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Stellar evolution with rotation

III. Meridional circulation with μ-gradients and non-stationarity

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Abstract. Models of stellar evolution with rotation are very much needed in order to interpret recent stellar properties, in particular for massive stars. For that we proceed to a detailed investigation of some critical physical processes in rotating stars:
1. The equation expressing the transport of angular momentum by circulation and diffusion is revised to also account for expansion and contraction in non-stationary models.
2. We examine the current expressions for the velocity $U(r)$ of the meridional circulation, also taking into account μ-gradients and horizontal turbulence. We show that there are no "μ-currents" but just meridional currents which can be sizeably influenced by the gradients of mean molecular weight μ. A proper account of the μ-gradients may reduce $U(r)$ by one or two orders of magnitude with respect to current expressions.
3. While the usual expressions for the meridional circulation would predict an infinite velocity at the edge of a radiative and semiconvective zone and an inverted circulation in a semiconvective zone, the present developments give a continuity of the solutions for the circulation.
4. The approximation of a stationary circulation, which is no longer valid after the main-sequence phase, is also removed and the case of a general equation of state is considered. We notice that in the stationary regime the horizontal fluctuations of μ represent some fixed fraction of the vertical μ-gradient. To first order, this fraction is not dependent on rotation, because the building of horizontal fluctuations by the circulation is compensated by the smoothing due to horizontal turbulence.

Key words: stars: rotation – stars: early-type – stars: interior

1. Introduction

The accumulation of observational evidences in favour of some additional mixing processes occurring during stellar evolution, particularly in massive stars (cf. Maeder 1995a), shows the necessity to deepen the study of stellar evolution with rotational mixing. Several models have been elaborated in the past (e.g. Endal and Sofia 1975; Schatzman and Maeder 1981) which often contained a large number of free parameters (cf. Pinsonneault et al. 1989). Progress has been achieved by taking into account the turbulence which is likely to arise from differential rotation (Zahn 1992); it is then possible to build a self-consistent picture of the transport of angular momentum and chemical elements, including the effects of meridional circulation and of turbulent diffusion.

In Paper I (Meynet and Maeder 1997) a number of evolutionary models with rotation were calculated, according to Zahn's prescriptions (1992). It was found that the μ-gradients developing during nuclear evolution are always too large to allow any significant mixing, despite the fact that the threshold for mixing imposed by Richardson's criterion was lowered by the inclusion of thermal effects (Maeder and Meynet 1996). In paper II (Maeder 1997) the modification of the Richardson criterion was considered when other sources of turbulence exist, such as semiconvection or horizontal turbulence. A new diffusion coefficient was derived with the assumption that the energy excess present in the shears is degraded by turbulence, which changes the local entropy gradient und consequently the local T- and μ-gradients. On their side Talon and Zahn (1997) treated the effect of horizontal mixing much as that of radiative damping, and they established an alternate form of the turbulent diffusivity. This latter prescription was applied to describe the rotational mixing of a $9M_\odot$ star (Talon et al. 1997).

In the present work we extend our theoretical approach by removing a few restrictive assumptions, in particular regarding the effects of μ-gradients on meridional circulation, as well as regarding the stationarity of the solutions. In Sect. 2 we re-examine the eulerian equation for the transport of angular momentum in non-stationary models and we recall in Sect. 3 the effect of a strong horizontal diffusion on the transport of chemical species. In Sect. 4 we express the thermal imbalance and the meridional velocity, allowing for non-stationarity and for a general equation of state. Special care will be taken in treating the large effects due to μ-gradients. Finally in Sect. 5 we discuss the results.

In an Appendix, we give the expression of the entropy change for a simple mixture of perfect gas and radiation.

2. Transport of the angular momentum

The equation describing the transport of angular momentum is necessary in order to follow-up the evolution of the angular velocity Ω in stars. In slightly different forms, this equation has been derived for example by Jeans (1929), Tassoul (1978), Chaboyer and Zahn (1992). Written in eulerian coordinates, the equation generally applies to a stationary star. This means that contraction or expansion occurring at any level in an evolving star are not accounted for, although these contribute most, in general, to the changes of the internal rotation. In the following we derive the equation applicable to evolving stars.

A mass element $dm = \rho r^2 \sin\,\vartheta d\vartheta d\varphi dr$ (in polar coordinates), located at the mass level M_r in the star, is conserved in a contraction or expansion, and therefore the lagrangian derivative of its angular momentum is $\rho r^2 \sin\vartheta d\vartheta d\varphi dr \frac{d}{dt}(r^2 \sin^2\theta\,\Omega)_{Mr}$. Using the relation between lagrangian and eulerian expressions, and applying the equation of continuity, we obtain the equivalent form

$$\rho \frac{d}{dt}\left(r^2\ \sin^2\ \vartheta\,\Omega\right)_{M_r} = \frac{\partial}{\partial t}\ \left(\rho r^2\ \sin^2\vartheta\,\Omega\right)_r + \boldsymbol{\nabla}\cdot\left[\rho r^2 \sin^2\vartheta\,\Omega \boldsymbol{u}\right]. \quad (2.1)$$

