the ensemble velocity $\boldsymbol{V}(V,\alpha_V,\beta_V)$. The Stokes parameter profiles are integrated in frequency. The rotation angle goes to zero when the modulus of the ensemble velocity goes to zero. This diagram is useful for experimental purposes since both the shift of the scattered profile and the rotation of the polarization plane are observable quantities.

We show in Fig. 4 the polarization degree for two altitudes above the limb, and for the atom at the limb. The degree of polarization rises to a maximum value for $V \approx 70$ km/s and then decreases. When limb darkening is taken into account the polarization degree is larger.

5. The effect of collisions

The presence of free electrons and protons in the plasma of the spicule leads us to consider the effect of collisions between the scatterer and the surrounding particles.

These implies that collisional rates have to be added in the statistical equilibrium equations for ρ^K_Q (Eq. (A1)). As an example we give the equations for the evolution of the populations ${}^{3d_{5/2}}\rho^0_0$ and ${}^{3p_{1/2}}\rho^0_0$

$$\begin{aligned} &{}^{3d_{5/2}}\rho^0_0\left[1+\frac{N_\mathrm{e}\alpha\left(3d_{5/2}\to3p_{3/2}\right)}{A\left(3d_{5/2}\to2p_{3/2}\right)}\right] \\ &= \frac{N_\mathrm{e}c^{(0)}\left(0,\frac{3}{2}\to\frac{5}{2}\right)\alpha\left(3p_{3/2}\to3d_{5/2}\right)}{A\left(3d_{5/2}\to2p_{3/2}\right)}\ {}^{3p_{3/2}}\rho^0_0 \\ &+\frac{3\sqrt{2}}{4}f_{\mathrm{H}\alpha}f_{\mathrm{Ly}\alpha}\phi^0_0 , \end{aligned}$$

Table 6. Values of the elastic collisional cefficients involved in the transitions for $T_{\rm e} = 10,000$ K and $N_{\rm e} = 10^{10}$ cm^{-3}. $\alpha^{\rm e}_{ij}$ refers to collisions with electrons and $\alpha^{\rm p}_{ij}$ with protons. $\alpha^{\rm T}_{ij} = \alpha^{\rm e}_{ij} + \alpha^{\rm p}_{ij}$.

Transition	$c^{(0)}(0, J_1 \to J_2)$	$\alpha^{\rm e}_{ij}$ cm^{3}s^{-1}	$\alpha^{\rm p}_{ij}$ cm^{3}s^{-1}	$\alpha^{\rm T}_{ij}$ cm^{3}s^{-1}
$2s_{1/2} \to 2p_{3/2}$	$\sqrt{2}/2$	0.380×10^{-4}	0.306×10^{-3}	0.344×10^{-3}
$2p_{3/2} \to 2s_{1/2}$	$\sqrt{2}$	0.190×10^{-4}	0.153×10^{-3}	0.172×10^{-3}
$2s_{1/2} \to 2p_{1/2}$	1	0.231×10^{-4}	0.292×10^{-3}	0.315×10^{-3}
$2p_{1/2} \to 2s_{1/2}$	1	0.231×10^{-4}	0.292×10^{-3}	0.315×10^{-3}
$3s_{1/2} \to 3p_{1/2}$	1	0.126×10^{-3}	0.171×10^{-2}	0.184×10^{-2}
$3p_{1/2} \to 3s_{1/2}$	1	0.126×10^{-3}	0.171×10^{-2}	0.184×10^{-2}
$3s_{1/2} \to 3p_{3/2}$	$\sqrt{2}/2$	0.226×10^{-3}	0.177×10^{-2}	0.2×10^{-2}
$3p_{3/2} \to 3s_{1/2}$	$\sqrt{2}$	0.113×10^{-3}	0.891×10^{-3}	0.1×10^{-2}
$3p_{1/2} \to 3d_{3/2}$	$\sqrt{2}/2$	0.143×10^{-3}	0.117×10^{-2}	0.131×10^{-2}
$3d_{3/2} \to 3p_{1/2}$	$\sqrt{2}$	0.719×10^{-4}	0.587×10^{-3}	0.658×10^{-3}
$3p_{3/2} \to 3d_{3/2}$	1	0.178×10^{-4}	0.386×10^{-3}	0.4×10^{-3}
$3d_{3/2} \to 3p_{3/2}$	1	0.178×10^{-4}	0.386×10^{-3}	0.4×10^{-3}
$3p_{3/2} \to 3d_{5/2}$	0.8165	0.144×10^{-3}	0.161×10^{-2}	0.176×10^{-2}
$3d_{5/2} \to 3p_{3/2}$	1.225	0.965×10^{-4}	0.108×10^{-2}	0.117×10^{-2}

