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ASTRONOMY AND ASTROPHYSICS

Effects of moderate rotation on stellar pulsation

I. Third order perturbation formalism

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Abstract. Interpretation of the available ground-based and forthcoming space observations of multiperiodic variable stars require accurate computations of oscillation frequencies. Typical rotational equatorial velocities of upper main sequence stars range between 50 and 200 km/s and the effect of rotation on the oscillation frequencies of these stars must be included. The rotation can still be considered as a perturbation, provided the expansion used in the perturbation method is carried out far enough to provide frequencies accurate enough to match that of the observations. For that purpose, we develop here a perturbation formalism for adiabatic oscillations of moderately rotating stars which is consistently valid up to third order in the rotation rate.

The formalism includes the case of near-degeneracy i.e. when the frequencies of two (or three) oscillation modes happen to be close to each other. This occurs systematically for $l = 0$ and $l = 2$ p-modes. Near-degeneracy leads to spherical harmonic mixing and affects frequencies. Consequences for mode identification are discussed.

Key words: stars: oscillations – stars: rotation – δ Scu – stars: variables: other

1. Introduction

Many pulsating stars in the Main Sequence band are of potential interest for asteroseismic sounding. Observers have devoted a considerable attention to δ Scuti and β Cephei stars for instance. Multisite campaigns of photometric observations have indeed revealed that these stars are multimode, nonradial oscillators. Modes are observed to be excited in a wide frequency range which covers modes with large amplitude both in outer layers and in the deep interior. These pulsators seem particularly suitable for determining the extent of the convective core and internal rotation rate, and thereby for probing poorly understood hydrodynamical processes occurring in deep stellar interiors; to name a few: convective overshoot, mixing and transport of angular momentum through the propagation of gravity waves (Schatzman, 1995) and the combined effect of differential rotation and turbulence (Zahn, 1992).

The seismological applications of these pulsating stars is however currently hindered by the problem of mode identification, that is identifying the detected frequencies as associated with specific oscillation modes. Unlike the Sun, these stars are pulsating with modes of low radial orders with frequencies which are far from beeing equidistant. The degree l of the modes is therefore difficult to infer from the observational frequencies without comparing with theoretical ones obtained from stellar models. Detecting all the modes of given degree in the observed frequency range would help in identifying unambiguously the modes. The δ Scuti stars which have been sofar monitored with ground based multisite campaigns, all show power spectra with ~ 5 up to ~ 20 frequencies (Michel et al., 1995; Breger, 1995) but it is clear that in all cases, only a fraction of low degree modes is being detected in the observed frequency ranges. Forthcoming space observations (COROT, Catala et al., 1995) with a much lower detection threshold may provide far richer power spectra. Still, since the power spectra will lack simple equidistant structure, we will not be able to separate mode identification from determination of model parameters. An iterative procedure will be needed. In any case, accurate computations of oscillation frequencies are necessary. For upper main sequence stars which usually are moderate or fast rotators, the effect of rotation has to be included.

Rotation modifies the structure of the star, changes conditions for wave propagation and thereby the frequencies of normal modes. First, rotation lifts mode degeneracy and leads to multiplets of progade, centroid and retrograde modes. If the rotational angular velocity, Ω, does not have latitudinal dependence, which will be assumed throughout this work, and rotation is slow enough, the multiplets show a Zeeman-like equidistant structure.

At faster rotation nonnegligible quadratic effects in Ω cause a departure from equidistant splitting. Modification of the frequency pattern in a power spectrum may be substantial to the extent that it jeopardizes the mode identification (Dziembowski and Goode, 1992, hereafter DG92). Also, for faster rotators, the

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position of the centroid frequency does not coincide with that of nonrotating models.

One effect, not included in DG92 is an overall centroid shift caused by modification of the spherical structure by the averaged centrifugal force. The shift was computed by Simon (1969), then by Saio (1981) for oscillation modes of polytropes. The centroid shift depends on radial mode order. Thus, in particular, period ratios of consecutive radial modes are affected (Soufi et al.,1995a,b). These ratios have been used as a guide in mode identification. Later in this paper we will discuss yet another important effect affecting period ratios.

Chlebowski (1978) calculated the quadratic effects of uniform rotation on oscillations of white dwarf models. Gough & Thompson (1990) evaluated effects of rotation up to Ω^2 on adiabatic p-mode frequencies in a realistic solar model. They allowed a radial dependence in the Ω-profile. Lee & Baraffe (1995) considered only uniform rotation but went further in the sense that they included nonadiabaticity of the oscillations. Latitudinal differential rotation was considered in DG92.

In the present paper, like in that of Gough & Thompson, we only allow radial dependence in Ω and consider adiabatic oscillations. The calculation cost for taking into account the latitudinal dependence is relatively high for a marginal reward. Our primary interest are Upper Main Sequence stars which have predominantly radiative interiors where significant latitudinal rotation is not expected. Indeed turbulent mixing in radiative interiors has been shown to be highly anisotropic i.e very efficient horizontally and much less efficient vertically (Zahn, 1992). As far as nonadiabatic effects are concerned, we believe that adding nonadiabatic corrections in the way described in Sect. 6 is accurate enough. The main innovation here is that we carry out the perturbation analysis up to Ω^3 i.e. one order higher than our predecessors. Futhermore, we explore consequences of near-degeneracy of modes coupled by rotation – an effect largely ignored in previous investigations.

Seismology of the Upper Main Sequence pulsators requires a careful treatment of the influence of rotation on oscillation frequencies because our small parameter, $\epsilon = \Omega/\omega$, is not that small in this case. At typical equatorial velocities, ~ 100 km/s, and oscillation periods from half to few hours we have $\epsilon \sim 0.1$. An accuracy of 10^{-3} in frequency calculation is needed for any meaningful effort in asteroseismology of these complicated multimode pulsators.

The paper is organized as follows. In Sect. 2, we outline the problem and the adopted procedure. In Sect. 3, we determine the equilibrium model which includes the second order as well as higher order effects of the symmetric distortion by rotation. Oscillation frequencies are derived in Sect. 4.

When the frequencies of two or more modes happen to be very close to each other, we are dealing with the case of near-degeneracy. The perturbation method requires modifications. Near-degeneracy is handdled in Sect. 5. Appendix A and B are associated with Sects. 4 and 5. The developement of the third order formalism does not present mathematical difficulties but calls for lenghty calculations. In a first reading, the reader may jump over these sections to go directly to Sect. 6 where a summary of the practical results is given together with some comments.

2. Computing frequencies of rotating stars: outline

In order to determine the oscillation frequencies of a rotating star, the first step proceeds in the same way as in the case of a non rotating star: one linearizes the stellar hydrodynamical equations about the equilibrium configuration of the rotating star (Unno et al., 1989).

The equilibrium state of a rotating star is characterized here by a velocity field $\boldsymbol{v_0} = \boldsymbol{\Omega} \times \boldsymbol{r} = \Omega r \sin\theta\, \boldsymbol{e_\varphi}$ where Ω denotes the angular velocity. The rotation axis of the star is chosen so as to coincide with the $\theta = 0$ axis of a spherical coordinates system (r, θ, φ) of an inertial frame of reference and $\boldsymbol{\Omega} = \Omega(r)\boldsymbol{e_z}$. The r-dependent rotation profile $\Omega(r)$ is written as :

$$\Omega(r) \;=\; \bar{\Omega}\,[1 + \eta(r)] \tag{1}$$

where $\bar{\Omega}$ is the mean rotation rate. A uniform rotation then corresponds to $\Omega = \bar{\Omega}$ and $\eta(r) = 0$.

The hydrostatic equilibrium obeys :

$$\nabla p = -\rho\nabla\phi + \rho\boldsymbol{F} \tag{2}$$

where p, ρ, and ϕ, are the pressure, density and gravitational potential, respectively, and $\boldsymbol{F}$ is the centrifugal acceleration :

$$\boldsymbol{F} = -\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r} = r\Omega^2 \sin\theta\, \boldsymbol{e}_s \tag{3}$$

where $\boldsymbol{e}_z$, and $\boldsymbol{e}_s = \sin\theta\, \boldsymbol{e}_r + \cos\theta\, \boldsymbol{e}_\theta$ are unit vectors in cylindrical coordinates (s, φ, z).

The centrifugal acceleration can be written in terms of the zeroth and second order Legendre polynomials $P_0(\cos\theta) = 1$ and $P_2(\cos\theta) = 3/2\cos^2\theta - 1/2$:

$$\boldsymbol{F} = F_r(r)\,[1 - P_2(\cos\theta)]\,\boldsymbol{e}_r + F_\theta(r)\frac{dP_2}{d\theta}\,\boldsymbol{e}_\theta \tag{4}$$

with

$$F_r(r) = \tfrac{2}{3}r\Omega^2 \text{ and } F_\theta(r) = -\tfrac{1}{3}r\Omega^2 \tag{5}$$

The effect of the centrifugal force on the equilibrium structure is twofold : on one hand, a spherically symmetric perturbation, which mainly modifies the gravity; on the other hand, θ-dependent perturbations, which generate a non-spherically symmetric distortion of the star, i.e. the rotating star is no longer a sphere but an oblate spheroid (cf Tassoul, 1978). As the centrifugal force distorts the equilibrium configuration of a rotating star, equilibrium pressure and all structural quantities of course depend on (r, θ).

Since we consider perturbation about an axisymmetric and steady configuration the Lagrangian fluid displacement $\boldsymbol{\xi}$ can be assumed of the form

$$\boldsymbol{\xi} \propto \exp[i(\omega t + m\varphi)] \tag{6}$$

where ω and m are the oscillation frequency in the inertial frame and the azimuthal order respectively (note that $m < 0$ corresponds to a prograde mode). The same dependence is assumed for the Eulerian perturbations of the structural properties which we denote with a prime, i.e. p', ρ' ....

The linearized equation of motion leads to the following equation of oscillation, (see also DG92),

$$\begin{aligned}\mathcal{L}\boldsymbol{\xi} \equiv L\boldsymbol{\xi} - \rho\,(\omega + m\Omega)^2\,\boldsymbol{\xi} + \\ 2i\rho\,(\omega + m\Omega)\,\Omega\,\boldsymbol{e}_z \times \boldsymbol{\xi} + \rho\left(\boldsymbol{\xi}.\boldsymbol{\nabla}\Omega^2\right) r\sin\theta\,\boldsymbol{e}_s = 0\end{aligned} \tag{7}$$

L is a linear operator acting on $\boldsymbol{\xi}$ and defined as :

$$L\boldsymbol{\xi} = \nabla p' + \rho'\left(\nabla\phi - \Omega^2 r\sin\theta\,\boldsymbol{e}_s\right) + \rho\nabla\phi' \tag{8}$$

Concomittantly, the linearized continuity, Poisson and adiabatic energy conservation equations are :

$$\begin{aligned}& p' = -p\,\Gamma_1\,\nabla.\boldsymbol{\xi}\ -\ \boldsymbol{\xi}.\nabla p \\ & \nabla^2\phi' = 4\pi G\,\rho' \qquad \rho' = -\boldsymbol{\nabla}.\left(\rho\,\boldsymbol{\xi}\right)\end{aligned} \tag{9}$$

with Γ_1 the adiabatic exponent.

Eqs. (7-9) supplemented with appropriate boundary conditions forms an eigenvalue system which can be solved for the eigenfrequency, ω, provided the equilibrium configuration of the rotating star ($\rho(r,\theta)$ ...) is known for a given rotation profile.