Assuming that angular momentum is transported only through advection (by the velocity field $\boldsymbol{u}$) and through turbulent diffusion (of viscosity ν), this time derivative is governed by

$$\rho \frac{d}{dt}\left(r^2 \sin^2\vartheta\,\Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\rho r^2 \sin^2\vartheta\,\Omega\right)_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^4 \sin^2\vartheta\, w_r\,\Omega\right) + \frac{1}{r\sin\vartheta}\ \frac{\partial}{\partial\vartheta}\left(\rho r^2 \sin^3\vartheta\ w_\vartheta\,\Omega\right) = \frac{\sin^2\vartheta}{r^2}\ \frac{\partial}{\partial r}\left(\rho\nu r^4 \frac{\partial\Omega}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\rho\nu\sin^3\vartheta\frac{\partial\Omega}{\partial\vartheta}\right). \quad (2.2)$$

We have explicited the velocity $\boldsymbol{u}$ in its radial and meridional components w_r et w_ϑ. But from now on we shall distinguish between the radial expansion or contraction of the star $\dot{r}$ and the components u_r and u_θ of the meridional circulation,

$$w_r = u_r + \dot{r}\,, \qquad w_\vartheta = u_\vartheta.$$

Assuming, as in Zahn (1992), that the rotation depends little on latitude, due to strong horizontal diffusion, we write

$$\Omega\,(r,\,\theta) = \overline{\Omega}(r) + \widehat{\Omega}\,(r,\,\theta)\,,$$

with $\widehat{\Omega} \ll \overline{\Omega}$, the horizontal average being taken over the angular momentum:

$$\overline{\Omega}\ = \frac{\int \Omega\sin^3\vartheta d\vartheta}{\int \sin^3\vartheta d\vartheta}.$$

With such a "shellular" rotation law, the meridional circulation is of quadrupolar type, as it would be for uniform rotation. One has then

$$u_r = U(r)P_2(\cos\vartheta)\,, \qquad u_\vartheta = V(r)\frac{dP_2(\cos\vartheta)}{d\vartheta}.$$

Multiplying equation (2.2) by $\sin\vartheta$ and integrating over ϑ, we obtain for the radial diffusion of angular momentum

$$\frac{\partial}{\partial t}\left(\rho r^2\overline{\Omega}\right)_r = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4\overline{\Omega}\left[U(r) - 5\dot{r}\right]\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\nu r^4\frac{\partial\overline{\Omega}}{\partial r}\right). \quad (2.3)$$

Note that the change in radius $\dot{r}$ of the given mass shell is now included in the eulerian formulation of this conservation equation. The characteristic timescale for the change of r is t_{KH} in hydrostatic models (in hydrodynamic ones it may even be shorter). The characteristic time associated to the transport of Ω by the circulation is (cf. Zahn 1992)

$$t_\Omega \approx t_{KH}\ \left(\frac{\Omega^2 R}{g_s}\right)^{-1} \quad (2.4)$$

where g_s is the gravity at the surface. Thus, we clearly have in general $t_\Omega \gg t_{KH}$ (except possibly in very external zones). This inequality means that the term in $\dot{r}$, which is due to the secular contraction expansion of the star, can be dominant with respect to $U(r)$; it must therefore be included in evolutionary models.

Another possibility is to use the lagrangian formulation, and to consider r as the coordinate linked to M_r through $dM_r = 4\pi\rho r^2 dr$:

$$\rho\frac{d}{dt}\left(r^2\overline{\Omega}\right)_{M_r} = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4\overline{\Omega}U(r)\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho\nu r^4\frac{\partial\overline{\Omega}}{\partial r}\right). \quad (2.5)$$

In that case the effects of expansion or contraction are automatically included; such a lagrangian treatment was applied for instance by Talon et al. (1997).

3. Transport of chemical elements

The transport of chemical species proceeds much like that of angular momentum: the concentration c_i (in mass) of a given element obeys an advection-diffusion equation similar to (2.2)

$$\frac{\partial\rho c_i}{\partial t}\ +\ \nabla\cdot\rho c_i\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho D_v r^2\frac{\partial c_i}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\rho D_h\frac{\partial c_i}{\partial\vartheta}\right) \quad (3.6)$$

where we have allowed for different diffusivities in the vertical and horizontal directions.

