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ASTRONOMY AND ASTROPHYSICS

Analytical relations concerning the collapse time in hierarchically clustered cosmological models

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Abstract. By means of numerical methods, we solve the equations of motion for the collapse of a shell of baryonic matter, made of galaxies and substructure falling into the central regions of a cluster of galaxies, taking into account effect of the dynamical friction. The parameters on which the dynamical friction mainly depends are: the peaks' height, the number of peaks inside a protocluster multiplied by the correlation function evaluated at the origin, the filtering radius and the nucleus radius of the protocluster of galaxies. We show how the collapse time τ of the shell depends on these parameters. We give a formula that links the dynamical friction coefficient η to the parameters mentioned above and an analytical relation between the collapse time and η. Finally, we obtain an analytical relation between τ and the mean overdensity $\overline{\delta}$ within the shell. All the analytical relations that we find are in excellent agreement with the numerical integration.

Key words: cosmology: theory – large scale structure of Universe – galaxies: formation

1. Introduction

The problem of the formation and evolution of clusters of galaxies has been one of the crucial topics of the last years (see *e.g.* Ryden & Gunn 1987, Colafrancesco et al. 1989, Antonuccio-Delogu 1992, Kaiser 1993, Colafrancesco & Vittorio 1993, Croft & Efstathiou 1994, Dutta & Spergel 1994, Mosconi et al. 1994, Sutherland & Dalton 1994, Efstathiou 1994 and Colafrancesco et al. 1995). It is well known that the formation of cosmic structures is strictly related to the evolution of the density perturbations: in the present *paradigm* of structure formation, it is generally assumed that cosmic structures of size $\sim$ R form preferentially around the local maxima of the primordial density field, once it is smoothed on the filtering scale R_f. These linear density fluctuations eventually evolve towards the nonlinear regime under the action of gravitational instability; they detach from the Hubble flow at *turn around* epoch t_m, given by:

$$t_m = \left[\frac{3\pi}{32G\rho_b}(1+\overline{\delta})\right]^{1/2}(1+z)^{3/2} \quad (1)$$

where ρ_b is the mean background density, z is the redshift and $\overline{\delta}$ is the mean overdensity within the nonlinear region. After the *turn around* epoch, the fluctuations start to recollapse when their overdensity defined by

$$\delta(\mathbf{x}) \equiv \frac{\rho(\mathbf{x})-\rho_b}{\rho_b} \quad (2)$$

reaches the value $\overline{\delta}=1$.
The evolution of the density fluctuations is described by the power spectrum, given by:

$$\mathcal{P}(k) \equiv \langle|\delta_{\mathbf{k}}|^2\rangle \quad (3)$$

with:

$$\delta_{\mathbf{k}} \equiv \int d^3k\, e^{-i\mathbf{kx}}\,\delta(\mathbf{x}) \quad (4)$$

Since the density field depends on the power spectrum, which in turn depends on the matter that dominates the universe, the mean characteristics of the cosmic structures depend on the assumed model. In this context the most successful model is the biased Cold Dark Matter (hereafter CDM) (see *e.g.* Kolb & Turner 1990; Peebles 1993; Liddle & Lyth 1993) based on a scale invariant spectrum of density fluctuations growing under gravitational instability. In such a scenario the formation of the structures occurs through a ”*bottom up* ” mechanism. A simple model that describes the collapse of a density perturbation is that by Gunn & Gott (1972, hereafter $\mathrm{GG72}$). The main assumptions of this model are: **(a)** the symmetry of the collapse is spherical; **(b)** the matter distribution is uniform in that region of space where the density exceeds the background density; **(c)** no tidal interaction exists with the external density perturbations and **(d)** there is no substructure (collapsed objects having sizes less than that of main perturbation).

Point **(d)** is in contradiction to the predictions of CDM models. It is well known that in a CDM Universe, an abundant production of substructures during the evolution of the fluctuations is predicted.

The problem of the substructures in a CDM Universe and its consequences for structure formation have been widely studied in previous papers (see e.g. Antonuccio-Delogu 1992, hereafter $\mathrm{A92}$; Antonuccio-Delogu & Atrio-Barandela 1992, hereafter $\mathrm{AA92}$; Antonuccio-Delogu & Colafrancesco 1994, hereafter

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$\mathrm{AC94}$, Del Popolo et al. 1996, Gambera 1997 and Del Popolo & Gambera 1997 hereafter $\mathrm{DG97}$). Being a rather recent topic in cosmology much work still has to be done.