One procedure, as in DG92, is to consider the effects of rotation on the oscillation frequencies and on the equilibrium structure as small perturbations. All equilibrium and oscillating quantities entering Eqs. (7-9) are thus expanded with respect to $\mu = \Omega/\omega_0$ and $\epsilon = \Omega R_0^{3/2}/(GM_0)^{1/2}$, where G is the gravitational constant, M_0 and R_0 are the stellar mass and radius of the model. The ratio μ can be interpreted as measuring the effect of Coriolis force on the frequencies; ϵ measures the ratio of the centrifugal acceleration to the gravity at the equator. The perturbation method requires that the rotation be slow enough compared with the oscillation period, i.e. $\epsilon << 1$. Note that the ratio $\mu/\epsilon = 1/\sigma_0$, where $\sigma_0 = \omega_0 R_0^{3/2}/(GM_0)^{1/2}$ is the dimensionless oscillation frequency: the effect of the Coriolis force decreases with increasing frequencies i.e. with increasing radial order for a fixed degree, l, of the mode. For frequencies of interest here, $\sigma_0 \sim 3-10$ implies that $\mu/\epsilon \sim 1$ for low radial order modes to $\mu/\epsilon << 1$ for high radial order modes and it is enough to consider only one small parameter denoted loosely as ϵ.

The frequency ω of a rotating star is obtained as the frequency of the non rotating star ω_0 (zeroth order) corrected for the rotational effect (higher orders), i.e. $\omega = \omega_0 + \delta\omega$.

The equilibrium quantities are expanded in terms of Legendre polynomials in the equations of internal structure. This yields the coefficients of order ϵ^2 in this expansion (DG92). Applied straightforwardly, this procedure cannot give the frequency correction arising from the spherically symmetric perturbation of pressure, density, and potential induced by the spherically symmetric part of the centrifugal acceleration (zeroth order of the Legendre expansion). With a slight modification, a similar procedure can nevertheless be adopted and is now outlined.

For a uniform rotation, the spherically symmetric part of the centrifugal acceleration acts purely as a modification of the gravity of the star. Included as part of an effective gravity, this contribution does not modify the form of the hydrostatic equilibrium equation. Following the approach described in Kippenhahn & Weigert (1994), a stellar model can therefore be built with an evolutionary code modified so as to include the rotation through an effective gravity. Note that the effect of rotation on the stellar evolution is only partially taken into account. At a given age, the spherically symmetric part of the centrifugal force modifies the structure of a model which then evolves differently than a nonrotating one. At a later age, the structure of the model therefore differs even further from that of a nonrotating model. However, not all effects of rotation– in particular the non-spherically symmetric distortion– are included and such a model does not fully represent the equilibrium model of a rotating star. Therefore, to avoid confusion, we refer to it as a *pseudo-rotating* model. This first step is detailed in the next section.

It is possible to investigate the oscillations of such a model in the same way as the nonrotating case by perturbing the hydrodynamical equations *about the pseudo-rotating model*. The resulting linearized system yields the oscillation frequencies as eigenvalues. An oscillation code can also easily be adapted in order to compute the frequencies of such a model. When the rotation depends on r, this effect can be included as well in much the same way. The oscillation frequencies of a pseudo-rotating model, $\tilde{\omega}$, however, include only the effect of the spherically symmetric part of the centrifugal acceleration. In order to obtain the contributions to the oscillation frequencies arising from the Coriolis force and the non-spherically symmetric part of the centrifugal force, we use a perturbation method. The zeroth order eigensystem is the eigensystem for a pseudo-rotating model but modified so as to include parts of the Coriolis and non-spherical distortion effects; this yields eigenfrequencies ω_0 of eigenmodes which are no longer m-degenerate, even at zeroth order. The perturbation method provides the correction to the frequency ω_0, ω_c (which depends on the structure of the *pseudo-rotating* model). This way of building our zeroth order eigensystem and the associated basis of eigenmodes enable us to take into account the third order effects of rotation and to treat the near-degenerate (i.e. close frequencies) cases while at the same time minimizing the calculation effort. This procedure is developed in Sect. 4.

The complete rotationally perturbed frequencies, $\omega_0 + \omega_c$, are valid up to third order in ϵ. It is important to note that the resulting frequencies of a rotating star are thereby conveniently obtained without any reference to a non rotating model.

3. Equilibrium structure of rotating models

With Eq. (4) and Eq. (5) for the centrifugal acceleration, the hydrostatic equation, Eq. (2), becomes :

$$\begin{aligned}\frac{\partial p}{\partial r} &= -\rho\frac{\partial \phi}{\partial r} + \rho F_r(r)\,(1-P_2) \\ \frac{\partial p}{\partial \theta} &= -\rho\frac{\partial \phi}{\partial \theta} - \rho\tfrac{1}{3}r^2\Omega^2\frac{dP_2}{d\theta}\end{aligned} \quad (10)$$

In order to solve Eq. (10), the equilibrium pressure is expanded in Legendre polynomials, viz.

$$p(r,\theta) = \tilde{p}(r) + \epsilon^2\, p_2 = \tilde{p}(r) + \epsilon^2\, p_{22}(r)\; P_2(\cos\theta) \quad (11)$$

Similar expansions are performed for the density ρ and the gravitational potential ϕ. Inserting these expansions into Eq. (10), one gets three equations for $p_{22}(r)$, $\rho_{22}(r)$, and $\phi_{22}(r)$ which will be discussed in Sect. 3.2, and equations for the expansion coefficients $\tilde{p}$, $\tilde{\rho}$, and $\tilde{\phi}$, which we consider next.

3.1. Spherically symmetric distortion: pseudo-rotating models

The spherically symmetric components of pressure, density, and gravitational potential in Eq. (10), $\tilde{p}$, $\tilde{\rho}$, and $\tilde{\phi}$ include the spherically symmetric perturbation induced by rotation. The pressure $\tilde{p}$ is obtained from the θ-independent part of Eq. (10a) i.e.:

$$\frac{d\tilde{p}}{dr} = -\tilde{\rho}\; g_e \quad (12)$$

with

$$g_e = \tilde{g} - F_r(r) = \frac{d\tilde{\phi}}{dr} - \tfrac{2}{3}r\Omega^2 \quad (13)$$

$$\tilde{g} = \frac{d\tilde{\phi}}{dr} = \frac{GM_r}{r^2} \quad (14)$$

and M_r is the mass contained in the shell of radius r.

Eq. (12) is the equation of hydrostatic equilibrium of a spherically symmetric ‘star’, and thus is similar to that of a non rotating star. In a first approximation, the other equations of internal structure are not modified, hence a usual evolutionary code only modified according to Eq. (12) yields the evolution and the structure of *pseudo-rotating* models. Of course, in order to compute a model at a given age, the profile of the rotation rate is needed which requires the knowledge of its temporal evolution. One simple assumption is local angular momentum conservation. Then only expansions and contractions of the fluid modify the r-dependence of the rotation rate from one evolutionary stage to the next one. A simple alternative is uniform rotation and global angular momentum conservation. While neither angular momentum evolution prescription is correct, they have the virtue of being simple. The reality lies somewhere between these two extreme cases, depending on how efficient is the transport of angular momentum and we hope that asteroseismology will eventually tell us what are the real stellar rotation profiles and angular momentum evolutions.

3.2. Non-spherically symmetric distortion

We proceed as in DG92: the non-spherically symmetric components of pressure and density are obtained from the $O(\epsilon^2)$ order equations arising from the expansion of Eq. (10). This yields the following relations :

$$p_{22} = -\tilde{\rho}r^2\bar{\Omega}^2 u_2; \quad \rho_{22} = \frac{\tilde{\rho}r\bar{\Omega}^2}{\tilde{g}}\left(\frac{d\ln\tilde{\rho}}{d\ln r}u_2 + b_2\right) \quad (15)$$

where u_2 and b_2 are defined as :

$$u_2 = \frac{\phi_{22}}{r^2\bar{\Omega}^2} + \tfrac{1}{3}\,(1+\eta_2) \text{ and } b_2 = \tfrac{1}{3}r\frac{d\eta_2}{dr} \quad (16)$$

with $\eta_2 = \eta(\eta+2)$ (see Eq. (1)).

The perturbation of the gravitational potential ϕ_{22} is obtained by integration of the perturbed Poisson equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_{22}(r)}{dr}\right) - \frac{6}{r^2}\phi_{22}(r) = 4\pi G\;\rho_{22}(r) \quad (17)$$

with the appropriate boundary conditions : $\phi_{22} \propto r^2$ at the center of the star, and $\phi_{22} \propto r^{-3}$ at the surface of the star. The relations (15) to (17) are identical to the relations (77) to (80) (with $k=1$) in DG92.

4. Third order perturbation formalism

Using expansions of type (11) for the equilibrium quantities, the oscillation system (7-9) is expanded up to ϵ^3 order. The energy, continuity and Poisson equations are obtained as:

$$p' = \tilde{p}' + p_2' \;; \qquad \rho' = \tilde{\rho}' + \rho_2' \;; \qquad \phi' = \tilde{\phi}' + \phi_2' \quad (18)$$

with

$$\begin{aligned}\tilde{p}' &= -\Gamma_1 p\;\nabla.\boldsymbol{\xi} - \boldsymbol{\xi}.\nabla p \\ p_2' &= -\left(\Gamma_1 p\right)_2\;\nabla.\boldsymbol{\xi} - \boldsymbol{\xi}.\nabla p_2\end{aligned} \quad (19)$$

and

$$\begin{aligned}\tilde{\rho}' &= -\nabla\cdot(\rho\boldsymbol{\xi})\;; \qquad \rho_2' = -\nabla\cdot(\rho_2\boldsymbol{\xi}) \\ \nabla^2\tilde{\phi}' &= 4\pi G\tilde{\rho}'\;; \qquad \nabla^2\phi_2' = 4\pi G\rho_2'\;;\end{aligned} \quad (20)$$

The oscillation equation, Eq. (7), then becomes:

$$\mathcal{L}\boldsymbol{\xi} \equiv (A+\epsilon B)\boldsymbol{\xi} + \epsilon^2(D+\epsilon C)\boldsymbol{\xi} + O(\epsilon^4) = 0 \quad (21)$$

with

$$\begin{aligned}A(\omega,\epsilon) &= L - \rho\,\hat{\omega}^2 \\ B(\omega,\epsilon) &= 2\rho\,\Omega\,\hat{\omega}\,K \\ D(\omega,\epsilon) &= L_2 - \rho_2\,\hat{\omega}^2 \\ C(\omega,\epsilon) &= 2\rho_2\,\Omega\,\hat{\omega}\,K\end{aligned} \quad (22)$$

where, for shortness, we define the operator $K\,\boldsymbol{.} = i\boldsymbol{e_z}\times\boldsymbol{.}$ and $\hat{\omega} \equiv \omega + m\Omega$. Note that $\hat{\omega}$ depends on r for non uniform rotations. The operator A represents the basic linear oscillation operator (which subsists in absence of rotation, then $\hat{\omega} = \omega$); the operators B, D respectively represent the effects of the Coriolis force and of non-spherically symmetric distortion. The operator

C shows that a coupling between the non-spherically symmetric distortion and the Coriolis force exists.