Splitting the concentration in its horizontal average and fluctuation on the isobar,

$$c_i\,(r,\,\vartheta) = \overline{c_i}\,(r) + \widetilde{c_i}\,(r)\ P_2\,(\cos\vartheta)$$

and assuming that $D_h \gg D_v$, Chaboyer and Zahn (1992) have shown that

$$\frac{\partial\widetilde{c_i}}{\partial t} + U\frac{\partial\overline{c_i}}{\partial r} = -\frac{6}{r^2}D_h\widetilde{c_i}. \quad (3.7)$$

Since the inverse of the molecular weight is a linear function of the mass concentrations, we have likewise

$$\frac{\partial\widetilde{\mu}}{\partial t}+U\frac{\partial\overline{\mu}}{\partial r}=-\frac{6}{r^2}D_h\widetilde{\mu} \tag{3.8}$$

and therefore, with $\Lambda=\widetilde{\mu}/\overline{\mu}$,

$$\frac{\partial\Lambda}{\partial t}+U\frac{\partial\ln\overline{\mu}}{\partial r}=-\frac{6}{r^2}D_h\Lambda. \tag{3.9}$$

Eqs. (3.7) and (3.9) have a stationary solution for long enough time $t\gg r^2/6D_h$. For (3.7) it is

$$\widetilde{c_i}=-\frac{r^2}{6D_h}U\frac{\partial\overline{c_i}}{\partial r}. \tag{3.10}$$

It can be used to evaluate the vertical flux of this chemical species, with the result that the vertical transport obeys a diffusion equation

$$\rho\left(\frac{d\overline{c_i}}{dt}\right)_{M_r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho(D_v+D_{eff})r^2\frac{\partial\overline{c_i}}{\partial r}\right) \tag{3.11}$$

with the effective diffusivity (Chaboyer & Zahn 1992)

$$D_{eff}=\frac{\mid rU(r)\mid^2}{30D_h}. \tag{3.12}$$

For simplicity, we have omitted in (3.11) the source and sink term due to nuclear reactions, as well as the effects of element separation through radiation and gravitational settling.

4. A revised expression for the velocity of meridional circulation

The velocity of meridional circulation is derived from the equation of energy conservation (cf. Mestel 1965)

$$\rho T\frac{dS}{dt}=\nabla\cdot(\chi\nabla T)+\rho\epsilon-\nabla\cdot\mathbf{F}_h \tag{4.13}$$

where S is the entropy per unit mass and χ the thermal conductivity. The term ϵ refers to the nuclear energy only. We include the flux $\mathbf{F}_h$ of thermal energy due to horizontal turbulence; it can be approximated by $\mathbf{F}_h=-D_h\rho T\nabla S$. D_h is the coefficient of horizontal turbulent diffusion, which we have introduced above in §3; it is large with respect to vertical diffusivity, a property which ensures shellular rotation and tends to smoothen the chemical composition over horizontal surfaces (cf. Zahn 1992).

The problem of meridional circulation has been studied for 3/4 of a century (Eddington 1925); it is rather surprising then that it is still in debate nowadays. However, there are four points which lead us to re-examine that old, important astrophysical problem.

1. The first point is rather minor. Usually, only the case of a perfect gas is considered, but for massive stars, in particular, a more general equation of state is needed, which is easy to implement.

2. In general the approximation of a stationary circulation is made, which is certainly valid for main sequence stars. However, in the shell H-burning phase, in the He-burning and subsequent phases, it is not applicable and we need to remove this hypothesis.

3. The presence of the term containing $\mathbf{F}_h$, the thermal flux due to the horizontal turbulence in expr. (4.13), introduces some changes which need to be accounted for.

4. Some effects of the gradients of the mean molecular weight μ, in particular horizontal homogeneities, have already been taken into account by Chaboyer and Zahn (1992) and Zahn (1992), who considered mainly stationary stellar models. However, some rather large effects due to the radial μ-gradients in the equation of energy conservation must also be considered in the case of evolutionary models.

In the following we try to be as short as possible and we mainly refer to Zahn (1992) whenever the developments are similar. All quantities are expanded linearly around their average on a level surface or isobar, for example

$$\begin{aligned} T\,(P,\theta) &= \overline{T}\,(P)+\widetilde{T}\,(P)\,P_2(\cos\vartheta) \\ T\,dS\,(P,\theta) &= \overline{T}\,d\overline{S}+\overline{T}\,d\widetilde{S}\,(P)\,P_2(\cos\vartheta) \end{aligned} \tag{4.14}$$

Later on, we shall need also a similar expansion for the divergence of the centrifugal force:

$$\nabla\left(\frac{1}{2}\Omega^2\nabla\left(r\sin\theta\right)^2\right)=2\overline{\Omega}^2+\frac{2}{3}r\frac{d\Omega^2}{dr}\left(1-P_2\left(\cos\theta\right)\right)$$

$$=2\overline{\Omega}^2+2\widetilde{\Omega}^2P_2(\cos\vartheta) \tag{4.15}$$

To express the non-stationarity we may develop (4.13) in

$$\rho T\frac{dS}{dt}=\overline{\rho}\overline{T}\frac{d\overline{S}}{dt}+\overline{\rho}\overline{T}\frac{d\widetilde{S}}{dt}P_2(\cos\vartheta) \tag{4.16}$$

and note that the horizontal average in lagrangian coordinates is (cf. Kippenhahn and Weigert 1990)

$$\overline{T}\frac{d\overline{S}}{dt}=-\overline{\epsilon}^{grav}, \tag{4.17}$$

where ϵ^{grav} is the release rate of gravitational energy.