The presence of substructure is very important for the dynamics of collapsing shells of baryonic matter made of galaxies and substructure of $10^6 M_\odot \div 10^9 M_\odot$, falling into the central regions of a cluster of galaxies. As shown by Chandrasekhar & von Neumann (1942, hereafter $\mathrm{CvN42}$; 1943), in the presence of substructure it is necessary to modify the equation of motion:

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2(t)} \tag{5}$$

since the graininess of mass distribution in the system induces dynamical friction that introduces a frictional force term.

Adopting the notation of $\mathrm{GG72}$ (see also their Eqs. 6 and 8) and remembering that $T_{c0}/2$ is the collapse time of a shell of baryonic matter in the absence of dynamical friction ($\mathrm{GG72}$), one can write:

$$T_{c0} = \frac{\pi \bar{\rho}_i \rho_{ci}^{1/2}}{H_i(\bar{\rho}_i - \rho_{ci})^{3/2}} \tag{6}$$

where ρ_{ci} is the critical density at a time t_i and $\bar{\rho}_i$ is the *average density* inside r_i at t_i. The equation of motion of a shell of baryonic matter in presence of dynamical friction (Kandrup 1980, hereafter $\mathrm{K80}$; Kashlinsky 1986 and $\mathrm{AC94}$), using the dimensionless time variable $\tau = \frac{t}{T_{c0}}$, can be written in the form:

$$\frac{d^2 a}{d\tau^2} = -\frac{4\pi G \rho_{ci}(1+\overline{\delta_i})}{a^2(t)} T_{co}^2 - \eta T_{c0}\frac{da}{d\tau} \tag{7}$$

where $\overline{\delta_i}$ is the overdensity within r_i, η is the coefficient of dynamical friction and $a(r_i, t)$ is the expansion parameter of the shell (see $\mathrm{GG72}$ Eq. 6), that can be written as:

$$a(r_i, t) = \frac{r(r_i, t)}{r_i} \tag{8}$$

Supposing that there is no correlation among random force and their derivatives, we have:

$$\eta = \frac{\int d^3F W(F) F^2 T(F)}{2\langle v^2 \rangle} \tag{9}$$

($\mathrm{K80}$), where $T(F)$ is the average *"duration"* of a random force impulse of magnitude F, $W(F)$ is the probability distribution of stochastic force (which for a clustered system is given in Eq. 37 of $\mathrm{AA92}$).

$\mathrm{DG97}$ solved Eq. (7) numerically, showing qualitatively how the expansion parameter $a(\tau)$ depends on the dynamical friction coefficient and how τ changes in the presence of dynamical friction, but without undertaking a more complete study of the dependence on the parameters.

The plan of this paper is as follows. In Sect. 2 we show how τ depends on the peaks' height ν_c, on the parameter Ξ that we define there as the correlation function at the origin multiplied by the total number of peaks inside a protocluster, on the filtering radius R_f and on the nucleus radius of the protocluster r_0. A more detailed description of these parameters is given below. In Sect. 3 we give an analytical relation between the dynamical friction coefficient and the collapse time:

$$\tau = f_1(\eta) \tag{10}$$

In Sect. 4 we give a semi-analytical relation that links the dynamical friction coefficient with the parameters on which it depends:

$$\eta = f_2(\nu_c, R_f, r_0, \Xi) \tag{11}$$

Linking these two relations we also find the dependence of the dimensionless collapse time τ on the parameters used:

$$\tau = f_1 \circ f_2 \equiv f_1[f_2(\nu_c, R_f, r_0, \Xi)] \tag{12}$$

In Sect. 5 we give a semi-analytical relation between τ and $\overline{\delta}$ and η:

$$\tau = f_3(\overline{\delta}, \eta) \tag{13}$$

Finally, in Sect. 6 we summarize our results and comment on their possible implications.

2. The collapse time

$\mathrm{DG97}$ showed how the expansion parameter $a(\tau)$ depends on the dynamical friction, solving Eq. (7) by means of a numerical method but not taking into account the parameters on which η depends.

Here we examine how the dynamical friction coefficient η varies according to the parameters and how the collapse time depends on them. We consider Eq. (9) and the functions $T(F)$ and $W(F)$, the former given by Chandrasekhar (1943), the latter (for clustered system) given by the so-called Holtsmark law (Holtsmark 1919):

$$\begin{aligned} W(F) &= \frac{2F}{\pi}\int_0^\infty dk k \sin(kF) \exp\left[-\frac{3}{2}N\left(\frac{G m_{typ}}{r_{typ}^2}k\right)^{3/2}\right] \\ &= 4\pi\,|\mathbf{F}|\, W\left(|\mathbf{F}|\right) \end{aligned} \tag{14}$$

where $W(F)dF$ is the probability for a test particle of experiencing a force in the range $F \div F + dF$, N is the total number of particles, m_{typ} is a typical particle mass and r_{typ} is a typical distance among the particles.