The operators L and L_2 acting on $\boldsymbol{\xi}$ are defined as:

$$\begin{aligned} L\boldsymbol{\xi} &= \nabla p' - \frac{\rho'}{\rho}\,\nabla p + \rho\nabla\phi' \\ &= \nabla p' + \rho'\,\left[\nabla\phi - \tfrac{2}{3}r\Omega^2\boldsymbol{e}_r\right] + \rho\nabla\phi' \end{aligned} \tag{23}$$

and

$$\begin{aligned} L_2\boldsymbol{\xi} &= \nabla p_2' + \frac{\rho'}{\rho}\,\left[\frac{\rho_2}{\rho}\nabla p - \nabla p_2\right] - \frac{\rho_2'}{\rho}\nabla p \\ &+\; \rho_2\nabla\phi' + \rho\nabla\phi_2' + \rho\,\boldsymbol{e_s} r\sin\theta\,\nabla\Omega^2.\boldsymbol{\xi} \end{aligned} \tag{24}$$

We stress that all equilibrium quantities entering Eqs. (19)-(24) must be understood as equilibrium quantities in a pseudo-rotating model. In particular, the pressure gradient ∇p in Eqs. (19)-(23) satisfies Eq. (12). As there is no ambiguity, we have dropped the tilde symbol on these quantities and will do so in the remaining part of the paper. Boundary conditions must be added to the set of Eqs. (18)-(24). The rotation does not modify the way of deriving these boundary conditions and we refer an interested reader to Unno et al., 1989.

Following Unno et al. (1989), we assume that the displacement eigenvector, $\boldsymbol{\xi}$, can be written in terms of a poloidal, $\boldsymbol{\xi}_p$, and a toroidal, $\boldsymbol{\xi}_t$, components, namely:

$$\boldsymbol{\xi} = \boldsymbol{\xi}_p + \boldsymbol{\xi}_t \tag{25}$$

with

$$\begin{aligned} \boldsymbol{\xi}_p &= \textstyle\sum_k r\,(y_k Y_k \boldsymbol{e_r} + z_k \nabla_H Y_k) \\ \boldsymbol{\xi}_t &= \textstyle\sum_k r\,\tau_k\,\boldsymbol{e_r}\times\nabla_H Y_k \end{aligned} \tag{26}$$

where, for shortness we use the subscript k as a single mode identifier standing for (n, l, m), where n is the radial order, l is the spherical harmonic degree and m is the azimuthal order. $Y_l^m(\theta,\varphi)$ is the spherical harmonic with the normalization $\int_0^{2\pi}\int_0^{\pi} |Y_l^m|^2\,\sin\theta d\theta d\varphi = 1$. The operator ∇_H stands for the horizontal divergence, i.e.

$$\nabla_H = \left(\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi\right) \tag{27}$$

Like the eigenfunction $\boldsymbol{\xi}$, the vector $\mathcal{L}\boldsymbol{\xi}$ in Eq. (21) can be split into poloidal and toroidal components. Due to orthogonality, each of the two components of Eq. (21) must be separately set to zero. However, the resulting equations mix the poloidal and toroidal components $\boldsymbol{\xi}_p, \boldsymbol{\xi}_t$. To solve these equations for the eigenmodes and eigenfrequencies, we adopt a perturbative approach. With our zeroth order eigensystem – which will be built in Sect. 4.1– we can assume a simplified expansion of the form:

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{\xi}_0 + \epsilon^2\boldsymbol{\xi}_c + O(\epsilon^4) \\ \omega &= \omega_0 + \epsilon^2\omega_c + O(\epsilon^4), \end{aligned} \tag{28}$$

where the first order (in ϵ) and third order terms are implicitly included in those of zeroth and second orders, respectively.

Note that this formalism enables to obtain eigenfrequencies with the required ϵ^3 accuracy without the computation of eigenfunction corrections at successive orders of ϵ. Throughout the remainder of the paper, the ordering parameter ϵ will be explicitly written only when necessary.

Because each of the operators in Eq. (22) depends on the frequency, they must also be expanded. The operator A for instance expands into:

$$A = A_0 + \epsilon^2 A_c + O(\epsilon^4) \tag{29}$$

and the operator A_0 is the operator A defined in Eq. (22) with the frequency ω set to ω_0. Similar expansions are performed for the operators $B,\ C,\ D$. Expansion of Eq. (21) up to ϵ^3 shows that the only perturbed operator which is needed is:

$$A_c + B_c = -2\,\omega_c\,A_I; \qquad A_I = \rho\,(\hat{\omega}_0 - \Omega K) \tag{30}$$

For later convenience, we also define the operator $\mathcal{L}_0$ such that:

$$\mathcal{L}_0 = A_0 + B_0 \tag{31}$$

The expansions (28-30) are inserted into Eq. (21). The fully expanded equation is given in Eq. (46) below. One has to solve first the zeroth order system for $\boldsymbol{\xi}_0,\ \omega_0$. This is done in Appendix A but the results are summarized in Sect. 4.1 below. In Sect. 4.2, we next solve the system for the eigenfrequency correction ω_c.

4.1. Zeroth order solution

The 'zeroth' order system forms an eigenvalue problem and yields a set of well defined eigenmodes $\boldsymbol{\xi}_0$ associated with well defined eigenfrequencies ω_0. More explicitely, as shown in Appendix A, a zeroth order mode $\boldsymbol{\xi}_0$ is defined by

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_{p0} + \epsilon\,\boldsymbol{\xi}_{t1} \tag{32}$$

The toroidal and poloidal components are obtained as solutions of the system (with associated boundary conditions):

$$\begin{aligned} \left[\mathcal{L}_0\boldsymbol{\xi}_{p0}\right]_{pol} &= 0 \\ \left[\mathcal{L}_0\boldsymbol{\xi}_0 + \epsilon^2 D_0\boldsymbol{\xi}_{p0}\right]_{tor} &= 0 \end{aligned} \tag{33}$$

The subscripts pol and tor mean the poloidal and toroidal components of the respective expression in brackets.

Each mode is characterized by a different set of quantum numbers (n, l, m). Because we include some rotation effects at zeroth order, the usual m-degeneracy occuring in the absence of rotation is already removed at zeroth order here. For each mode of given (n, l, m) indices, the solution of Eq. (33) respectively yields:

• the poloidal part $\boldsymbol{\xi}_{p0}$ which is accurate to $O(\epsilon)$ inclusively and which, as in the case of a non rotating model, is characterized by a single spherical harmonic:

$$\boldsymbol{\xi}_{p0} = r\,\left(y\,Y_l^m + z\,\nabla Y_l^m\right) \tag{34}$$

• the associated toroidal component $\boldsymbol{\xi}_t$ which is accurate up to ϵ^3.

$$\boldsymbol{\xi}_{t1} = r\,\frac{\bar{\Omega}}{\hat{\omega}_0}\,\left(\tau_{l+1}\,\boldsymbol{e_r}\times\boldsymbol{\nabla}Y_{l+1}^m + \hat{\tau}_{l-1}\,\boldsymbol{e_r}\times\boldsymbol{\nabla}Y_{l-1}^m\right) \tag{35}$$

The poloidal radial eigenfunctions, y and z in Eq. (34) are solutions of differential equation system Eqs.(93) of Appendix A. The explicit formulae for toroidal radial eigenfunctions $\tau, \hat\tau$ in terms of y and z are given in Eqs.(96).

With respect to the scalar product defined as:

$$< \boldsymbol{a}|\boldsymbol{b} >= \int \boldsymbol{d^3x}\;(\boldsymbol{a}^*.\boldsymbol{b}) \quad (36)$$

we find that the orthogonality relation involves only the poloidal components of the eigenfuntions; the condition of hermiticity of the operators is automatically satisfied for the toroidal part of the system by construction of the toroidal part of the eigenfunctions $\tau, \hat\tau$.

The orthogonality relation is given by:

$$< \boldsymbol{\xi}_{p0q}|\rho[(\hat\omega_{0k} + \hat\omega_{0q}) - 2\Omega\; K]\boldsymbol{\xi}_{p0k} >= 2\,\omega_{0k}^{(0)}\; I_k\; \delta_{q,k} \quad (37)$$

where $\delta_{q,k}$ is the Kronecker delta for the three indices n, l, m; I_k is the poloidal oscillatory moment of inertia of mode k i.e. dropping the subscript k

$$I =< \boldsymbol{\xi}_{p0}|\rho\boldsymbol{\xi}_{p0} >= \int dr \rho r^4\; (y^2 + \Lambda z^2) \quad (38)$$

with $\Lambda = l(l+1)$. Eq. (37) with $k = q$ defines the frequency $\omega_0^{(0)}$ (see Appendix A). With this definition, one notes that the eigenfrequency ω_0 of Eq. (33) can be rewritten as

$$\omega_0 = \omega_0^{(0)} + m\bar\Omega\,(C_L - 1 - J_1) \quad (39)$$

where

$$\begin{aligned} C_L &= \frac{1}{I}\int dr\rho r^4\;(2yz + z^2) \\ J_1 &= \frac{1}{I}\int dr\rho r^4\;\eta\;(y^2 + (\Lambda-1)z^2 - 2yz) \end{aligned} \quad (40)$$

The second term in Eq. (39) is the usual first order splitting term due to Coriolis force in a classical perturbative approach (DG92).

As we already mentioned, including parts of the effects of Coriolis force in the zeroth order system has the great advantage of our being able to solve the eigenvalue problem up to cubic order without having to solve the successive equations for the eigenfunctions at each order. Another advantage is that it takes partially into account effects of near-degeneracy and thereby simplifies the treatment of near-degenerate cases (see Sect. 5). The price to pay, however, is that the eigenmodes are no longer orthogonal with respect to the scalar product Eq. (36) i.e.

$$\begin{aligned} < \boldsymbol{\xi}_{0q} \mid \rho\boldsymbol{\xi}_{0k} > &= < \boldsymbol{\xi}_{p0q} \mid \rho\boldsymbol{\xi}_{p0k} > + < \boldsymbol{\xi}_{t1q} \mid \rho\boldsymbol{\xi}_{t1k} > \\ &= < \boldsymbol{\xi}_{p0q} \mid \rho\boldsymbol{\xi}_{p0k} > + O(\epsilon^2) \end{aligned} \quad (41)$$

The poloidal scalar product is given by:

$$P_{kq} \equiv < \boldsymbol{\xi}_{p0q} \mid \rho\boldsymbol{\xi}_{p0k} >= \delta_{m_q,m_k}\;\delta_{l_q,l_k}\;I_{kq} \quad (42)$$

with

$$I_{kq} = \int dr\;\rho r^4\;\;(y_k y_q + \Lambda_k z_k z_q) \quad (43)$$

Using the orthogonality relation Eq. (37), the inertia term can also be rewritten as:

$$I_{kq} = \delta_{n_q,n_k}\; I_k + \frac{\overline{\Omega}}{\overline{\omega}_0}(1 - \delta_{n_q,n_k})\;\mathcal{F}_{Lkq} + O(\epsilon^2) \quad (44)$$

where

$$\begin{aligned} \Lambda_k &= l_k(l_k+1); \quad \overline{\omega}_0 = \frac{\omega_{0k} + \omega_{0q}}{2} \\ \mathcal{F}_{kq} &= \int dr\rho r^4\;\;[\;(1+\eta)(y_k z_q + y_q z_k + z_k z_q) \\ &\qquad - \eta(y_k y_q + \Lambda_k z_k z_q)\;] \end{aligned} \quad (45)$$

The toroidal scalar product $< \boldsymbol{\xi}_{t1q} \mid \rho\boldsymbol{\xi}_{t1k} >$ is given in Appendix A.