From now on, we concentrate on the horizontal perturbations, and proceed to linearize Eq. (4.13), putting for convenience the horizontal diffusion term on the left hand side:

$$\overline{\rho}\,\overline{T}\left[\frac{d\widetilde{S}}{dt}-D_h\widetilde{S}\nabla^2\right]P_2(\cos\vartheta)=$$

$$\langle\overline{\rho\chi}\frac{d\overline{T}}{dP}\Big(-4\pi G\overline{\rho}+2\overline{\Omega}^2\Big)+\overline{\rho}\frac{d}{dP}\left(\overline{\rho\chi}\frac{d\overline{T}}{dP}\right)\overline{g}^2+\overline{\rho}(\overline{\epsilon}+\overline{\epsilon}^{grav})\rangle$$

$$+\left[\left(\widetilde{\rho\chi}\frac{d\overline{T}}{dP}+\overline{\rho\chi}\frac{d\widetilde{T}}{dP}\right)\left(-4\pi G\overline{\rho}+2\,\overline{\Omega}^2\right)\right.$$

$$+\overline{\rho\chi}\,\frac{d\overline{T}}{dP}\left(-4\pi G\widetilde{\rho}+2\widetilde{\Omega}^2\right)+\overline{\rho}\frac{d}{dP}\left(\overline{\rho\chi}\frac{d\overline{T}}{dP}\right)2\overline{g}\,\widetilde{g}$$

$$+\overline{\rho}\frac{d}{dP}\left(\widetilde{\rho\chi}\frac{d\overline{T}}{dP}+\overline{\rho\chi}\,\frac{d\widetilde{T}}{dP}\right)\overline{g}^2+\overline{\chi}\widetilde{T}\nabla^2$$

$$+\widetilde{\rho}\frac{d}{dP}\left(\overline{\rho\chi}\frac{d\overline{T}}{dP}\right)\overline{g}^2+\widetilde{\rho\epsilon}\Big]\,P_2(\cos\vartheta). \quad (4.18)$$

The horizontal average $<>$ is zero on a level surface, which means that the star is in average radiative equilibrium including nuclear and gravitational energy production. The second last term involving $\widetilde{\rho}$ in square brackets was missing in Zahn (1992), as pointed out by Urpin et al. (1995).
Since

$$\overline{\rho\chi}\,\frac{d\overline{T}}{dP}=\frac{L}{4\pi GM_*} \quad (4.19)$$

with $M_*=M\left(1-\left(\Omega^2/2\pi G\rho_m\right)\right)$, we get

$$\overline{T}\left[\frac{d\widetilde{S}}{dt}+D_h\frac{6}{r^2}\widetilde{S}\right]= \quad (4.20)$$

$$\left\{2\left[\frac{L}{M_*}\left(1-\frac{\overline{\Omega}^2}{2\pi G\overline{\rho}}\right)-(\overline{\epsilon}+\overline{\epsilon}^{grav})\right]\frac{\widetilde{g}}{\overline{g}}+\frac{L}{M_*}\frac{\widetilde{\Omega}^2}{2\pi G\overline{\rho}}\right\}$$

$$+\left[\frac{\overline{g}^2}{4\pi G}\frac{L}{M_*}\frac{d}{dP}\left(\frac{d\widetilde{T}}{d\overline{T}}+\frac{\widetilde{\rho\chi}}{\overline{\rho\chi}}\right)-(\overline{\epsilon}+\overline{\epsilon}^{grav})\left(\frac{d\widetilde{T}}{d\overline{T}}+\frac{\widetilde{\rho\chi}}{\overline{\rho\chi}}\right)\right.$$

$$\left.+\frac{\widetilde{\rho\epsilon}}{\overline{\rho}}-\frac{\widetilde{\rho}(\overline{\epsilon}+\overline{\epsilon}^{grav})}{\overline{\rho}}-\frac{6\overline{\chi}\widetilde{T}}{r^2\overline{\rho}}-\frac{L}{M_*}\frac{\overline{\Omega}^2}{2\pi G\overline{\rho}}\frac{\widetilde{\rho}}{\overline{\rho}}\right].$$

The terms are ordered in the same way as in Zahn (1992) to facilitate the comparison. Note in particular the term $\overline{\epsilon}^{grav}$ associated with $\overline{\epsilon}$. The last term in $\widetilde{\rho}/\overline{\rho}$ will be neglected later on because it is of the order of $\Omega^2/G\rho_m$ smaller than the other terms in $\widetilde{\rho}/\overline{\rho}$.
The derivative d/dP is replaced by $-\left(1/\overline{\rho g}\right)\frac{d}{dr}$ and we introduce the auxiliary variables $\Theta=\widetilde{\rho}/\overline{\rho}$ and $\Lambda=\widetilde{\mu}/\overline{\mu}$. By using a general equation of state (cf. Kippenhahn and Weigert 1990) of the form