According to $\mathrm{K80}$:

$$W(|\mathbf{F}|) = \frac{1}{2\pi^2\,|\mathbf{F}|}\int_0^\infty dk k \sin(k\,|\mathbf{F}|) A_f(k) \tag{15}$$

with:

$$A_f(k) = \lim_{N\to\infty} A_N(\mathbf{k}) = A_{nor}\exp(\mathcal{F}) \tag{16}$$

The original expression for $\mathcal{F}$ given by $\mathrm{K80}$ has been modified by $\mathrm{AA92}$ to take into account clustering, and turns out to be given by:

$$\mathcal{F} = \left[-\left\{A_{uncl}(k) + A_{cl}(k)\left[1 + \frac{\Sigma_{cl}(k)}{2A'_{2,cl}}\right]\right\}\right] \tag{17}$$

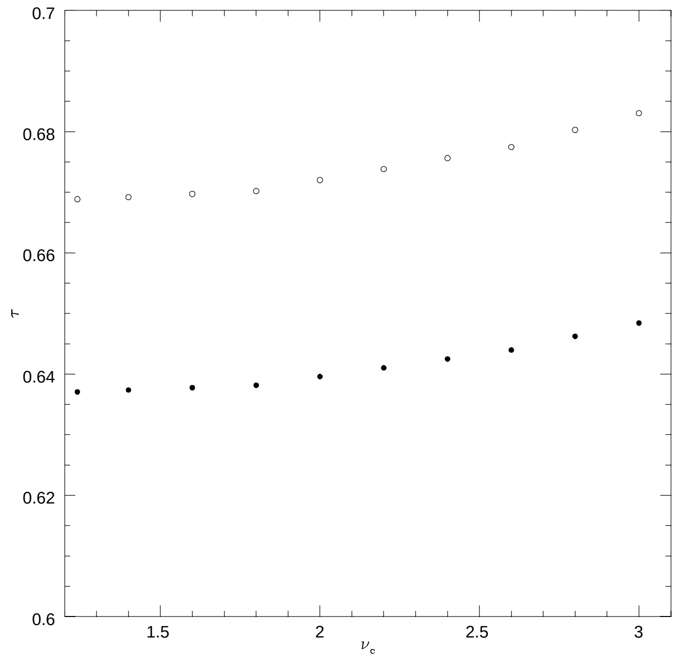

Fig. 1. Collapse time τ of a shell of matter made of galaxies and substructure when dynamical friction is taken into account, versus ν_c We assume a nucleus radius of $r_0 = 1h^{-1}$Mpc and a filtering radius $R_f = 0.74h^{-1}$Mpc. *Open circles:* $\Xi = 10^3$; *filled circles:* $\Xi = 10^2$.

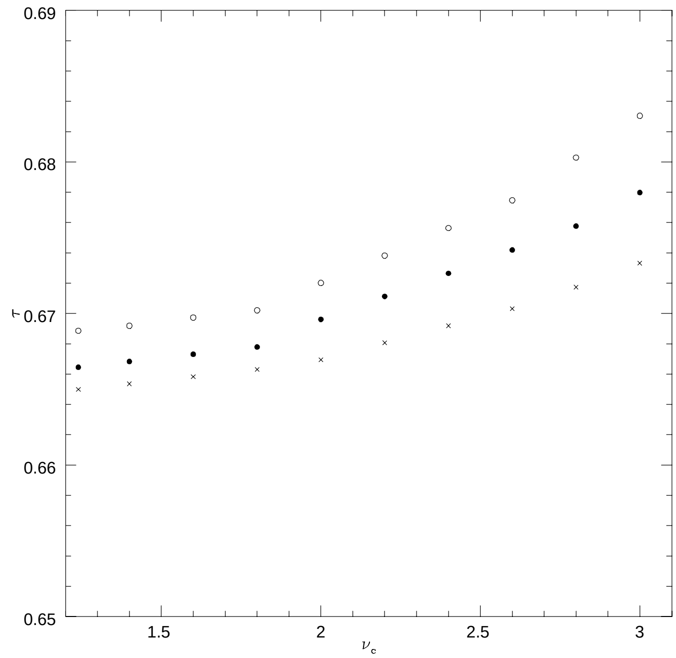

Fig. 2. Collapse time τ of a shell of matter made of galaxies and substructure when dynamical friction is taken into account, versus ν_c We assume a nucleus radius of $r_0 = 1h^{-1}$Mpc and a fixed correlation $\Xi = 10^3$. *Open circles:* $R_f = 0.74h^{-1}$ Mpc; *filled circles:* $R_f = 0.65h^{-1}$ Mpc; *crosses:* $R_f = 0.55h^{-1}$ Mpc.

where A_{nor}, $A_{cl}(k)$, $A_{uncl}(k)$, $A'_{2,(cl)}(k)$ and $\Sigma_{cl}(k)$ are given in $\mathrm{AA92}$ respectively by the Eqs. (21), (29), (31), (32) and (36).