4.2. Correction to eigenfrequencies

We are now able to proceed with the perturbation procedure. Inserting the expansions Eqs. (28)-(30) into Eq. (21) yields:

$$\begin{aligned} \mathcal{L}\boldsymbol{\xi} &= \mathcal{L}_0\boldsymbol{\xi}_0 + \epsilon^2\mathcal{L}_0\boldsymbol{\xi}_c \\ &+ \epsilon^2[-2\omega_c A_I\boldsymbol{\xi}_0 + (D_0 + \epsilon C_0)\boldsymbol{\xi}_0] + O(\epsilon^4) = 0 \end{aligned} \quad (46)$$

Due to our choice of basis eigenfunctions, $\mathcal{L}_0\boldsymbol{\xi}_0$ does not quite vanish (Appendix A), i.e.

$$\mathcal{L}_0\boldsymbol{\xi}_0 = \epsilon^2\;T_0\boldsymbol{\xi}_0 \quad (47)$$

with

$$T_0\boldsymbol{\xi}_0 = (B_0\boldsymbol{\xi}_{t1})_{pol} - \big(D_0\boldsymbol{\xi}_{p0}\big)_{tor} \quad (48)$$

Eq. (46) becomes:

$$\begin{aligned} \mathcal{L}\boldsymbol{\xi} &= \epsilon^2[\mathcal{L}_0\boldsymbol{\xi}_c - 2\omega_c A_I\boldsymbol{\xi}_0 \\ &\quad +(T_0 + D_0 + \epsilon C_0)\boldsymbol{\xi}_0] + O(\epsilon^4) = 0 \end{aligned} \quad (49)$$

Eq. (49) for mode k is satisfied by imposing

$$< \boldsymbol{\xi}_{0q}|\mathcal{L}\boldsymbol{\xi}_k >= 0 \quad (50)$$

The correction to the eigenfrequency, ω_c, of mode k is obtained from Eq. (50) for $k = q$. Dropping the subscript k, we get:

$$\frac{1}{2I} < \boldsymbol{\xi}_0|\mathcal{L}_0\boldsymbol{\xi}_c > -\omega_c\,\omega_0^{(0)} + \frac{H}{2I} + O(\epsilon^2) = 0 \quad (51)$$

where we have defined

$$H \;\equiv\; < \boldsymbol{\xi}_0|(T_0 + D_0 + C_0)\boldsymbol{\xi}_0 > \quad (52)$$

The first term in Eq. (51) is $O(\epsilon^2)$, hence can be neglected. Indeed,

$$< \boldsymbol{\xi}_0|\mathcal{L}_0\boldsymbol{\xi}_c >=< \mathcal{L}_0\boldsymbol{\xi}_0|\boldsymbol{\xi}_c >= O(\epsilon^2) \quad (53)$$

where the first equality arises from the hermitian property of a oscillating rotating system (Lynden-Bell & Ostriker, 1967) and the second equality comes from Eq. (47).

The correction to the eigenfrequency is therefore:

$$\omega_c = \frac{H}{2I\omega_0^{(0)}} \tag{54}$$

The H term can be simplified. From the zeroth order system, one has (Appendix A):

$$\begin{aligned} &< \boldsymbol{\xi}_0| \quad (B_0\boldsymbol{\xi}_{t1})_{pol} >= \\ &\qquad < \boldsymbol{\xi}_{t1}|(\rho\hat{\omega}_0^2 - B_0)\boldsymbol{\xi}_{t1} > - < \boldsymbol{\xi}_{p0}|D_0\boldsymbol{\xi}_{t1} > \end{aligned} \tag{55}$$

Because

$$< \boldsymbol{\xi}_0|(D_0\boldsymbol{\xi}_{p0})_{tor} >=< \boldsymbol{\xi}_{t1}|D_0\boldsymbol{\xi}_{p0} > \tag{56}$$

we obtain (neglecting $O(\epsilon^4)$ terms):

$$H =< \boldsymbol{\xi}_{t1}|(\rho\hat{\omega}_0^2 - B_0)\boldsymbol{\xi}_{t1} > + < \boldsymbol{\xi}_{p0}|(D_0 + C_0)\boldsymbol{\xi}_{p0} > \tag{57}$$

Finally, the correction for the eigenfrequency up to ϵ^3 for a given mode is written in a very compact form as:

$$\omega_c = \omega^T + \omega^D + \omega^C + O(\epsilon^4) \tag{58}$$

where

$$\begin{aligned} \omega^T &= \frac{1}{2I\omega_0^{(0)}} < \boldsymbol{\xi}_{t1}|(\rho\hat{\omega}_0^2 - B_0)\boldsymbol{\xi}_{t1} > \\ \omega^D &= \frac{1}{2I\omega_0^{(0)}} < \boldsymbol{\xi}_{p0}|D_0\boldsymbol{\xi}_{p0} > \\ \omega^C &= \frac{1}{2I\omega_0^{(0)}} < \boldsymbol{\xi}_{p0}|C_0\boldsymbol{\xi}_{p0} > \end{aligned} \tag{59}$$

The first contribution arises from Coriolis second order effect; the last two correct the frequency for the fact that the oscillation does not occur about a spherically symmetric object but about an objet which is an oblate spheroid: not only the deformation of the structure itself must be taken into account but also the additional Coriolis deviation of the motion that distortion generates. Detailed expressions for $\omega^T, \omega^D, \omega^C$ are given in Appendix B.

Once ω_c is known, the eigenfrequency is simply given by $\omega = \omega_0 + \omega_c + O(\epsilon^4)$ where ω_0 includes the first order correction due to rotation as well as the second order correction due to spherically symmetric distortion as discussed in the previous section. An alternative expression for ω can obtained by collecting contributions of same order ϵ, i.e.:

$$\omega = \omega_0 + \omega_c + O(\epsilon^4) = \omega_0 + \omega_2 + \omega_3 + O(\epsilon^4) \tag{60}$$

where the frequency correction ω_c has been separated into second and third order terms:

$$\begin{aligned} \omega_2 &= \omega_0 \left(\frac{\bar{\Omega}}{\omega_0}\right)^2 (J_2^T + J_2^D) \\ \omega_3 &= m\,\bar{\Omega}\left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \; (J_3^T + J_3^D + J_3^C) \\ &+ \; m\,\omega_2\,\frac{\bar{\Omega}}{\omega_0}\,(C_L - 1 - J_1) \end{aligned} \tag{61}$$

The expressions for the J's are given in Appendix B. The second order correction arises both from the non-spherically symmetric distortion of the star, J_2^D, and from the Coriolis force, J_2^T. The third order correction includes additional contributions due to the coupling between the non-spherically symmetric distortion and the Coriolis force, J_3^C and due to inertia. The explicit cubic contribution affects only the ($m \neq 0$) components of a mode. For these modes, the second order correction, ω_2 also includes some third order effects, although implicitly i.e. through the eigenfunctions. If only second order corrections are required, one can obtain ω_2, free of cubic contamination, by setting $m = 0$ in Eq. 94 and $\alpha = 0$ and $h_1 = 0$ (defined in Eq. 94) in J_2^T, J_2^D.

4.3. *Correction to eigenfunctions*

We assume that the corrections to the eigenfunction $\boldsymbol{\xi}_{ck}$ of mode k are of the usual form:

$$\boldsymbol{\xi}_{ck} = \sum_{j\neq k} \alpha_{kj}\; \boldsymbol{\xi}_{0j} \tag{62}$$

and the coefficients of the expansion, α_{kj} are obtained by requiring that Eq. (50) be satisfied for any q. Because the eigenmodes are not orthogonal with respect ot the scalar product Eq. (36), we do not obtain individual equations for the different α coefficients when $q \neq k$. Instead, they are solution of the linear system:

$$\begin{aligned} &\alpha_{kq} \; + \; \frac{1}{I_q\gamma_{kq}} \; \textstyle\sum_{j\neq k,q} \alpha_{kj} \; (\omega_{0k} - \omega_{0j}) \; I_{kj} \; = \\ &\qquad \frac{1}{I_q\gamma_{kq}} \left(\frac{H_{qk}}{\omega_{0k} - \omega_{0q}} \; - \omega_{ck} \; I_{kq} \right) \end{aligned} \tag{63}$$

with

$$\begin{aligned} \gamma_{kq} \;&=\; \omega_{0k} - \omega_{0q} + 2\omega_{0q}^{(0)} \\ H_{qk} \;&=\; < \boldsymbol{\xi}_{0q}|(T_0 + D_0 + C_0)\boldsymbol{\xi}_0)_k > \end{aligned} \tag{64}$$

A detailed expression for H_{kq} is given in Appendix B. Note that this coefficient describes couplings of eigenmodes. It involves terms of ϵ^2 and ϵ^3 order.

Our purpose here is not to solve Eq. (63) for the α coefficients, but to emphasize in this equation the existence of the denominator $(\omega_{0k} - \omega_{0q})$ which in the case of close frequencies may lead to large values of the perturbation, if $H_{kq} \neq 0$, and thereby invalidate the approach. Existence of close frequencies requires the use of degenerate perturbation theory, as developed in Quantum Mechanics. We describe our application in the next section. Here, we recall that in the context of stellar oscillations the effect of near-degeneracy was first discussed by Chandrasekhar & Lebovitz (1962), who invoked it to explain nonradial oscillations in β Cep stars.

5. Near-degeneracy

For the range of stellar models we are interested in, our numerical applications reveal that near-degeneracy occurs for quite a number of modes. As we show later, rotation couples ($H_{kq} \neq 0$) modes with the same m and l differing by 0 or 2. In the latter case, the normal modes are no longer associated with a single spherical harmonic.

Let us consider a set of N resonant modes coupled by rotation. Near-degeneracy is taken into account by assuming that the zeroth order part of the eigenfunction is a mixture of the eigenfunctions of the N modes determined by Eq. (33) i.e.:

$$\boldsymbol{\xi}_t = \sum_k \mathcal{A}_k \,\boldsymbol{\xi}_{0k} + \epsilon^2 \,\boldsymbol{\xi}_c; \quad k = 1, N \tag{65}$$

with

$$\boldsymbol{\xi} = \sum_k \mathcal{A}_k \,\boldsymbol{\xi}_{0k}; \quad \boldsymbol{\xi}_c = \sum_{j \neq 1,N} \alpha_j \,\boldsymbol{\xi}_{0j} \quad (k = 1, N) \tag{66}$$

The second order correction $\boldsymbol{\xi}_c$ is composed of all the nonresonant modes. The relative contributions, say $\mathcal{A}_2/\mathcal{A}_1$ is not known *a priori* and must be determined.

The eigenfunction $\boldsymbol{\xi}_t$ is associated with an eigenvalue ω ($\hat{\omega} = \omega + m\Omega$), and must satisfy the oscillation equation Eq. (21). We then write:

$$\sum_k \mathcal{A}_k \left[\mathcal{L}\boldsymbol{\xi}_{0k} - \mathcal{L}_{0,k}\,\boldsymbol{\xi}_{0k} + T_{0,k}\,\boldsymbol{\xi}_{0k}\right] + \epsilon^2 \,\mathcal{L}\boldsymbol{\xi}_c = 0 \tag{67}$$

where $\mathcal{L}_{0,k}$ and $T_{0,k}$ are defined in Eq. (31) and Eq. (48) with $\omega = \omega_{0k}$. The projection of Eq. (67) onto the resonant mode $\boldsymbol{\xi}_{0q}$ yields

$$< \boldsymbol{\xi}_{0q} | \mathcal{L}\boldsymbol{\xi}_t >= \sum_k \mathcal{A}_k \left[H_{qk} - \mathcal{Q}_{qk}\right] + O(\epsilon^4) = 0 \tag{68}$$

where H_{qk} was defined in Eq. (64) and

$$\mathcal{Q}_{qk} \equiv$$

$$(\omega - \omega_{0k}) < \boldsymbol{\xi}_{0q} | (\rho + \rho_2) \left[(\hat{\omega} + \hat{\omega}_{0k}) - 2\Omega K\right] \boldsymbol{\xi}_{0k} > \tag{69}$$

Calculations show that the ϵ^2 terms in $\mathcal{Q}_{qk}$ may be neglected. Then Eq. (68) reduces to the following linear system:

$$\mathcal{A}_k \,\beta_{kk} + \sum_{q \neq k} \mathcal{A}_q \,\beta_{kq} = 0 \tag{70}$$

where

$$\begin{aligned} \beta_{kk} &= (\omega_k - \omega)(\omega - \omega_k + 2\omega_{0k}^{(0)})\, I_k \\ \beta_{kq} &= H_{kq} + (\omega_{0k} - \omega)(\omega - \omega_{0q}) P_{kq}; \quad q \neq k \end{aligned} \tag{71}$$

The frequencies $\omega_0^{(0)}, \omega_{k,q}$ are given respectively by Eqs. (39), (60) and ω_0 is the zeroth order eigenfrequency. The coefficients I_k, P_{kq}, H_{kq} were defined in Eqs. (38), (42), (64) respectively.