$$\frac{d\rho}{\rho}=\alpha\frac{dP}{P}-\delta\frac{dT}{T}+\varphi\frac{d\mu}{\mu}, \quad (4.21)$$

we get for the subsonic perturbations on the isobar

$$\frac{\widetilde{\rho}}{\rho}=-\delta\frac{\widetilde{T}}{\overline{\overline{T}}}+\varphi\frac{\widetilde{\mu}}{\overline{\overline{\mu}}}, \quad (4.22)$$

with $\delta=-\left(\partial\ln\rho/\partial\ln T\right)_{P,\mu}$ and $\varphi=\left(\partial\ln\rho/\partial\ln\mu\right)_{P,T}$. For a mixture of perfect gas and radiation, $\delta=\left(4-3\beta\right)/\beta$ and $\varphi=1$. Thus one has

$$\varphi\Lambda-\delta\frac{\widetilde{T}}{\overline{\overline{T}}}=\Theta. \quad (4.23)$$

The horizontal linear expansion of the radiative conductivity and nuclear energy generation rate can be written

$$\frac{\widetilde{\chi}}{\overline{\chi}} = -\chi_T\frac{\widetilde{T}}{\overline{\overline{T}}}+\chi_\mu\frac{\widetilde{\mu}}{\overline{\mu}}$$

$$= -\chi_T\frac{\Theta}{\delta}+\left(\chi_\mu+\frac{\varphi}{\delta}\chi_T\right)\Lambda \quad (4.24)$$

and

$$\frac{M}{L}\frac{\widetilde{\rho\epsilon}}{\overline{\rho}}=\frac{\overline{\epsilon}}{\epsilon_m}\left[\left(1-\frac{\epsilon_T}{\delta}\right)\Theta+\left(\frac{\varphi}{\delta}\epsilon_T+\epsilon_\mu\right)\Lambda\right], \quad (4.25)$$

where the indices T and μ refer to the derivatives with respect to T and μ, while $\epsilon_m(r)$ is the internal average $\epsilon_m(r)=L(r)/M(r)$. The term $\epsilon_m(r)$ refers indeed to the sum of nuclear and gravitational energy.
The fluctuation of gravity may be calculated by solving the perturbed Poisson equation, as explained in Zahn (1992); an approxmiate expression is

$$\frac{\widetilde{g}}{\overline{g}}\simeq\frac{4}{3}\left(\frac{\Omega^2r^3}{GM}\right). \quad (4.26)$$

The density fluctuation is drawn from the hydrostatic equation

$$\Theta=\frac{1}{3}\frac{r^2}{\overline{g}}\frac{d\Omega^2}{dr}\quad\text{and from (4.15)}\quad\widetilde{\Omega}^2=-\frac{\overline{g}}{r}\Theta. \quad (4.27)$$

These expressions imply that the horizontal variations of ρ and g are functions of Ω^2. We may also write expression (4.18) by separating the terms which depend on Ω^2, either explicitely or through $\widetilde{g}/\overline{g}$ or Θ, from the terms which depend on the horizontal variation of μ:

$$\overline{T}\left[\frac{d\widetilde{S}}{dt}+\frac{6}{r^2}D_h\widetilde{S}\right]=\frac{L}{M_*}\left(E^\star_\Omega+E_\mu\right) \quad (4.28)$$

(we use an asterisc here because we anticipate that this will not be the final form of E_Ω).
Without entering into details, we get, by defining $f_\epsilon=\overline{\epsilon}/(\overline{\epsilon}+\overline{\epsilon}^{grav})$, H_T the temperature scale height, ρ_m the average density inside the mass M_r

$$E^\star_\Omega = 2\left[1-\frac{\overline{\Omega^2}}{2\pi G\overline{\rho}}-\frac{(\overline{\epsilon}+\overline{\epsilon}^{grav})}{\epsilon_m}\right]\frac{\widetilde{g}}{\overline{g}}$$

$$-\frac{\rho_m}{\overline{\rho}}\left\{\frac{r}{3}\frac{d}{dr}\left[H_T\frac{d}{dr}\left(\frac{\Theta}{\delta}\right)-\chi_T\Theta+\left(1-\frac{1}{\delta}\right)\Theta\right]\right.$$

$$\left.-\frac{2H_T}{r}\left(\frac{\Theta}{\delta}\right)+\frac{2}{3}\Theta\right\}$$

$$-\frac{(\overline{\epsilon}+\overline{\epsilon}^{grav})}{\epsilon_m}\left[H_T\frac{d}{dr}\left(\frac{\Theta}{\delta}\right)+(f_\epsilon\epsilon_T-\chi_T)\left(\frac{\Theta}{\delta}\right)\right.$$

$$\left.+\left(2-f_\epsilon-\frac{1}{\delta}\right)\Theta\right]. \quad (4.29)$$

This writing has been chosen in order to facilitate the comparison with Zahn (1992) with whom we notice some slight differences, mainly due to the use of a more general equation of state.
In the case of uniform rotation $\Theta=0$, and only the first term in $\widetilde{g}/g$ remains in E_Ω. It is then positive, meaning that angular momentum is transported inwards, except close enough to the surface, as pointed out already by Gratton (1945) and Öpik

(1951). When evolutionary effects are taken in account, the term $-(\rho_m/\overline{\rho})\,\{\ldots[\ldots]$ is also positive, and is the dominant one; the 2nd and 3rd terms in $-(\rho_m/\overline{\rho})\,\{\}$ can be significant near the surface, whereas all terms in $-\overline{\epsilon}/\epsilon_m$ are completely negligible in general.