Here, we want to remind that $\Sigma_{cl}(k)$ is a linear function of the correlation function $\xi(r)$ and that the general expression for $\Sigma_{cl}(k)$, adopting the notation of $\mathrm{AA92}$ is given by:

$$\Sigma_{cl}(\mathbf{k}) = \int_{\Re^3} d^3\mathbf{r}_1 \int_{\Re^3} d^3\mathbf{r}_2 \exp\left(i\frac{Gm_{av}\mathbf{k}\hat{\mathbf{r}}_1}{r_1^2}\right) \cdot \exp\left(i\frac{Gm_{av}\mathbf{k}\hat{\mathbf{r}}_2}{r_2^2}\right)\tau_{cl}(\mathbf{r}_1)\tau_{cl}(\mathbf{r}_2)\xi(|\mathbf{r}_1 - \mathbf{r}_2|) \quad (18)$$

where masses are measured in units of solar mass $M_\odot$ and distances in h^{-1} Mpc, so that k will be measured in unit of $h^{-1}(\mathrm{Mpc})^2/GM_\odot$, with m_{av} the average mass of the substructure. Since we have $m_{av} \geq 10^6 M_\odot >> 1M_\odot$ for all the cases that we consider in this paper, we can adopt the asymptotic expansion of Σ_{cl} demonstrated in the appendix of $\mathrm{AA92}$:

$$\Sigma_{cl}(k) \propto \xi(0) \quad (19)$$

(see $\mathrm{AA92}$ for details). Σ_{cl} does not depend on $\xi(r)$, but only on $\xi(0)$ (that is $\xi(r)$ calculated at the origin).

We solve Eq. (9) and the other equations related for an outskirts shell of baryonic matter with $\overline{\delta} = 0.01$ inside the spherical regions (protocluster), for different values of ν_c, R_f, r_0 and Ξ (where $\Xi \equiv N_{tot} \cdot \xi(0)$), the latter quantity being better defined in Sect. 4.

After having determined η solving numerically Eq. (9), we get τ as a function of ν_c, R_f, r_0 and Ξ solving Eq. (7). We perform these calculations for different set of values of ν_c, R_f, r_0 and Ξ inside the following intervals:

$$\begin{aligned} 1.2 \leq \nu_c &\leq 3.2 \\ 0.5\ h^{-1}\mathrm{Mpc} \leq\ r_0 &\leq 10\ h^{-1}\mathrm{Mpc} \\ 10^{-3}\ h^{-1}\mathrm{Mpc} \leq R_f &\leq 1\ h^{-1}\mathrm{Mpc} \\ 20 \leq \Xi &\leq 2 \cdot 10^3 \end{aligned} \quad (20)$$

The results that we have obtained are shown in Figs. $1\div4$. Before commenting upon the figures, we want to remark that the dependence of τ on $\overline{\delta}$ is qualitatively shown in Fig. 5 by $\mathrm{AC94}$. We observe that for $\overline{\delta} > 10^{-2}$ the collapse time in the presence of dynamical friction is always larger than in the unperturbed case but the magnitude of the deviation is negligible for larger $\overline{\delta}$, whilst for $\overline{\delta} \leq 10^{-2}$ the deviations increase steeply with lower $\overline{\delta}$. Then, having considered $\overline{\delta} = 0.01$, the estimate we get for τ in Sect. 4 must be considered as a lower limit.

In Fig. 1 we show the collapse time in the presence of dynamical friction, versus the peaks' height, for different values of Ξ. In this picture, we show how τ grows for larger values of ν_c and for larger values of Ξ. Similarly, in Fig. 2 we note how τ increases for larger values of ν_c and of R_f. The slope of the curves confirm our prediction on the behaviour of the collapse of a shell of baryonic matter falling into the central regions of a cluster of galaxies in the presence of dynamical friction: the