The existence of nontrivial solutions of Eq. (70) for $\mathcal{A}_k$ implies the following equation for the eigenvalues, ω.

$$det(\beta_{ij}) = 0; \quad i, j = 1, N \tag{72}$$

where det is the determinant of the β_{ij} matrix. The case of no degeneracy is recovered for $N = 1,\ \beta_{11} = 0$. The apparent second solution of $\beta_{11} = 0$ is redundant, it corresponds to the complex conjugate of the mode $(l, n, -m)$.

The frequency differences $\omega_{k,q} - \omega$ remain small compared with $2\omega_{0p}$ and may be neglected in front of this term in β_{kk}. Similarly, the second term in the coefficient β_{kq} in Eq. (71) does not contribute to eigenfrequencies at the level of $O(\epsilon^3)$ and the coupling term H_{kq} then becomes symmetric in k and q. The linear system Eq. (70) then simplifies to:

$$\mathcal{A}_k \,(\omega - \omega_k) + \sum_{q \neq k} \mathcal{A}_q \,\mathcal{H}_{kq}\, R_{kq} = 0; \quad (k, q = 1, N) \tag{73}$$

where we have defined

$$\mathcal{H}_{kq} = \frac{H_{kq}}{2J}; \qquad J = \sqrt{\omega_{0k}^{(0)}\ \omega_{0q}^{(0)}\ I_k I_q} \tag{74}$$

and

$$R_{qk} = \left(\frac{\omega_{0k}^{(0)}}{\omega_{0q}^{(0)}}\ \frac{I_k}{I_q}\right)^{1/2} \sim \sqrt{\frac{I_k}{I_q}} \tag{75}$$

We consider the most frequent case of two interacting resonant modes. The condition (72) then reduces to

$$(\omega_k - \omega)(\omega_q - \omega) - \mathcal{H}^2 = 0 \tag{76}$$

where we dropped subscripts for $\mathcal{H}$.

The solutions of (76), $\omega_\pm$, provide the desired eigenfrequencies i.e.

$$\begin{aligned} \omega_\pm &= \bar{\omega} \pm \sqrt{(\frac{\Delta\omega}{2})^2 + \mathcal{H}^2} + O(\epsilon^4) \\ \bar{\omega} &= \frac{\omega_k + \omega_q}{2}; \quad \Delta\omega = \omega_k - \omega_q \end{aligned} \tag{77}$$

The coupling term, $\mathcal{H}$, can be written as:

$$\mathcal{H} = \delta_{l_k,l_q}\,\mathcal{H}_1 + \delta_{l_k,l_q+2}\,\mathcal{H}_{2,kq} + \delta_{l_k,l_q-2}\,\mathcal{H}_{2,qk} \tag{78}$$

where the quantities $\mathcal{H}_1, \mathcal{H}_{2,kq}$ are lengthy expressions involving the eigenfunctions (Appendix B). The $\mathcal{H}$'s are proportional to integrals of $Y_k Y_q P_2$, which vanish whenever the azimuthal indices differ, $m_k \neq m_q$. Regardless of the frequency separation, near-degenerate coupling therefore only occurs between modes with either the same degree l (and different radial orders) or with degrees which differ by 2. The first case involves modes in avoided crossing, a familiar phenomenon occuring when the convecting core recedes during main sequence evolution (see Unno et al., 1989 and references therein). Photometrically detectable cases from ground-based observations are likely to be $(l_k, l_q) = (1, 1)$ and $(l_k, l_q) = (2, 2)$, perhaps even $(l_k, l_q) = (3, 3)$. However this concerns only some specific modes in a given frequency spectrum for a given star at a given age. In the second case, on the other hand, all pairs of modes $(l = 0, 2)$ and $(l = 1, 3)$ with respective radial orders $n, n-1$ have close enough frequencies (small separations) that they are systematically coupled if the rotation is rapid enough.

The eigenfrequencies corrected for near-degeneracy depend on the frequency separation $\Delta\omega$. Note that $\Delta\omega$ includes all effects of rotation except those due to rotational coupling. Let us define the parameter $s = \frac{\Delta\omega}{2\mathcal{H}}$, which measures the ratio of the proximity of the frequencies to the strength of the coupling. Resonances play a role whenever s is of order unity or smaller. With no loss of generality, we consider the case $\omega_k > \omega_q$ and $s > 0$. The associated eigenfunctions are given by:

$$\begin{aligned} \boldsymbol{\xi}_+ &= \mathcal{A}_k \left(\boldsymbol{\xi}_{0k} - R_{qk}(s - \sqrt{s^2+1})\, \boldsymbol{\xi}_{0q}\right) \\ \boldsymbol{\xi}_- &= \mathcal{A}_q \left(\boldsymbol{\xi}_{0q} + R_{kq}(s - \sqrt{s^2+1})\, \boldsymbol{\xi}_{0k}\right) \end{aligned} \tag{79}$$

where the amplitudes $\mathcal{A}_k, \mathcal{A}_q$ are arbitrary and can be set by normalizing the eigenfunctions to 1 at surface. Note that the eigenfunctions involve the ratio of inertia of the two modes, R_{kq} (Eq. 75).

Away from resonance, when $\Delta\omega$ is large and $s >> 1$, one recovers the case of two uncoupled modes:

$$\begin{aligned} \omega_+ &= \omega_k + O(\frac{1}{s^2}) = \omega_k + O(\epsilon^4) \\ \omega_- &= \omega_q + O(\frac{1}{s^2}) = \omega_q + O(\epsilon^4) \end{aligned} \tag{80}$$

associated with the eigenfunctions

$$\boldsymbol{\xi}_+ = \boldsymbol{\xi}_{0k} + O(\tfrac{1}{s}); \quad \boldsymbol{\xi}_- = \boldsymbol{\xi}_{0q} + O(\tfrac{1}{s}) \tag{81}$$

For near-degenerate modes, $\omega_k - \omega_q \sim \mathcal{H}$ and $s \sim 1$. Then, the coupling is efficient and the mode $\boldsymbol{\xi}_+$ ($\boldsymbol{\xi}_-$) is contaminated by mode $\boldsymbol{\xi}_{0q}$ ($\boldsymbol{\xi}_{0k}$). The frequency separation is:

$$\omega_+ - \omega_- = \sqrt{(\Delta\omega)^2 + 4\mathcal{H}^2} > |\Delta\omega| = |\omega_k - \omega_q| \tag{82}$$

Hence, the resonance pushes apart frequencies which get too close to each other. The distribution of frequencies in an observed power spectrum can therefore be much different than one would expect, when near-degeneracies are overlooked.

For the extreme case $s = O(\epsilon)$ ($\Delta\omega = O(\epsilon^3)$), Eq. (77) shows that the frequencies depart considerably from the corresponding isolated mode frequencies. In this case, the eigenfunctions become:

$$\begin{aligned} \boldsymbol{\xi}_+ &= \mathcal{A}_k(\boldsymbol{\xi}_{0k} + R_{qk}(1-s)\, \boldsymbol{\xi}_{0q}) + O(s^2) \\ \boldsymbol{\xi}_- &= \mathcal{A}_q(\boldsymbol{\xi}_{0q} - R_{kq}(1-s)\, \boldsymbol{\xi}_{0k}) + O(s^2) \end{aligned} \tag{83}$$

From Eq. (83), it can be seen that the eigenfunctions of the two modes are superpositions of the uncoupled eigenfunctions with nearly equal amplitudes. If one of the modes was radial, it is no longer even approximately radial. This was the case considered by Chandrasekhar & Lebovitz (1962). Even then, the nonradial nature of some β Cep pulsation modes was firmly established. However, no excitation mechanism was envisioned for radial nor for nonradial modes but it was believed that radial modes were more easily driven.

In general, if coupled modes have different l values, the surface amplitude is given by a superposition of the two spherical harmonics. This has important consequences when visibility arguments are used as hints for a mode identification. For the same intrinsic amplitude, $l = 0$ modes are expected to have larger observed amplitudes than $l = 2$ modes due to geometric averaging effects. However, if the frequencies are close, both modes will have similar observed amplitudes as they both include contributions from $l = 0$ and $l = 2$ spherical harmonics in almost equal proportions. Also, nominally $l = 3$ modes may become more easily detectable mainly through their $l = 1$ contamination.

In cases of nearly equal contributions from different ls, problems of l-value assignement arise. We still need three quantum numbers to identify a mode for asteroseismological applications. For that purpose, we must make sure that the frequency and its derivative are continuous functions of the stellar parameters. The l value must therefore be assigned in such a way that the rotational resonant coupling does not change the ordering of frequencies. In all cases studied so far such an assignement agrees with the one based on relative contributions to mode inertia.

One piece of bad news is that this coupling may limit applicability of the mode identification method based on two-colour photometry. Positions corresponding to such modes in the light to colour amplitude ratio *versus* phase difference diagrams depend on the inclination of rotation axis i. These diagrams are our best tools for determining the l-values of observed modes. If we do not know i, which is almost always the case, these diagrams are useless for coupled modes. For instance, a mode composed of $l = 0$ and $l = 2$ components may appear in the $l = 0$, 1 or 2 domain of the diagram, depending on the value of i. Fortunately the $m = \pm 1$ and ± 2 are not affected. The second piece of bad news is that the interaction with $l = 2$ modes has a significant effect on the $l = 0$ period ratios. They are quite sensitive to Ω and are therefore less useful as a hint for mode identification.

A final word concerns triple mode near-degenerate couplings. These are frequent enough that they must be taken into account. Triple mode coupling will occur for instance when two $l = 2$ modes undergo an avoided-crossing. They then couple with each other as well as with the neighboring radial mode. Frequencies of three near-degenerate modes are solutions of a cubic equation easily derived from Eq. (73).

6. Summary

In Sects 2 to 5 we derived formulae for calculations of frequency shifts caused by rotation for the case when the angular velocity Ω depends only on radius. Our formulae are accurate up to ϵ^3, where ϵ is the ratio of the mean rotation rate, $\bar{\Omega}$, to oscillation frequency ω. The formulae are very lengthy and the longest ones are given in the Appendices. Because applying these formulae is involved, we provide here a guide after some comments.

One may wonder whether it would not be better to abandon the perturbation approach which leads to complicated expressions and instead rely on two-dimensional calculations of models and their oscillations. After all, at equatorial velocities exceeding some 150 km/s, even our third order theory is likely to be inadequate in certain circumstances. We feel perturbation theory calculations are still useful, because coding the formulae is in fact quite straightforward — far easier than deriving them. In contrats, it is highly nontrivial to achieve a 10^{-3} precision in frequency calculations with a 2-D hydrocode, were one available. Undoubtedly, the use of such codes will ultimately be unavoidable, but then it will be very helpful to have a code based on the perturbational approach for comparisons at moderate equatorial velocities where both are valid.