For E_μ we have likewise

$$E_\mu = \frac{\rho_m}{\overline{\rho}}\left\{\frac{r}{3}\frac{d}{dr}\left[H_T\frac{d}{dr}\left(\frac{\varphi}{\delta}\Lambda\right)-\left(\chi_\mu+\frac{\varphi}{\delta}\chi_T+\frac{\varphi}{\delta}\right)\Lambda\right]\right.$$
$$\left. - \frac{2H_T}{r}\frac{\varphi}{\delta}\Lambda\right\} + \frac{\overline{\epsilon}+\overline{\epsilon}^{grav}}{\epsilon_m}\left[H_T\frac{d}{dr}\left(\frac{\varphi}{\delta}\Lambda\right)\right.$$
$$\left. + \left(f_\epsilon\epsilon_\mu + f_\epsilon\frac{\varphi}{\delta}\epsilon_T - \chi_\mu - \frac{\varphi}{\delta}\chi_T - \frac{\varphi}{\delta}\right)\Lambda\right]. \qquad (4.30)$$

Let us now turn to the first member of our equation of energy conservation (4.28). Keeping only the first order term in the horizontal fluctuations, we can write

$$\overline{T}\,\frac{d\widetilde{S}}{dt} = \overline{T}\,\frac{\partial\widetilde{S}}{\partial t} + U(r)\,\overline{T}\,\frac{\partial\overline{S}}{\partial r} \qquad (4.31)$$

where $U(r)$ has been defined in Sect. 2: it is the amplitude of the radial component of the meridional velocity, $u_r = U(r)P_2(\cos\vartheta)$.

We will see in the Appendix that in a medium of varying composition the entropy of mixing must be taken into account. In the simplest case, where the stellar material can be approximated by a simple mixture of H, He with a fixed abundance of metals, the entropy of mixing may be expressed in terms of the molecular weight only, and then

$$dS = C_P\left[\frac{dT}{T} - \nabla_{\rm ad}\frac{dP}{P} + \Phi(P,T,\mu)\frac{d\mu}{\mu}\right]. \qquad (4.32)$$

Since there are no pressure fluctuations on the isobar, we have (cf. 4.11)

$$\widetilde{S} = C_P\left[\frac{\widetilde{T}}{\overline{T}} + \Phi\frac{\widetilde{\mu}}{\overline{\mu}}\right] = C_P\left[\left(\frac{\varphi}{\delta}+\Phi\right)\Lambda - \frac{\Theta}{\delta}\right]. \qquad (4.33)$$

Likewise, for the entropy gradient we get

$$\frac{\partial\overline{S}}{\partial r} = \frac{C_P}{H_P}\left(\nabla_{ad} - \nabla - \Phi\nabla_\mu\right) \qquad (4.34)$$

with ∇_{ad} and ∇ the adiabatic and actual T-gradients, and $\nabla_\mu = d\ln\overline{\mu}/d\ln P$. At this point, we replace the time derivative of Λ by the expression derived in §3 (3.9):

$$\frac{\partial\Lambda}{\partial t} = \frac{U}{H_P}\nabla_\mu - \frac{6}{r^2}D_h\Lambda \qquad (4.35)$$

to cast (4.28) in its final form

$$-\frac{C_P}{\delta}\frac{\partial\Theta}{\partial t} + U(r)\frac{C_P\overline{T}}{H_P}\left(\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right)$$
$$= \frac{L}{M_\star}(E^\star_\Omega + E_\mu) + \frac{6}{r^2}C_P\overline{T}D_h\left(\frac{\Theta}{\delta}\right).$$
$$= \frac{L}{M_\star}(E_\Omega + E_\mu), \qquad (4.36)$$

where we have replaced $E^\star_\Omega$ by

$$E_\Omega = E^\star_\Omega + \frac{2H_T}{r}\frac{\rho_m}{\overline{\rho}}\frac{D_h}{K}\left(\frac{\Theta}{\delta}\right). \qquad (4.37)$$

The amplitude of the meridional velocity $U(r)$ can thus be written as

$$U(r) = \frac{P}{\overline{\rho g}C_P\overline{T}\left[\nabla_{\rm ad} - \nabla + (\varphi/\delta)\nabla_\mu\right]}$$
$$\times\left\{\frac{L}{M_\star}(E_\Omega + E_\mu) + \frac{C_P}{\delta}\frac{\partial\Theta}{\partial t}\right\}. \qquad (4.38)$$