$$
\begin{aligned}
&{}^{3p_{1/2}}\rho^0_0 \Bigg[1 \\
&+ \frac{N_{\rm e}\alpha\left(3p_{1/2} \to 3d_{3/2}\right) + N_{\rm e}\alpha\left(3p_{1/2} \to 3s_{1/2}\right)}{A\left(3p_{1/2} \to 2s_{1/2}\right) + A\left(3p_{1/2} \to 1s_{1/2}\right)}\Bigg] \\
&= \frac{N_{\rm e}c^{(0)}\left(0, \frac{3}{2} \to \frac{1}{2}\right)\alpha\left(3d_{3/2} \to 3p_{1/2}\right)}{A\left(3p_{1/2} \to 2s_{1/2}\right) + A\left(3p_{1/2} \to 1s_{1/2}\right)}\, {}^{3d_{3/2}}\rho^0_0 \\
&+ \frac{N_{\rm e}c^{(0)}\left(0, \frac{1}{2} \to \frac{1}{2}\right)\alpha\left(3s_{1/2} \to 3p_{1/2}\right)}{A\left(3p_{1/2} \to 2s_{1/2}\right) + A\left(3p_{1/2} \to 1s_{1/2}\right)}\, {}^{3s_{1/2}}\rho^0_0 \\
&+ \frac{\sqrt{2}}{4} f_{{\rm Ly}\beta}\,.
\end{aligned}
\tag{43}
$$

In these equations we have neglected the collisional rates due to inelastic collisions, with $\Delta n = 1$, since they are very small compared to those due to the elastic ones, with $\Delta n = 0$, (see Tables 5 and 6). The factor $c^{(0)}(K, J_1 \to J_2)$ is defined by Sahal et al. (1996)

$$
\begin{aligned}
&c^{(0)}(K, J_1 \to J_2) = \\
&(2J_2 + 1)(-1)^{1+J_1+J_2+K} \begin{Bmatrix} K & J_1 & J_1 \\ 1 & J_2 & J_2 \end{Bmatrix}\,.
\end{aligned}
\tag{44}
$$

In order to find the solution for ρ^K_Q , we solve symbolically the equations in the $n = 3$ subsystem with the collisional rates taken into account. This is possible through Mathematica. The Stokes parameters Q and U are proportional to the tensors ${}^{3d_{5/2}}\rho^2_0$, ${}^{3d_{3/2}}\rho^2_0$ and for these tensors we are led to the following approximate result

$$
\begin{aligned}
&{}^{3d_{5/2}}\rho^2_0 \left[1 + \frac{N_e\alpha\left(3d_{5/2} \to 3p_{3/2}\right)}{A\left(3d_{5/2} \to 2p_{3/2}\right)}\right] \\
&= \frac{3}{5}\sqrt{\frac{7}{2}} f_{{\rm H}\alpha}\phi^2_0\, {}^{2p_{3/2}}\rho^0_0\,,
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&{}^{3d_{3/2}}\rho^2_0 \Bigg[1 \\
&+ \frac{N_{\rm e}\left[\alpha\left(3d_{3/2} \to 3p_{3/2}\right) + \alpha\left(3d_{3/2} \to 3p_{1/2}\right)\right]}{A\left(3d_{3/2} \to 2p_{3/2}\right) + A\left(3d_{3/2} \to 2p_{1/2}\right)}\Bigg] \\
&= \frac{7\sqrt{3}}{10} f_{{\rm H}\alpha}\phi^2_0\, {}^{2p_{1/2}}\rho^0_0\,.
\end{aligned}
$$

For electronic densities less than $N_{\rm e} = 10^{14}$ cm^{-3}, one can just consider relaxation in the $n = 2$ subsystem and write the populations as

$$
{}^{2p_{3/2}}\rho^0_0 \left[1 + \frac{N_{\rm e}\alpha\left(2p_{3/2} \to 2s_{1/2}\right)}{A\left(2p_{3/2} \to 1s_{1/2}\right)}\right] = \frac{1}{2} f_{{\rm Ly}\alpha}\,, \tag{46}
$$

$$
{}^{2p_{1/2}}\rho^0_0 \left[1 + \frac{N_{\rm e}\alpha\left(2p_{1/2} \to 2s_{1/2}\right)}{A\left(2p_{1/2} \to 1s_{1/2}\right)}\right] = \frac{\sqrt{2}}{4} f_{{\rm Ly}\alpha}\,,
$$