In most of previous treatments of the effect of rotation on oscillations, perturbation methods are valid for cases of no degeneracy. We point out that even at very low equatorial velocities, such an approach is inadequate in certain cases. Near-degeneneracies, hence coupling by rotation, of the $l = 0$ and $l = 2$ modes occur systematically and are most important for p-modes of higher orders. A similar, though less severe problem, occurs for the $l = 1$ and 3 pairs. Occasionally, close pairs with same l exist. This phenomenon– called avoided crossing– occurs as a consequence of the development of a g-mode propagation zone outside the shrinking convective zone. We provide formulae for evaluation of frequencies for all these cases as well as formulae for the relative amplitudes of the components. Modes composed of different spherical harmonic components with comparable amplitudes at the surface present a problem in asteroseismic diagnosis. As we just discussed at the end of previous section, they are difficult to identify in observed frequency spectra.

Now we give a guide to our formulae, assuming that the reader has standard stellar evolution and oscillation codes. In the evolution code, the spherically-symmetric effect of centrifugal force must be included and assumptions concerning angular momentum evolution must be explicitely introduced. In our code, we assume uniform rotation and global angular momentum conservation, but this is easy to change. Modification of the stellar structure equations consists only in adding a -$2\Omega^2 r/3$ term to the gravity. The corresponding modifications of the equations for adiabatic oscillations are given in Eqs.(93) of Appendix A. Eqs.(93) take also into account linear effects of rotation (terms involving asimuthal order m). Thus, they yield dimensionless frequencies, $\sigma = \omega/\sqrt{GM/R^3}$, which are already accurate up to $O(\epsilon)$.

The corresponding modifications in the equations for nonadiabatic oscillations are easy to introduce. Throughout this paper we use adiabatic approximation, however the nonadiabatic corrections are not negligible and we add them to frequencies assuming that the rotation does not change them.

For cases when near-degeneracy effects are negligible the formulae for ω accurate up to $O(\epsilon^3)$ are given by Eqs.(62) and (63). In Appendix B, we give expressions for all the J's ocurring in Eq. (63) in terms of radial eigenfunctions obtained as solutions of Eqs. (93) as well the radial eigenfunctions for the toroidal components of the displacement given in Eqs.(96).

Eq. (77) gives an explicit expression for frequencies in the two-mode degeneracy case. The relative contributions of the two components, R_{kq} for the two modes are given in Eqs. (79). If the number of near-degenerate modes is higher than two then we have to rely on the numerical solution of the linear equation system (Eq. (70)). Our rule of mode assignment in the case of degeneracy is that the coupling does not change ordering of frequencies. Then, for fixed mode, ω is always a continous function of parameters of stellar models.

In a subsequent paper we will present results of the formalism developed here for δ Scuti and β Cephei stars. We will determine the ranges of applicability of our formalism in terms of equatorial rotational velocity. It will cover all low degree modes that may be excited in these two types of pulsating stars.

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Appendix A

This appendix details how we build our zeroth order eigensystem. We recall first that for any vector $\boldsymbol{f}$ which can be decomposed into poloidal and toroidal parts i.e.

$$\boldsymbol{f} \;=\; \sum_k (f_{rk} Y_k \boldsymbol{e_r} + f_{Hk} \boldsymbol{\nabla_H} Y_k + f_{Tk} \boldsymbol{e_r} \times \boldsymbol{\nabla_H} Y_k) \quad \text{(A1)}$$

the condition $\boldsymbol{f} = 0$ is satisfied whenever

$$\begin{aligned} \int d\underline{\Omega}\;\; Y_q^*\; \boldsymbol{e_r}.\boldsymbol{f} \;&=\; 0 \\ \int d\underline{\Omega}\;\; Y_q^*\; \boldsymbol{\nabla_H}.\boldsymbol{f} \;&=\; 0 \\ \int d\underline{\Omega}\;\; Y_q^*\; \boldsymbol{e_r}.\boldsymbol{\nabla} \times \boldsymbol{f} \;&=\; 0 \end{aligned} \quad \text{(A2)}$$

with $d\underline{\Omega} = r^2 \sin\theta \; d\theta d\varphi$.

In building the zeroth order eigensystem, we assume an eigenfunction of the form (Unno et al., 1989):

$$\begin{aligned} \boldsymbol{\xi}_0 \;&=\boldsymbol{\xi}_{p0} + \boldsymbol{\xi}_{t1} \\ \boldsymbol{\xi}_{p0} \;&= \sum_k (r y_k Y_k \boldsymbol{e_r} + r z_k \boldsymbol{\nabla_H} Y_k) \\ \boldsymbol{\xi}_{t1} \;&= \sum_k \frac{\overline{\Omega}}{\hat{\omega}} r \tau_k \; \boldsymbol{e_r} \;\times \boldsymbol{\nabla_H} Y_k \end{aligned} \quad \text{(A3)}$$

The poloidal part $\boldsymbol{\xi}_{p0}$ is obtained in such a way that it decouples from the toroidal part, that-is by applying Eq. (85a,b) with

$$\boldsymbol{f} = A_0 \; \boldsymbol{\xi}_0 + B_0 \; \boldsymbol{\xi}_{p0} \quad \text{(A4)}$$

This is equivalent to writing:

$$[\mathcal{L}_0 \boldsymbol{\xi}_{p0}]_{pol} = 0 \quad \text{(A5)}$$

with $\mathcal{L}_0$ defined in Eq. (31).

The toroidal part is obtained by applying Eq. (85c) with

$$\boldsymbol{f} = (A_0 + B_0)\boldsymbol{\xi}_0 + D_0 \boldsymbol{\xi}_{p0} \quad \text{(A6)}$$

This is equivalent to writing:

$$[\mathcal{L}_0 \boldsymbol{\xi}_0 + D_0 \boldsymbol{\xi}_{p0}]_{tor} = 0 \quad \text{(A7)}$$

and results in the following relation:

$$(B_0 \boldsymbol{\xi}_0)_{tor} + (D_0 \boldsymbol{\xi}_{p0})_{tor} = \rho \hat{\omega}_0^2 \; \boldsymbol{\xi}_{t1} \quad \text{(A8)}$$

A.1. Poloidal eigenfunctions

Eq. (85a,b) with Eq. (87) along with the appropriate linearized continuity, energy and Poisson equations, Eqs. (19)-(20), yield the components y, z of $\boldsymbol{\xi}_0$ (see Eq. (34)). It is convenient to define the dimensionless variables y_t, v and w as :

$$y_t = \frac{1}{g_e r}\left(\tilde{\phi}' + \tilde{p}'/\tilde{\rho}\right) \qquad v = \frac{\tilde{\phi}'}{g_e r} \qquad w = \frac{1}{g_e}\frac{d\tilde{\phi}'}{dr} \tag{A9}$$

with g_e defined in Eq. (13). One obtains:

$$\begin{aligned} r\frac{dy}{dr} &= (V_g - 3 + h_1)\, y + (\zeta - V_g)\, y_t + V_g v \\ &= \lambda - 3y + \Lambda z \\ r\frac{dy_t}{dr} &= \left(C_r\hat{\sigma}^2 - A + \delta_{l0}U\right) y \\ &+ (A + 1 - U - \chi - h_1)\, y_t - Av \\ r\frac{dv}{dr} &= (1 - U - \chi)\, v + w \\ r\frac{dw}{dr} &= \frac{UA}{1-\sigma_r} y + \frac{V_g U}{1-\sigma_r} y_t + \frac{\Lambda - V_g U}{1-\sigma_r} v - (U + \chi)\, w \end{aligned} \tag{A10}$$

where $\sigma^2 = \dfrac{\omega^2 R^3}{GM}$ is the square of the dimensionless eigenfrequency and

$$\begin{aligned} \hat{\sigma} &= \sigma + m\,\sigma_\Omega; \qquad \sigma_\Omega = \frac{\Omega}{(GM/R^3)^{(1/2)}} \\ A &= \frac{1}{\Gamma_1}\frac{d\ln\tilde{p}}{d\ln r} - \frac{d\ln\tilde{\rho}}{d\ln r},\ V = -\frac{d\ln\tilde{p}}{d\ln r}, \\ V_g &= V/\Gamma_1,\ U = \frac{d\ln M_r}{d\ln r},\ \sigma_r = F_r(r)/\tilde{g}, \\ C &= \frac{(r/R)^3}{(M_r/M)},\ C_r = \frac{C}{1-\sigma_r}, \\ \lambda &= V_g(y - y_t + v) \\ \chi &= \frac{F_r(r)}{g_e}\left(U - 3 - \frac{d\ln\Omega^2}{d\ln r}\right) \\ \alpha &= 2\,m\,\frac{\sigma_\Omega}{\hat{\sigma}} \quad \Lambda = l(l+1) \\ \zeta &= \frac{\Lambda}{\Lambda - \alpha}\,\frac{\Lambda}{C\hat{\sigma}^2},\ h_1 = \frac{\Lambda\,\alpha}{\Lambda - \alpha} \end{aligned} \tag{A11}$$

with R, and M being the radius and the mass of the pseudo-rotating model and $F_r(r)$ defined in Eq. (5). The component z is related to y and y_t by:

$$\Lambda z - \zeta y_t - h_1 y = 0 \tag{A12}$$

For radial modes ($l = 0$), the system Eq. (93) reduces to the first two equations with $y_t = \tilde{p}'/(g_e r p)$ and $v = 0$. The eigensystem Eq. (93) closely resembles the eigensystem obtained for a non rotating star (Unno et al., 1989). The effect of rotation occurs through χ and σ_r, σ_Ω. One indeed recovers the non rotating oscillating system by setting $\Omega = 0$. Together with boundary conditions, the system Eq. (93) provides the eigenfrequency, ω_0, and the components y, z for each mode of pseudo-rotating models.

A.2. Toroidal eigenfunctions

Eq. (85c) with Eq. (89) yields an expression for the radial parts $\tau, \hat{\tau}$ of the toroidal component of the mode viz. for a mode k

$$\begin{aligned} \tau_{k+1} &= i\frac{\beta_{k+1}}{(\Lambda_{k+1} - \alpha_k)}\left(2P_k + 3m\,\frac{\bar{\Omega}}{\hat{\omega}_0}d_k\right) \\ \hat{\tau}_{k-1} &= i\frac{\beta_k}{(\Lambda_{k-1} - \alpha_k)}\left(2\hat{P}_k + 3m\frac{\bar{\Omega}}{\hat{\omega}_0}\,d_k\right) \end{aligned} \tag{A13}$$

with

$$\begin{aligned} P_k &= (1+\eta)\,(l_k + 2)\,(-y_k + l_k z_k) \\ \hat{P}_k &= (1+\eta)\,(l_k - 1)\,(y_k + (l_k + 1) z_k) \end{aligned} \tag{A14}$$

and

$$\begin{aligned} \beta_k &= \sqrt{\frac{(l_k^2 - m^2)}{4l_k^2 - 1}} \\ d_k &= (v_k + y_k - C\hat{\sigma}_k^2 z_k)\,\left(\frac{d\ln\rho}{d\ln r}u_2 + b_2\right) + \lambda_k u_2 \end{aligned} \tag{A15}$$

For brevity, we use τ_{k+1} as shorthand for τ_{n_k, l_k+1, m_k}; similarly $\hat{\tau}_{k-1}$ stands for $\hat{\tau}_{n_k, l_k-1, m_k}$. On the other hand, subscript k applied to y, z, d means the whole set (l_k, n_k, m_k). Λ_k and α_k are defined in Eq. (94).