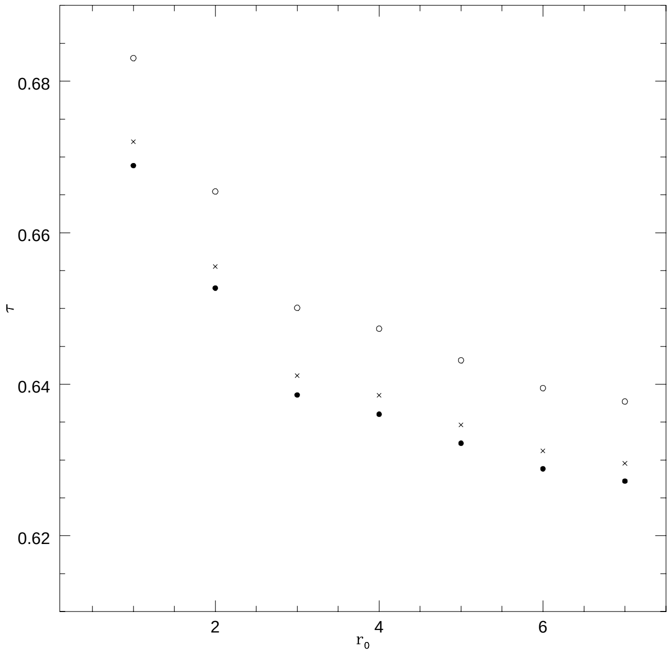

Fig. 3. Collapse time τ versus r_0. We assume a filtering radius $R_f = 0.74h^{-1}$ Mpc. and a total number of peaks of substructure $\Xi = 10^3$. *Open circles:* $\nu_c = 3$; *crosses:* $\nu_c = 2$; *filled circles:* $\nu_c = 1.24$.

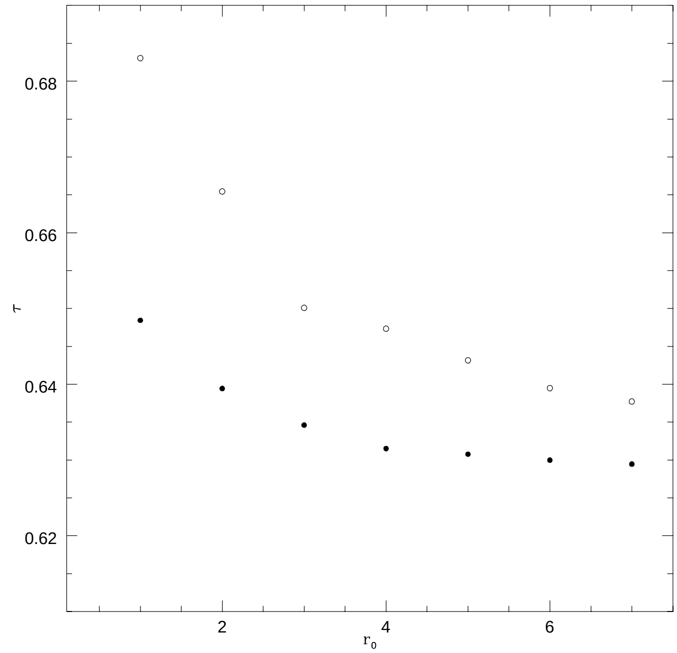

Fig. 4. Collapse time τ of a shell of matter made of galaxies and substructure when dynamical friction is taken into account, versus r_0 We assume a filtering radius $R_f = 0.74h^{-1}$ Mpc and a peaks' height $\nu_c = 3$. *Open circles:* $\Xi = 10^3$; *filled circles:* $\Xi = 10^2$.

dynamical friction slows down the collapse (as $\mathrm{DG97}$ had already shown) and the effect, as we are showing in Figs. 1 and 2, increases as Ξ, R_f, ν_c grow.
Here we want to remind you that we are considering only the peaks of the local density field with central height ν larger than a critical threshold ν_c. This latter quantity is chosen to be the threshold at which r_{peak} $(\nu \geq \nu_c) << l_{av}$ where r_{peak} is the typical size of the peaks and l_{av} is the average peak separation (see also Bardeen et al. 1986).
In Figs. 3 and 4 we show how the collapse time varies with the nucleus radius of the protocluster r_0. Note how τ grows as r_0 decreases: the smaller the nucleus of the protocluster, the larger the time of collapse in the presence of dynamical friction; besides we show how this effect increases for larger values of both ν_c and Ξ.

3. How the collapse time depends on the dynamical friction

Like $\mathrm{DG97}$, we have solved Eq. (7) numerically and calculated the collapse time τ for different values of the dynamical friction coefficent η. However, $\mathrm{DG97}$ showed the dependence of τ on η qualitatively, whilst we have found also an analytical relation $\tau = f_1(\eta)$ that links τ with η in the range of values $0 \leq \eta \leq 3.1$ (see also Gambera 1997). The results of our calculations are shown in Fig. 5, where we report the collapse time τ versus the dynamical friction coefficient. We have determined the function $\tau = f_1(\eta)$ through a least squares fit method. Our best solution is given by:

$$\tau = a_1\eta^2 + a_2\eta + a_3 \tag{21}$$

with:

$a_1 = 0.00595923,$
$a_2 = 0.0493460,$
$a_3 = 0.492164,$

Eq. (21) and our results for $0 < \eta < 3.1$ from the numerical integration of Eq. (7) are in excellent agreement. As a matter of fact, the correlation coefficient value obtained is: $r^2 = 1.000$.