For higher densities one has to consider efficient inelastic collisional rates from the ${}^{1s_{1/2}}\rho^0_0$ level which is by far the most populated. The relevant collisional rates, for a temperature $T_{\rm e} = 10^4$ K, are

$$
\alpha(1s \to 2p) = 0.202 \times 10^{-12}\ {\rm cm^3\ s^{-1}}
$$

$$
\alpha(2p \to 1s) = 0.279 \times 10^{-7}\ {\rm cm^3\ s^{-1}}
$$

One can show that:

$$
\Phi^2_Q = A_{3d} f_{{\rm H}\alpha} f_{{\rm Ly}\alpha}\phi^2_Q \left(\frac{63}{25} f_1 g_1 + \frac{7}{4} f_2 g_2 - \frac{7}{25} f_1 g_2\right)\,,
$$

with

$$
f_1 = \left[1 + \frac{N_{\rm e}\alpha\left(2p_{3/2} \to 2s_{1/2}\right)}{A\left(2p_{3/2} \to 1s_{1/2}\right)}\right]^{-1}\,,
$$

$$f_2 = \left[1 + \frac{N_e \alpha\left(2p_{1/2} \to 2s_{1/2}\right)}{A\left(2p_{1/2} \to 1s_{1/2}\right)}\right]^{-1} ,$$

$$g_1 = \left[1 + \frac{N_e \alpha\left(3d_{5/2} \to 3p_{3/2}\right)}{6A_{3d}}\right]^{-1} ,$$

$$g_2 = \left[1 + \frac{N_e\left[\alpha\left(3d_{3/2} \to 3p_{3/2}\right) + \alpha\left(3d_{3/2} \to 3p_{1/2}\right)\right]}{6A_{3d}}\right]^{-1} .$$

This expression reduces to the preceding one for Φ_Q^2 (Eq. 15) when $N_e = 0$ that is $f_1 = f_2 = g_1 = g_2 = 1$.

We can introduce U_{col}, Q_{col} the Stokes parameters when collisions are considered. For the angle of rotation of the polarization plane, we find

$$\tan 2\gamma = \frac{U_{col}}{Q_{col}} = \frac{-\iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \frac{3}{4\pi}\,\mathrm{Re}\,\Phi_1^2}{\iint F(v_X)F(v_Y)\mathrm{d}v_X\mathrm{d}v_Y \frac{3}{4\pi}\left(\sqrt{\frac{3}{8}}\Phi_0^2 - \frac{1}{2}\,\mathrm{Re}\,\Phi_2^2\right)} = \frac{U}{Q} . \quad (47)$$

The rotation angle γ is unaffected by collisions due to the fact that the Stokes parameters Q and U are equally affected by collisions. Isotropic collisions depolarize, which is an effect well identified in the literature. This is shown in Fig. 5 displaying the polarization degree at the same altitude $h = 10,000$ km and for three different densities $N_e = 10^{10}$, 10^{11}, 10^{12} cm^{-3}. The central density corresponds to a model of spicules given in the literature (Heritschi & Mouradian 1992). At this density the polarization degree is reduced by a factor 2.

6. Conclusion

The first basic assumption of this Paper is to model the solar Hα incident radiation on the spicule by a negative exponential profile of width $\Delta\lambda_{D_i} = 1.41$ Å. Due to the fact that we use a gaussian profile, the average on the shifted velocity distribution of the scatterer can be performed nearly analytically. An important result is that limb darkening enhances the polarization degree together with the fact that isotropic collisions decrease the degree itself. We show that the polarization degree is maximum in a range of anisotropic velocities compatible with the real velocity of matter in spicules.

The order of the polarization degree is 1.5% for atoms situated at an altitude of $10,000$ km and for an electron density of $N_e \approx 10^{11}$ cm^{-3} (see Fig. 5). A consequence of the theory is to find a rotation of the polarization direction independent of collisions. Up to now, the radiation scattered by the matter in the spicule is used to determine the velocity of escape projected onto the line of sight. The measurement of both the shift of the profile and the angle γ gives information on the azimuth of the spicules, which provides a step forward towards the complete determination of the angular geometry of spicules.

The theory we have built is very general and may be improved by the introduction of a weak static magnetic field. This will be done in future work.

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Appendix A: statistical equilibrium equations

The statistical equilibrium equations for the atomic density matrix from Bommier (1977, Eq. (III-39); see also Bommier & Sahal-Bréchot 1978, Eq. (36)) and referred to in Sect. 2.2 are given by (see Table A1).