A.3. Useful relations

Because of the way our basis of zeroth order eigenfunctions is built, $\mathcal{L}_0\boldsymbol{\xi}_0$ vanishes only up to order ϵ. Indeed,

$$\begin{aligned} \mathcal{L}_0\boldsymbol{\xi}_0 &= [\mathcal{L}_0\boldsymbol{\xi}_{p0}]_{pol} + [\mathcal{L}_0\boldsymbol{\xi}_{t1}]_{pol} + [\mathcal{L}_0\boldsymbol{\xi}_0]_{tor} \\ &= \epsilon^2\left((B_0\boldsymbol{\xi}_{t1})_{pol} - (D_0\boldsymbol{\xi}_{p0})_{tor}\right) \end{aligned} \tag{A16}$$

From the hermiticity of the operators, we have:

$$\begin{aligned} < \boldsymbol{\xi}_0|(B_0\boldsymbol{\xi}_{t1})_{pol} > &= < \boldsymbol{\xi}_{p0}|B_0\boldsymbol{\xi}_{t1} > = < B_0\boldsymbol{\xi}_{p0}|\boldsymbol{\xi}_{t1} > \\ &= < (B_0\boldsymbol{\xi}_{p0})_{tor}|\boldsymbol{\xi}_{t1} > \end{aligned} \tag{A17}$$

From the toroidal part of the oscillating equation Eq. (91), one can write:

$$\begin{aligned} &< \boldsymbol{\xi}_0|(B_0\boldsymbol{\xi}_{t1})_{pol} > \\ &\quad = < (\rho\hat{\omega}_0^2 - B_0)\boldsymbol{\xi}_{t1}|\boldsymbol{\xi}_{t1} > - < D_0\boldsymbol{\xi}_{p0}|\boldsymbol{\xi}_{t1} > \\ &\quad = < \boldsymbol{\xi}_{t1}|(\rho\hat{\omega}_0^2 - B_0)\boldsymbol{\xi}_{t1} > - < \boldsymbol{\xi}_{p0}|D_0\boldsymbol{\xi}_{t1} > \end{aligned} \tag{A18}$$

A.4. Orthogonality

The Hermiticity properties of the system require

$$< \boldsymbol{\xi}_{0q}|\mathcal{L}\boldsymbol{\xi}_{0k} >=< \mathcal{L}\boldsymbol{\xi}_{0q}|\boldsymbol{\xi}_{0k} > \quad \text{(A19)}$$

where the operator $\mathcal{L}$ (Eq. 21) is defined with ω_{0k} (resp. ω_{0q}) when applied to $\boldsymbol{\xi}_{0k}$ (resp.$\boldsymbol{\xi}_{0q}$). This yields the orthogonality relation Eq. (37). The detailed expression is:

$$\begin{aligned} \int dr \quad \rho r^4 \ [(\hat\omega_{0k}+\hat\omega_{0q})(y_k y_q+\Lambda_k z_k z_q) \\ -2m\Omega(y_q z_k + y_k z_q + z_k z_q)] = 2\ \omega_{0k}^{(0)}\ I_k\ \delta_{q,k} \end{aligned} \quad \text{(A20)}$$

where

$$\begin{aligned} \omega_0^{(0)} &= \frac{1}{I} < \boldsymbol{\xi}_{p0}|A_I \boldsymbol{\xi}_{p0} >= \frac{1}{I} < \boldsymbol{\xi}_{p0}|\rho[\hat\omega_0 - \Omega\ K]\boldsymbol{\xi}_{p0} > \\ &= \frac{1}{I}\int dr\ \rho r^4\ \left[\hat\omega_0(y^2+\Lambda z^2) - m\Omega(2yz+z^2)\right] \end{aligned} \quad \text{(A21)}$$

The toroidal scalar product is given by:

$$\begin{aligned} &< \boldsymbol{\xi}_{t1q}\mid \rho\boldsymbol{\xi}_{t1k} >= \int \boldsymbol{d^3x}\ \rho\ \left(\boldsymbol{\xi}^*_{t1q}.\boldsymbol{\xi}_{t1k}\right) \\ &= \delta_{mq,mk}\ (\frac{\bar\Omega}{\omega_0})^2 \\ &\left[\delta_{lq,lk}\ K_{1kq} + \delta_{l_k,l_{q+2}}\ K_{2,k,q} + \delta_{l_k,l_{q-2}}\ K^*_{2,q,k}\right] \end{aligned} \quad \text{(A22)}$$

where the quantities K are defined in Appendix 2 (below).

Appendix B

In this appendix, we derive expressions for the various coefficients entering the eigenfrequencies. The coupling coefficient $\mathcal{H}_{kq}$ defined in Eq. (74) and Eq. (64) can be rewritten as :

$$\mathcal{H}_{kq} = \mathcal{T}_{kq} + \mathcal{D}_{kq} + \mathcal{C}_{kq} + \delta\omega_{kq}\ \mathcal{E}_{kq} + O(\epsilon^4) \quad \text{(B1)}$$

where

$$\begin{aligned} \mathcal{T}_{kq} &= \frac{1}{2J} < \boldsymbol{\xi}_{p0k}|(\overline{B}_0+\overline{D}_0)\boldsymbol{\xi}_{t1q} > \\ \mathcal{D}_{kq} &= \frac{1}{2J} < \boldsymbol{\xi}_{p0k}|\overline{D}_0\boldsymbol{\xi}_{p0q} > \\ \mathcal{C}_{kq} &= \frac{1}{2J} < \boldsymbol{\xi}_{p0k}|\overline{C}_0\ \boldsymbol{\xi}_{p0q} > \end{aligned} \quad \text{(B2)}$$

where the operators are defined as in Eq. (22) with the frequency $\omega = \overline{\omega_0} = (\omega_{0k}+\omega_{0q})/2$. It can be shown that the last term in $\mathcal{H}_{kq}$ is always negligible when the desired frequency accuracy is $O(\epsilon^3)$. In the explicit expressions of the coefficients entering $\mathcal{H}_{kq}$, below, only $\overline{\omega_0}$ appears and so we drop the bar over $\overline{\omega_0}$. Similarly, σ_0^2 must be understood as $(\sigma_{0k}^2+\sigma_{0q}^2)/2$. For shortness also, the subscripts k,q labelling the coefficients and representing modal indices are omitted in symmetric expressions whenever there is no ambiguity. The azimuthal indices are not specified, because couplings only occur between modes with the same m values. The reason is that the scalar products which define the various coupling contributions all generate either angular integrals over products of two spherical harmonics i.e. a Kronecker delta or integrals over products of $Y_k Y_q P_2$ (see Eq. (123)).

B.1. 'Coriolis' contribution $\mathcal{T}_{kq}$

The scalar product $\mathcal{T}$ is straighforwardly computed with the use of Eqs. (15), (24), (30), (34), (35) for the eigenfunctions and the definition of the Coriolis and distortion operators. Recall that some effects of distortion are already included in the toroidal zeroth order system. So distortion indirectly contributes through the toroidal part of the eigenfunctions (i.e. through the components τ). As a result, one obtains:

$$\mathcal{T}_{kq} = \delta_{l_q,l_k}\ T^{(1)} + \delta_{l_k,l_q+2}\ T^{(2)}_{kq} + \delta_{l_k,l_q-2}\ T^{(2)*}_{qk} \quad \text{(B3)}$$

The quantities $T^{(j)}$ can be formally split into quadratic and cubic order contributions:

$$\begin{aligned} T^{(1)} &= \left(\frac{\overline\Omega}{\omega_0}\right)^2\ \left(\omega_0\ K_1 - m\bar\Omega\ K_3\right) \\ T^{(2)}_{kq} &= \left(\frac{\overline\Omega}{\omega_0}\right)^2\ \left(\omega_0\ K_{2kq} - m\overline\Omega\ K_{4kq}\right) \end{aligned} \quad \text{(B4)}$$

with

$$\begin{aligned} K_1 &= \frac{\omega_0}{2J}\int dr\rho r^4\ \left(\Lambda_{k+1}\tau^*_{k+1}\tau_{q+1} + \Lambda_{k-1}\hat\tau^*_{k-1}\hat\tau_{q-1}\right) \\ K_{2kq} &= \frac{\omega_0}{2J}\int dr\rho r^4\ \Lambda_{k-1}\ \hat\tau^*_{k-1}\tau_{q+1} \\ K_3 &= \frac{\omega_0}{J}\int dr\rho r^4\ (1+\eta)\ \left(\tau^*_{k+1}\tau_{q+1} + \hat\tau^*_{k-1}\hat\tau_{q-1}\right) \\ K_{4kq} &= \frac{\omega_0}{J}\int dr\rho r^4\ (1+\eta)\ \hat\tau^*_{k-1}\tau_{q+1} \end{aligned} \quad \text{(B5)}$$

The case $k \neq q$ represents a coupling between near-degenerate modes. Eq(108) shows that the coupling occurs between modes either with same degree l (then different radial orders) or with degrees that differ by 2. On the other hand, the coefficient, $\mathcal{T}_{kk}$, yields the frequency correction, ω^T, of an isolated mode (Eq. 59)

$$\omega^T = \left(\frac{\bar\Omega}{\omega_0}\right)^2\ \left(\omega_0 J_2^T - m\bar\Omega J_3^T + m\bar\Omega J_2^T(C_L - 1 - J_1)\right) \quad \text{(B6)}$$

with

$$\begin{aligned} J_2^T &= \frac{1}{2I}\ \int dr\rho r^4\ \left(\Lambda_{k+1}|\tau_{k+1}|^2 + \Lambda_{k-1}|\hat\tau_{k-1}|^2\right) \\ J_3^T &= \frac{1}{I}\int dr\rho r^4\ (1+\eta)\ \left(|\tau_{k+1}|^2 + |\hat\tau_{k-1}|^2\right) \end{aligned} \quad \text{(B7)}$$

B.2. Non-spherically symmetric distortion $\mathcal{D}_{kq}$

The coefficient $\mathcal{D}_{kq}$ (Eqs. 107-22-24) is first turned into an expression symmetric in the eigenfunctions $\boldsymbol{\xi}_{0q}, \boldsymbol{\xi}_{0k}$ (dropping the 0 subscripts):

$$\begin{aligned} \mathcal{D}_{kq} = \quad & \frac{1}{2J}\int \boldsymbol{d^3x}\ [\ (\Gamma_1 p)_2 \boldsymbol{\nabla}.\boldsymbol{\xi}^*_k\ \boldsymbol{\nabla}.\boldsymbol{\xi}_q \\ &+\ \rho_2[\boldsymbol{\xi}_q.\boldsymbol{\nabla}\phi^{'*}_k + \boldsymbol{\xi}^*_k.\boldsymbol{\nabla}\phi^{'*}_q] \\ &-\ \frac{1}{\rho}\boldsymbol{\nabla}p_2.(\boldsymbol{\xi}_q\rho^{'*}_k + \boldsymbol{\xi}^*_k\rho^{'}_q) \\ &+\ \rho r^2 W - \rho_2\hat\omega_0^2\boldsymbol{\xi}^*_k.\boldsymbol{\xi}_q\ ] \end{aligned} \quad \text{(B8)}$$

$$W = y_k y_q Y_k^* Y_q \left(P_2 w_1 + (1 - P_2) \frac{2}{3} r \frac{d\Omega^2}{dr} \right) - \frac{1}{3} r \frac{d\Omega^2}{dr} \left(y_k z_q Y_k^* \boldsymbol{\nabla}_H Y_q + y_q z_k Y_q \boldsymbol{\nabla}_H Y_k^* \right) . \boldsymbol{\nabla}_H P_2 \quad \text{(B9)}$$

with P_2 being the second order Legendre polynomial and

$$w_1 = \frac{d}{dr}\left(\frac{1}{\rho}\right)\frac{dp_{22}}{dr} + \frac{1}{\rho}\frac{d}{dr}\left(\frac{\rho_{22}}{\rho}\right)\frac{dp}{dr} \quad \text{(B10)}$$