4. How the collapse time depends on the parameters

As already shown by $\mathrm{AC94}$, it is possible to write down the dynamical friction coefficient η as the sum of two terms:

$$\eta = \eta_0 + \eta_{cl} \tag{22}$$

where η_0 is the coefficient of dynamical friction of an unclustered distribution of field particles whilst η_{cl} takes into account the effect of clustering. We rewrite Eq. (22) as:

$$\eta = \eta_0 + \eta_{cl} = \eta_0 \left(1 + \frac{\eta_{cl}}{\eta_0}\right) \tag{23}$$

where, as demonstrated by $\mathrm{A92}$, the ratio $\frac{\eta_{cl}}{\eta_0}$ depends only on Ξ and r_0 whilst η_0 is given by $\mathrm{A92}$:

$$\eta_0 = \frac{3.33}{\pi} \frac{\left[G\langle m\rangle_{av} p(\gamma, \nu_c) c_p\right]^{1/2}}{\langle n\rangle_{av} R_{cl}^3} \log(1.36 \langle n\rangle_{av}^{2/3} R_{cl}^2 \pi^{2/3}) \frac{1}{a^{3/2}} \tag{24}$$

and, for a fixed value of $\overline{\delta}$, depends only on ν_c and R_f. In the previous formula, c_p is defined in Bower (1991), whilst

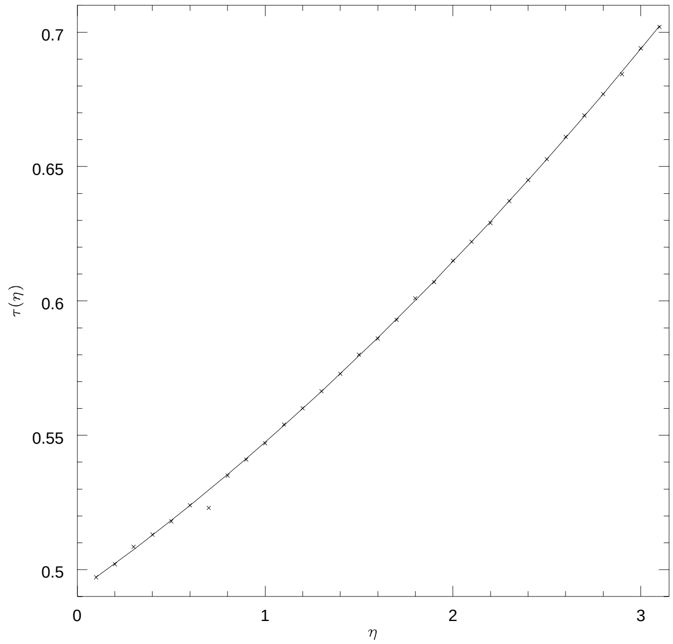

Fig. 5. Collapse time τ of a shell of matter made of galaxies and substructure when dynamical friction is taken into account, versus η.

$p(\gamma, \nu_c)$, $\langle m \rangle_{av}$ and $\langle n \rangle_{av}$ are defined in AC94. Therefore, we rewrite Eq. (23) as:

$$\eta = \eta_0(\nu_c, R_f)\left(1 + \frac{\eta_{cl}}{\eta_0}(r_0, \Xi)\right) \quad (25)$$

where the parameters on which η depends are the following:

- ν_c is the peaks' height;
- R_f is the filtering radius;
- r_0 is the parameter of the power-law density profile. A theoretical work (Ryden 1988) suggests that the density profile inside a protogalactic dark matter halo, before relaxation and baryonic infall, can be approximated by a power-law:

$$\rho(r) = \frac{\rho_0 r_0^p}{r^p} \quad (26)$$

  where $p \approx 1.6$ on a protogalactic scale.
- Ξ is the product $N_{tot} \cdot \xi(0)$ where N_{tot} is the total number of peaks inside a protocluster and $\xi(0)$ is the correlation function calculated in $r = 0$. AA92 have demonstrated that in the hypothesis $m_{av} \gg 1 M_\odot$, where m_{av} is the average mass of the subpeaks, the dependence of the dynamical friction coefficient on N_{tot} and $\xi(r)$ may be expressed as a dependence on a single parameter that we define as:

$$\Xi \equiv N_{tot} \cdot \xi(0) \quad (27)$$

The analytical relation $\eta = f_2(\nu_c, R_f, r_0, \Xi)$, that links the dynamical friction coefficient with the parameters on which it depends, can be rewritten as the product of two functions:

$$\eta = f_2(\nu_c, R_f, r_0, \Xi) = f_2'(\nu_c, R_f) \cdot f_2''(r_0, \Xi) \quad (28)$$

where

$$\eta_0 = f_2'(\nu_c, R_f) \quad (29)$$

and

$$1 + \frac{\eta_{cl}}{\eta_0} = f_2''(r_0, \Xi) \quad (30)$$

With a least square method, we find the best function $\eta = f_2(\nu_c, R_f, r_0, \Xi)$. First, we find an analytical relation between the dynamical friction coefficient in the absence of clustering η_0 and the parameters R_f and ν. We obtain:

$$\begin{aligned} \eta_0 \;=\; & b_1 + b_2\nu + b_3 R_f + b_4 \nu R_f + b_5 \nu^2 \\ & + b_6 R_f^2 + b_7 \nu R_f^2 + b_8 \nu^2 R_f^2 + b_9 \nu^2 R_f^2 \end{aligned} \quad (31)$$

with:

$$\begin{aligned} b_1 &= 1.00655066, & b_2 &= -0.01224778, \\ b_3 &= 0.03836632, & b_4 &= -0.03337465, \\ b_5 &= 0.00288929, & b_6 &= -0.02622193, \\ b_7 &= 0.04374888, & b_8 &= 0.00690435, \\ b_9 &= 0.00860054, \end{aligned}$$

We perform a χ^2 test between the values of η_0 obtained from the last equation and the values of η_0 calculated integrating Eq. (24). Our result for the range $1.2 \le \nu_c \le 3.2$ is excellent: $\chi^2 \approx 10^{-8}$.
We do the same job for the quantity $f_2''(r_0, \Xi)$, finding:

$$f_2'' = c_1 + c_2 r_0 + c_3 \Xi + c_4 \Xi r_0 + c_5 r_0^2 + c_6 \Xi r_0^2 \quad (32)$$

with:

$$\begin{aligned} c_1 &= 2.26516888, & c_2 &= -0.09679056, \\ c_3 &= 0.00054051, & c_4 &= -0.0001255044, \\ c_5 &= 0.00850948, & c_6 &= 0.000009336723 \end{aligned}$$

An analogue χ^2 test gives $\chi^2 \approx 10^{-8}$ for the range $20 \le \Xi \le 2 \cdot 10^3$.
The function $\eta = f_2(\nu, R_f, r_0, \Xi)$ is given by the product of Eqs. (31) and (32). These contain 54 terms.

$$\begin{aligned} \eta \;=\; & f_2(\nu, R_f, r_0, \Xi) = \\ =\; & (b_1 + b_2\nu + b_3 R_f + b_4 \nu R_f + b_5 \nu^2 + b_6 R_f^2 + b_7 \nu R_f^2 \\ +\; & b_8 \nu^2 R_f^2 + b_9 \nu^2 R_f^2) \cdot (c_1 + c_2 r_0 + c_3 \Xi \\ +\; & c_4 \Xi r_0 + c_5 r_0^2 + c_6 \Xi r_0^2) \end{aligned} \quad (33)$$

However, we have also found an empirical formula with only 13 terms that represents a good approximation. We performed a χ^2 test between the results obtained from the 13-term equation and the results obtained from the numerical integration. The result is $\chi^2 = 2.17 \cdot 10^{-3}$. The same test performed on the 54-term equation gives $\chi^2 \approx 10^{-8}$. Our 13-term equation reads as:

$$\begin{aligned} \eta \;\approx\; & d_1 + d_2\nu + d_3 R_f + d_4 r_0 + d_5 \Xi + d_6 \nu R_f \\ +\; & d_7 \nu^2 + d_8 R_f^2 + d_9 r_0^2 + d_{10} r_0 \Xi \\ +\; & d_{11} \nu R_f^2 + d_{12} \nu^2 R_f + d_{13} r_0^2 \Xi \end{aligned} \quad (34)$$

with:

$$\begin{aligned} d_1 &= 2.280000, & d_2 &= -0.027743, \\ d_3 &= 0.086906, & d_4 &= -0.097425, \\ d_5 &= 5.440520, & d_6 &= -0.075599, \\ d_7 &= 0.006545, & d_8 &= -0.059397, \\ d_9 &= 0.008565, & d_{10} &= 1.26 \cdot 10^{-4}, \\ d_{11} &= 0.099099, & d_{12} &= 0.015640, \\ d_{13} &= 9.39 \cdot 10^{-6} \end{aligned}$$