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Table A1. Appendix A

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{\alpha_1 J_1}\rho^{k_1}_{q_1} = & - \sum_{\alpha_2 J_2 < \alpha_1 J_1} {}^{\alpha_1 J_1}\rho^{k_1}_{q_1}\; A\left(\alpha_1 J_1 \to \alpha_2 J_2\right) \\
& - \sum_{\alpha_2 J_2 < \alpha_1 J_1} \sum_{k_1' q_1' K Q} {}^{\alpha_1 J_1}\rho^{k_1'}_{q_1'}\; \frac{c^3 u_\nu}{8\pi h \nu^3}\; \phi^K_Q\; 3\sqrt{(2k_1+1)\,(2k_1'+1)\,(2K+1)}\; (-1)^{k_1-q_1} \begin{pmatrix} K & k_1 & k_1' \\ Q & -q_1 & q_1' \end{pmatrix} \\
& \times (-1)^{1+J_1-J_2+k_1}\, \frac{1+(-1)^{k_1+k_1'+K}}{2} \begin{Bmatrix} J_1 & J_1 & K \\ 1 & 1 & J_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_1' & K \\ J_1 & J_1 & J_1 \end{Bmatrix} (2J_1+1)\, A\left(\alpha_1 J_1 \to \alpha_2 J_2\right) (-1)^K \\
& - \sum_{\alpha_2 J_2 > \alpha_1 J_1} \sum_{k_1' q_1' K Q} {}^{\alpha_1 J_1}\rho^{k_1'}_{q_1'}\; \frac{c^3 u_\nu}{8\pi h \nu^3}\; \phi^K_Q\; 3\sqrt{(2k_1+1)\,(2k_1'+1)\,(2K+1)}\; (-1)^{k_1-q_1} \begin{pmatrix} K & k_1 & k_1' \\ Q & -q_1 & q_1' \end{pmatrix} \\
& \times (-1)^{1+J_1-J_2+k_1}\, \frac{1+(-1)^{k_1+k_1'+K}}{2} \begin{Bmatrix} J_1 & J_1 & K \\ 1 & 1 & J_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_1' & K \\ J_1 & J_1 & J_1 \end{Bmatrix} (2J_2+1)\, A\left(\alpha_2 J_2 \to \alpha_1 J_1\right) \\
& + \sum_{\alpha_2 J_2 > \alpha_1 J_1} {}^{\alpha_2 J_2}\rho^{k_1}_{q_1}\; (-1)^{1+J_1+J_2+k_1} \begin{Bmatrix} k_1 & J_1 & J_1 \\ 1 & J_2 & J_2 \end{Bmatrix} (2J_2+1)\, A\left(\alpha_2 J_2 \to \alpha_1 J_1\right) \qquad \text{(A1)} \\
& + \sum_{\alpha_2 J_2 > \alpha_1 J_1} \sum_{k_2 q_2 K Q} {}^{\alpha_2 J_2}\rho^{k_2}_{q_2}\; \frac{c^3 u_\nu}{8\pi h \nu^3}\; \phi^K_Q\; 3\sqrt{(2k_1+1)\,(2k_2+1)\,(2K+1)}\; (-1)^{k_1-q_1} \begin{pmatrix} K & k_1 & k_2 \\ Q & -q_1 & q_2 \end{pmatrix} \\
& \times \begin{Bmatrix} 1 & J_1 & J_2 \\ 1 & J_1 & J_2 \\ K & k_1 & k_2 \end{Bmatrix} (2J_2+1)\, A\left(\alpha_2 J_2 \to \alpha_1 J_1\right) (-1)^K \\
& + \sum_{\alpha_2 J_2 < \alpha_1 J_1} \sum_{k_2 q_2 K Q} {}^{\alpha_2 J_2}\rho^{k_2}_{q_2}\; \frac{c^3 u_\nu}{8\pi h \nu^3}\; \phi^K_Q\; 3\sqrt{(2k_1+1)\,(2k_2+1)\,(2K+1)}\; (-1)^{k_1-q_1} \begin{pmatrix} K & k_1 & k_2 \\ Q & -q_1 & q_2 \end{pmatrix} \\
& \times \begin{Bmatrix} 1 & J_1 & J_2 \\ 1 & J_1 & J_2 \\ K & k_1 & k_2 \end{Bmatrix} (2J_1+1)\, A\left(\alpha_1 J_1 \to \alpha_2 J_2\right)\ .
\end{aligned}
$$