The detailed expression of $\mathcal{D}_{kq}$ involves derivatives of density which tend to magnify surface effects and can introduce inaccuracies in the computation of the frequencies when the atmosphere is poorly described. This is particularly true for high overtones which are more concentrated toward the outer layers. Density derivatives are therefore eliminated by means of integrations by parts and with the use of the following relations:

$$\frac{d\Gamma_1 p\lambda}{dr} = -\rho\, g\, q; \qquad q = c - y_t h_1$$
$$c = y\,(C\hat{\sigma}^2 + 4 - U) - \Lambda z - w \quad \text{(B11)}$$

As a result of integrations by parts, one finally obtains:

$$\mathcal{D}_{kq} = \omega_0 \left(\frac{\bar\Omega}{\omega_0}\right)^2 \left(\delta_{l_k,l_q}\, \mathcal{D}_1 + \mathcal{Q}_{kq2}\,(\mathcal{D}_2 + 2m\frac{\overline{\Omega}}{\omega_0}\mathcal{D}_3) \right) \quad \text{(B12)}$$

with

$$\mathcal{D}_1 = \frac{\omega_0}{2J}\int dr\rho r^4\, y_k y_q b_2$$
$$\mathcal{D}_2 = \frac{\omega_0}{2J}\int dr\rho r^2\, \frac{1}{2}(D_{qk} + D_{kq}) \quad \text{(B13)}$$
$$\mathcal{D}_3 = \frac{\omega_0}{2J}\int dr\rho r^2\, (D_{3kq} + D_{3qk}) d_1$$

where

$$D_{3kq} = \lambda_k y_{t_q} + \frac{1}{2} y_k y_{t_q}(U-4) + \frac{1}{2} C\sigma_0^2\, \left(y_k y_q + (\bar\Lambda - 3) z_k z_q\right)\, \frac{d\ln\Omega}{d\ln r} - \frac{1}{\Lambda_q}(\bar\Lambda - 3)(z_q + y_q) C\sigma_0^2 (y_k - 2z_k) + \frac{1}{\Lambda_q}(\bar\Lambda - 3)(z_q + y_q)\frac{A}{V_g}\lambda_k + (\bar\Lambda - 3) z_k y_{2q} \quad \text{(B14)}$$

$$D_{qk} = -(d_1 F_1 + d_2 F_2 + r^2 b_2 F_3 + r^2 b_3\, y_k s_q)$$

and

$$F_1 = y_k y_q a_1 + \lambda_k a_2 + a_3 y_k + a_4 z_k + a_5 z_k z_q + a_6 y_{tq}$$
$$F_2 = y_k y_q (U - C\sigma_0^2) + y_q[2w_k + (4 - 2U)s_k - q_k] + z_q[\Lambda_q(v_k - y_k) + (\Lambda_k - 6) y_{tk}] - C\sigma_0^2(\bar\Lambda - 3)\, z_k z_q$$
$$F_3 = C\sigma_0^2(y_q y_k + z_q z_k(\bar\Lambda - 3)) + y_q s_k (U + 6) + (\Lambda_k - \Lambda_q + 6) z_k y_q + (6 - 2\bar\Lambda) z_k v_q - 2 y_q w_k - s_k \lambda_q \left(\frac{\partial \ln\Gamma_1}{\partial \ln\rho}\right)_p \quad \text{(B15)}$$

The a_j quantities in F_1 above are defined as:

$$a_1 = 6 - U\left(U + 2\,\frac{d\ln\rho}{d\ln r}\right) + C\sigma_0^2\, U$$
$$a_2 = -2U y_q$$
$$a_3 = -2U w_q + 2\Lambda_q v_q + (U - 4)(c_q + 2\Lambda_q z_q) + s_q \psi \quad \text{(B16)}$$
$$a_4 = 2\Lambda_k w_q + \Lambda_k(4 - U) s_q + 2 y_{tq}(\bar\Lambda - 3)(4 - U) - (\Lambda_k - \Lambda_q + 6)\, y_q C\sigma_0^2 + 2(\bar\Lambda - 3) U \delta_{q,0} y_q$$
$$a_5 = C\sigma_0^2\,(U - 6)(\bar\Lambda - 3)$$
$$a_6 = 2\delta_{lq,0}(1 - \delta_{lk,0})\,(\bar\Lambda - 3)\left(y_k - 2z_k - \frac{A}{V_g}\frac{\lambda_k}{C\sigma_0^2}\right)$$

For shortness, we have defined:

$$\psi = 6 + U(U-3) + (4-U)(1-U)$$
$$\bar\Lambda = \frac{\Lambda_k + \Lambda_q}{2}$$
$$s = y - y_t + v; \qquad \lambda = V_g(y - y_t + v)$$
$$d_1 \equiv r^2 u_2;\;\; d_2 \equiv r\frac{dr^2 u_2}{dr};\;\; b_2 = \tfrac{1}{3} r \frac{d\eta_2}{dr};\;\; b_3 = \tfrac{1}{3} r^2 \frac{d^2\eta_2}{dr^2} \quad \text{(B17)}$$

We recall that d_1, d_2 arise from the non-spherically symmetric distortion of the equilibrium model and must be obtained by numerical integration of the system Eq. (17).

Integration over $\theta,\ \varphi$ imposes selection rules upon the degrees of the near-degenerate modes which can couple. The azimuthal order m must be the same and one has

$$\mathcal{Q}_{kq2} = \int \sin\theta\, d\theta\, d\varphi\, Y_k^* Y_q P_2$$
$$= \delta_{l_k,l_q}\mathcal{Q}_{kk2} + \frac{3}{2}\delta_{l_k,l_q+2}\,\beta_k\beta_{q+1} + \frac{3}{2}\delta_{l_k,l_q-2}\,\beta_{k+1}\beta_q \quad \text{(B18)}$$
$$\mathcal{Q}_{kk2} = \frac{3}{2}\,(\beta_{k+1}^2 + \beta_k^2) - \frac{1}{2} = \frac{\Lambda_k - 3m^2}{4\Lambda_k - 3}$$

The diagonal coefficient $\omega^D \equiv \mathcal{D}_{kk}$ is:

$$\omega^D = \left(\frac{\bar\Omega}{\omega_0}\right)^2 \left(\, \omega_0 J_2^D + m\bar\Omega J_3^D + m\bar\Omega J_2^D\,(C_L - 1 - J_1)\,\right) \quad \text{(B19)}$$

with

$$J_2^D = J_{2D,1} - Q_{kk2}\, J_{2D,2}$$
$$J_3^D = -Q_{kk2}\, J_{3D,1} \quad \text{(B20)}$$

and

$$J_{2D,1} = \frac{1}{2I}\int dr\, \rho r^4\, b_2\, y^2$$
$$J_{2D,2} = \frac{1}{2I}\int dr\, \rho r^2\left(d_1F_1 + d_2F_2 + r^2 b_2 F_3 + r^2 b_3 F_4\right)$$
$$J_{3D,1} = \frac{1}{I}\int dr\, \rho r^2\, D_{3kk} \quad \text{(B21)}$$

B.3. Distortion and Coriolis coupling: $\mathcal{C}_{kq}$

From its definition Eq. (107), it is straightforward to obtain:

$$\mathcal{C}_{kq} = \frac{m}{J}\int dr\, r^4\, \Omega\omega_0\, \rho_{22} \left[\delta_{l_k,l_q}\, z_k z_q + (y_k z_q + y_q z_k + 3 z_k z_q)\, \mathcal{Q}_{kq2}\right] \quad \text{(B22)}$$

The density perturbation ρ_{22} is replaced by its expression Eq. (15). Density derivatives are removed with an integration by parts and one obtains:

$$\begin{aligned} \mathcal{C}_{kq} &= m\bar{\Omega}\left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \left(\delta_{l_k,l_q}\, \mathcal{C}_1 + \mathcal{Q}_{kq2}\, \mathcal{C}_2\right) \\ \mathcal{C}_j &= \frac{\omega_0}{J}\int dr\rho r^2\,(1+\eta) \left[d_1 f_j - (d_2 - r^2 b_2) g_j\right] \;; \quad (j=1,2) \end{aligned} \quad \text{(B23)}$$

with

$$\begin{aligned} f_1 &= C\sigma_0^2\,(U - 2 - \frac{d\ln\Omega}{d\ln r})\, z_k z_q \\ &- (y_k y_{tq} + y_q y_{tk}) + \frac{A}{V_g}(\lambda_k z_q + \lambda_q z_k) \\ f_2 &= (U - 4 - \frac{d\ln\Omega}{d\ln r})(y_k y_{tq} + y_q y_{tk}) \\ &+ 3C\sigma_0^2\, z_k z_q\,(U - 2 - \frac{d\ln\Omega}{d\ln r} - \frac{2}{3}\bar{\Lambda}) \\ &+ (3\frac{A}{V_g} - C\sigma_0^2)(\lambda_k z_q + \lambda_q z_k) \\ &- 2C\sigma_0^2\, y_k y_q + \frac{A}{V_g}(y_k\lambda_q + y_q\lambda_k) \\ g_1 &= C\sigma_0^2\, z_k z_q \\ g_2 &= C\sigma_0^2\,(y_k z_q + y_q z_k + 3 z_k z_q) \end{aligned} \quad \text{(B24)}$$

The diagonal coefficient is obtained with $k = q$ i.e.

$$\begin{aligned} \omega^C &= \mathcal{C}_{kk} = m\bar{\Omega}\left(\frac{\bar{\Omega}}{\omega_0}\right)^2\, J_3^C \\ J_3^C &= J_{3C,1} + \mathcal{Q}_{kk2}\, J_{3C,2} \\ J_{3C,j} &= \frac{1}{I}\int dr\rho r^2\,(1+\eta) \left[d_1 f_j - (d_2 - r^2 b_2) g_j\right] \end{aligned} \quad \text{(B25)}$$

B.4. Full coupling coefficient $\mathcal{H}_{kq}$

Collecting the different contributions Eqs. 108, 117, 128, one finally gets for the coupling term:

$$\mathcal{H} = \delta_{l_k,l_q}\mathcal{H}_1 + \delta_{l_k,l_q+2}\,\mathcal{H}_{2,kq} + \delta_{l_k,l_q-2}\,\mathcal{H}^*_{2,qk} \quad \text{(B26)}$$

where

$$\begin{aligned} \mathcal{H}_1 &= \omega_0\left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \left(\mathcal{H}_1^{(1)} + m\frac{\bar{\Omega}}{\omega_0}\,\mathcal{H}_1^{(2)}\right) \\ \mathcal{H}_{2,kq} &= \omega_0\left(\frac{\bar{\Omega}}{\omega_0}\right)^2 \left(\mathcal{H}_{2kq}^{(1)} + m\frac{\bar{\Omega}}{\omega_0}\,\mathcal{H}_{2kq}^{(2)}\right) \end{aligned} \quad \text{(B27)}$$

with

$$\begin{aligned} \mathcal{H}_1^{(1)} &= K_1 + \mathcal{D}_1 + \mathcal{Q}_{2kk}\mathcal{D}_2 \\ \mathcal{H}_1^{(2)} &= K_3 + \mathcal{C}_1 + \mathcal{Q}_{2kk}\,(\mathcal{C}_2 - 2\mathcal{D}_3) \end{aligned} \quad \text{(B28)}$$

and

$$\begin{aligned} \mathcal{H}_{2kq}^{(1)} &= -K_{2kq} + \frac{3}{2}\,\beta_k\beta_{q+1}\mathcal{D}_2 \\ \mathcal{H}_{2kq}^{(2)} &= -K_{4kq} + \frac{3}{2}\,\beta_k\beta_{q+1}(\mathcal{C}_2 - 2\mathcal{D}_3) \end{aligned} \quad \text{(B29)}$$

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