In a stationary situation, as is usually considered, the term $\partial\Theta/\partial t$ is zero, and one has $\overline{\epsilon}^{grav} = 0$ and $f_\epsilon = 1$ in the expressions for E_Ω and E_μ. If, in addition, one takes a perfect gas with $\delta = 1$, one finds again the expressions given by Zahn (1992), except for the horizontal diffusion term which has been added here to E_Ω and, even more importantly, for the gradient of molecular weight ∇_μ which appears in the superadiabatic gradient. The latter now takes the form which enters in the Ledoux criterion for convective instability and in the Brunt-Väisälä frequency:

$$N^2 = \frac{g\delta}{H_P}\left[\nabla_{\rm ad} - \nabla + (\varphi/\delta)\nabla_\mu\right]. \qquad (4.39)$$

In case of stationarity, one would also have from (3.9)

$$\Lambda = -\frac{r^2}{6D_h}\,U\,\frac{\partial\ln\mu}{\partial r} \qquad (4.40)$$

If as supposed by Zahn (1992) and by Chaboyer and Zahn (1992), there is some proportionality between $U(r)$ and D_h, we get

$$\Lambda = -\frac{1}{6C_H}\frac{\partial\ln\mu}{\partial\ln r} \qquad (4.41)$$

with a factor C_H of the order of unity. This means that, in a situation of equilibrium, the horizontal fluctuations of μ are some fixed fraction of the vertical μ-gradient. It is noteworthy that this fraction does not depend on rotation to the first order. This can be understood in the following way: a larger rotation leads to a faster meridional circulation, thus building larger horizontal μ-fluctuations; however, at the same time, the larger rotation creates a stronger horizontal turbulence, which smoothens the horizontal μ-fluctuations, and the two effects, as far as they both depend on $U(r)$, tend to compensate each other.

However, the hypothesis of stationarity is not applicable after main-sequence evolution, because the timescales of the variations of entropy, i.e. the Kelvin-Helmholtz timescale and the evolutionary timescale, may be of the same order of magnitude, or the second one may even be shorter than the first one in the advanced stages. In that case the term in $\partial\Theta/\partial t$ will play an important role; it may then be derived from the time derivative of the Ω^2-gradient (see 4.27).

Before leaving this section, it may be useful to spell out the revised expression for E_Ω:

$$E_\Omega = 2\left[1-\frac{\overline{\Omega^2}}{2\pi G\overline{\rho}}-\frac{(\overline{\epsilon}+\overline{\epsilon}^{grav})}{\epsilon_m}\right]\frac{\widetilde{g}}{\overline{g}}$$
$$-\frac{\rho_m}{\overline{\rho}}\left\{\frac{r}{3}\frac{d}{dr}\left[H_T\frac{d}{dr}\left(\frac{\Theta}{\delta}\right)-\chi_T\Theta+\left(1-\frac{1}{\delta}\right)\Theta\right]\right.$$
$$\left.-\frac{2H_T}{r}\left(1+\frac{D_h}{K}\right)\left(\frac{\Theta}{\delta}\right)+\frac{2}{3}\Theta\right\}$$
$$-\frac{(\overline{\epsilon}+\overline{\epsilon}^{grav})}{\epsilon_m}\left[H_T\frac{d}{dr}\left(\frac{\Theta}{\delta}\right)+(f_\epsilon\epsilon_T-\chi_T)\left(\frac{\Theta}{\delta}\right)\right.$$
$$\left.+\left(2-f_\epsilon-\frac{1}{\delta}\right)\Theta\right]. \qquad (4.42)$$

Finally, let us stress that the meridional velocity is insensitive to the specific dependence of entropy on the chemical composition, as illustrated by the absence of Φ in (4.38). This result can be easily extended to the general case where S is a function of the concentration of more than two species. A term containing Φ is remaining only in $\overline{\epsilon}^{grav}$,

$$\overline{\epsilon}^{grav} = -C_P\frac{d\overline{T}}{dt}+\frac{\delta}{\rho}\frac{d\overline{P}}{dt}-\Phi\frac{C_P\overline{T}}{\overline{\mu}}\frac{d\overline{\mu}}{dt} \qquad (4.43)$$

but such a μ-contribution is usually neglected in stellar evolution.

The origin of the term ∇_μ in expression (4.38) for the velocity of the meridional circulation is interesting to trace back. This term is coming from the horizontal fluctuations $\widetilde{S}$ of entropy in (4.28), which depend on $\widetilde{T}$ an in turn on $\widetilde{\mu}$ through the equation of state. Then through the advection/diffusion equation (3.8) for the μ-variations, the horizontal fluctuations $\widetilde{S}$ also depend on the vertical μ-gradient ∇_μ. Athough the effect of horizontal smoothing has been taken in account in the derivation of (4.28), the result is independent of D_h; however the role of that smoothing is crucial, for it prevents the fluctuations of μ to become too large, thus allowing this linear treatment of the meridional circulation.