The function $\tau = f_1 \circ f_2$ is given by:

$$f_1 \circ f_2 = a_1 \left[f_2(\nu, R_f, r_0, \Xi) \right]^2 + a_2 \left[f_2(\nu, R_f, r_0, \Xi) \right] + a_3 \quad (35)$$

that is:

$$\tau = a_1 \left[\eta_0 \left(1 + \frac{\eta_{cl}}{\eta_0} \right) \right]^2 + a_2 \left[\eta_0 \left(1 + \frac{\eta_{cl}}{\eta_0} \right) \right] + a_3 \quad (36)$$

where the values of a_n are given in Sect. 2 and the function f_2 is given by Eq. (33) or by Eq. (34).

5. A semi-analytical relation between τ and $\overline{\delta}$

Our aim is to find a semi-analytical relation $\tau = f_3(\overline{\delta}, \eta)$ for the intervals $10^{-4} \leq \overline{\delta} \leq 10^{-2}$ and $0 \leq \eta \leq 3.1$.
The first step is the determination of a function $\tau = g(\overline{\delta})$ for a fixed value of η. We solve Eq. (7) for $10^{-4} \leq \overline{\delta} \leq 10^{-2}$ and $\eta = 0.01$ and by the least square method we find the function $\tau = g(\overline{\delta})$:

$$\tau = e_1 \overline{\delta}^2 + e_2 \overline{\delta} + e_3 \quad (37)$$

with:

$$\begin{aligned} e_1 &= 5678.65, \\ e_2 &= 45.3429, \\ e_3 &= -0.01591, \end{aligned}$$

The value of the correlation coefficient between this function and the numerical integration is $r^2 = 1.000$. With this method we find $\tau = g(\overline{\delta})$ for several values of η inside the interval $0 \leq \eta \leq 3.1$. We can write the dimensionless collapse time as the product:

$$\tau = g(\overline{\delta}) \cdot h(\eta) \quad (38)$$

where the function $h(\eta)$ can be written as

$$h(\eta) = K \cdot [1 + f(\eta)] \quad (39)$$

and, for $10^{-4} \leq \overline{\delta} \leq 10^{-2}$ and $0 \leq \eta \leq 3.1$, K is constant: $K = -1494.7705898$. So:

$$\tau = F(\overline{\delta}, \eta) = K \cdot [1 + f(\eta)] \cdot g(\overline{\delta}) \quad (40)$$

A χ^2 test performed between the values obtained for τ from the numerical integration and the values obtained from Eq. (7) gives the result: $\chi^2 \sim 10^{-4}$.

6. Conclusions and discussion

In Sect. 2 of this work we showed in a quantitative way how the collapse time τ of a shell of baryonic matter made of galaxies and substructure depends on some parameters. When one of the parameters Ξ or R_f or ν_c increases, the collapse time grows. It means that the effects of the presence of dynamical friction should be more evident in the outer regions of rich clusters of galaxies. Besides, we show how the collapse time of an infalling shell increases with decreasing values of r_0, and becomes very large for $r_0 \leq 2h^{-1}$ Mpc (see Fig. 4). As a consequence, the slowing down of the collapse of an outer shell within a cluster of galaxies owing to the dynamical friction is more remarkable in the clusters with nucleus of little dimension.

In Sect. 3 of this paper we give an analytical relation that links the dimensionless collapse time τ with the coefficient of dynamical friction η. This relation is in excellent agreement with the numerical integration of Eq. (7) for $0 \leq \eta \leq 3.1$ (see Fig. 5). Then, we find an analytical relation between η and the parameters on which it depends for the value $\overline{\delta} = 10^{-2}$ (we remind that the effects of the dynamical friction are negligible for $\overline{\delta} > 10^{-2}$). This is Eq. (28), that can be considered as a *"low order"* approximation to a more realistic situation of an outer shell of a cluster of galaxies with $\overline{\delta} \leq 10^{-2}$. We also find an empirical formula, Eq. (34), that is a good approximation of Eq. (33). Moreover, the dependence of the dimensionless collapse time on ν_c, r_0, Ξ and R_f is shown (Eq. (35)).
Finally, we give an analytical relation that links τ with η and $\overline{\delta}$. Here, we wish to stress the usefulness of an analytical relation like $\tau = K \cdot [1 + (f_1 \circ f_2)] \cdot g(\overline{\delta})$. This is a powerful tool to estimate the effect of the dynamical friction in the outer regions of clusters of galaxies (Gambera et al. in preparation) and to compare the observational data with the theoretical ones. It is a good method to test how important the role of the dynamical friction is in the collapse of the clusters of galaxies.

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