5. Discussion

Expression (4.38), which is the main result of this work, shows that a μ-gradient can significantly reduce the velocity of the meridional circulation, since ∇_μ can be larger than $(\nabla_{\rm ad}-\nabla)$ by an order of a magnitude or even more.

Mestel (1963) was the first to take into account the gradient of chemical composition and he called "μ-currents" the contribution of the term E_μ to the meridional circulation (cf. 4.38). There are two differences here with his original treatment. First we no longer assume uniform rotation, but solve explicitely for the changes in the rotation profile. In other words, not only E_μ, but also E_Ω evolves to stop the circulation in a star which does not lose angular momentum (Zahn 1992). And second the μ-gradient now enters in the entropy gradient: it acts there to reduce the strength of the meridional velocity (4.38).

Let us also note that the meridional circulation is generally studied in a purely radiative medium, i.e. with $(\nabla_{\rm ad}-\nabla)>0$. In a semiconvective region, the usual expression without ∇_μ would predict an inverted circulation, going down at the pole and ascending at the equator in an uniformly rotating star. Also, it would predict an infinite velocity at the transition between the radiative and semiconvective zones. In case of alternance of radiative and semiconvective layers in massive stars, we would have a really strange situation! With expression (4.38) such an unphysical situation does not occur. Circulation behaves similarily in radiative and semiconvective zones, keeping its direction and the continuity of its amplitude.

Detailed numerical models are now being developed to quantitatively estimate the importance of the various effects studied in this work and to examine their consequences on stellar evolution and in particular for the surface abundance enrichments.

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Appendix A: entropy for a mixture of perfect gases and radiation

From basic physics, the entropy of a mixture of perfect gases in molar fraction x_i is given by

$$S = N\mathcal{R}\left(\frac{5}{2}\ln T-\ln P\right)+S_{mix} \qquad (A1)$$

with

$$S_{mix} = -N\mathcal{R}\sum_i x_i\ln x_i + x_e\ln x_e) \qquad (A2)$$

where N is the number of moles, with

$$x_i = \frac{\mu X_i}{A_i} \qquad (A3)$$

for the ions of species i and

$$x_e = \mu\sum_i\frac{X_iZ_i}{A_i} \qquad (A4)$$

for the electrons with the assumption of complete ionisation. We call μ the mean molecular weight, X_i the mass fraction of the ions of type i with atomic mass A_i and electron number Z_i. Let us consider a mixture of hydrogen, helium and metals in respective mass fractions X, Y and Z, where Z is taken to be constant (case of H-burning).Then the entropy may be expressed in terms of μ only. Developing the various terms, we get

$$d\,S_{mix} = N\mathcal{R}B\,\frac{d\mu}{\mu} \qquad (A5)$$

with

$$B=-\frac{\mu}{80}\left\{(12Z-48)\ln\left(64-48\mu+12Z\mu\right)\right.$$
$$+(32-23Z)\ln\left(-16+32\mu-23Z\mu\right)+5Z\ \ln\ 5Z$$
$$\left.+(16+6Z)\ln\left(32+16\mu+6Z\mu\right)\right\}. \quad \text{(A6)}$$

We identify the heavy elements with oxygen and the corresponding molar fraction is thus $x=\mu Z/16$. For the sake of consistency, we must also take

$$\mu=\frac{16}{20X+12-3\,Z}. \quad \text{(A7)}$$

For a binary mixture with only H and He, we get

$$B=\frac{-\mu}{5}\left[-3\ln\left(4-3\mu\right)+2\ln\left(2\mu-1\right)\right.$$
$$\left.+\ln\left(2+\mu\right)\right]. \quad \text{(A8)}$$

For a mixture of perfect gases and radiation, we are looking for an expression of the form

$$dS=\left(\frac{\partial S}{\partial T}\right)_{P\mu}dT+\left(\frac{\partial S}{\partial P}\right)_{T\mu}dP+\left(\frac{\partial S}{\partial \mu}\right)_{TP}d\mu \quad \text{(A9)}$$

with $P=P_{gas}+P_{rad}$
and $S=S_{gas}+S_{rad}$.
We have for one mole

$$\left(\frac{\partial S}{\partial \mu}\right)_{TP}=\left(\frac{\partial S_{gas}}{\partial \mu}\right)_{TP}=\left(\frac{\partial S_{gas}}{\partial \mu}\right)_{T\,P_{gas}}=\frac{B}{\mu}. \quad \text{(A10)}$$

At constant μ, the usual expression of TdS is (cf. Kippenhahn and Weigert 1990)

$$TdS=C_PdT-\frac{\delta dP}{\rho} \quad \text{(A11)}$$

For variable μ, we get with $\Phi=B/C_P$

$$dS=C_P\left[\frac{dT}{T}-\nabla_{ad}\frac{dP}{P}+\Phi\frac{d\mu}{\mu}\right] \quad \text{(A12)}$$

We see that the entropy changes also depend on the changes of μ through the entropy of mixing, which is often not taken into account. Nevertheless, this term is not responsible for the term ∇_μ in (4.38) as seen above.

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