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Nonrotating astronomical relativistic reference frames

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Abstract. The target of this paper is a discussion of the meaning of the Newtonian concept of a reference frame showing no rotation with respect to a set of distant celestial objects in the framework of general relativity. Because of the properties of Newtonian absolute space and time and the existence of global inertial reference systems in Newton's theory the theoretical construction of such a globally nonrotating reference frame is obvious. This definitively no longer is the case in a relativistic framework. There exist no global inertial reference systems and one faces the necessity to give the notion of nonrotating frames a rigorous meaning.

Considering possible definitions of nonrotating reference frames in both Newtonian and relativistic physics, we conclude that the concept of relative spatial rotation between reference systems plays a fundamental role in defining nonrotating astronomical reference frames. It turns out that the property of two relativistic reference systems to show no spatial rotation relative to each other, being a coordinate-dependent concept, has some properties which cannot be interpreted within our "Newtonian common sense". As an example, we discuss two versions of a geocentric reference system, both of which can be considered to show no rotation relative to distant celestial objects at the present level of observational accuracy, but differing by a time-dependent rotation of considerable amplitude. Applying the obtained results to the recently elaborated formalisms for constructing relativistic astronomical reference systems, we describe relative spatial rotations between the galactic, barycentric, geocentric reference systems and the reference system of an observer.

We find a certain parallel between the concept of simultaneity (synchronization of clocks) and the concept of reference system showing no rotation relative to distant celestial objects. Both notions are absolute in Newtonian physics and become coordinate-dependent in the relativistic framework, representing, therefore, a mathematical convention rather than a physically meaningful phenomena.

Key words: relativity – reference systems

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1. Introduction

The accuracy of astronomical observations has reached a level where relativity plays a crucial role to theoretically describe the observables and to get physically and astronomically meaningful information from the observations. When processing observational data one wants usually to convert observables into a set of parameters which are believed to be physically meaningful and convenient (for example, [nearly] constant in time). In typical astronomical applications examples of such parameters are coordinates and velocities of celestial bodies in a well-defined reference system (RS) at a given moment of time, their masses, geocentric positions of Earth-bound observing sites, etc. Both Newtonian and relativistic physics allow one to model physical phenomena in any reference system, and to use, therefore, different sets of parameters. However, in Newtonian physics postulating the existence of an absolute three-dimensional Euclidean space and absolute time there exists a class of preferred global reference systems - the inertial reference systems - where equations of motion do not contain inertial forces. On the other side, general relativity theory abandons the Newtonian concepts of absolute space and time, and, in general, there is no reason to distinguish a class of preferred reference systems. In principle, all possible reference systems covering the region of space-time under consideration are equivalent. From a practical point of view, however, some coordinate systems might be preferred purely by matter of convenience.

In this paper under *reference system* we mean a purely mathematical construction (a chart) giving "names" to space-time events. In contrast to this a *reference frame* is some materialization of a reference system. In astronomy usually the materialization is realized by a form of catalogue (or ephemeris) containing positions of some celestial objects relative to the reference system under consideration. It is very important to understand, however, that any reference frame (a catalogue, an ephemeris, etc.) is defined only through a well-defined reference system, which has been used to construct physical models of observations.

It is believed that the aim of classical (Newtonian) astrometry is to construct an inertial celestial reference frame, that is, a catalogue containing celestial coordinates, parallaxes and proper motions of a number of reference celestial objects in a global inertial reference system. An important feature of such an inertial reference frame in the Newtonian framework is that it shows no overall rotation with respect to [a set of] celestial objects distant enough for their proper motions to be negligible for a reasonably long interval of time and a given accuracy of observations. The idea to construct a reference frame showing no rotation with respect to distant objects being quite natural and intuitively clear from the Newtonian point of view has a non-trivial content in relativistic physics. The reason is that we have to consider several possibly accelerated relativistic reference systems related to each other by nontrivial generalized Lorentz transformations affecting in contrast to the Galilean transformations of Newtonian mechanics also the orientation of spatial axes of the reference systems.

The Newtonian concept of a nonrotating reference system is discussed in Sect. 2. In Sect. 3 we outline briefly principal features of modern relativistic theory of astronomical reference systems. Section 4 is devoted to a discussion of the notion of spatial rotation of two relativistic reference systems. In Sect. 5 we consider the rotational state of motion of reference systems constituting a hierarchy of relativistic astronomical reference systems. In Sect. 6 the hierarchy of relativistic reference systems constructed under the assumption of isolateness of the solar system is discussed from the viewpoint of rotations between reference systems. In Sect. 7 we discuss the effects of the Galaxy on the results of astrometric observations of extragalactic sources performed within the solar system. In Concluding Remarks we outline the results of this paper and discuss a parallel between the concepts of simultaneity (clock synchronization) and reference frames showing no rotation relative to distant celestial objects.

2. Nonrotating reference systems in Newtonian physics

All experimental data which were at the disposal for the founders of Newtonian mechanics lent themselves to be interpreted within their accuracy in the framework of the following model: (1) space is three-dimensional and Euclidean, and time is one-dimensional; both space and time exist independently of any matter and observers, and have once and forever established properties; (2) there exists a class of preferred inertial reference systems, in which the laws of physics are the same in all moments of time and look especially simple: in any of these systems a free particle (that is a particle not affected by any force) moves uniformly and along a straight line. According to Galileo's principle of relativity all reference systems in uniform rectilinear motion with respect to an inertial one are themselves inertial. Since there is no preferred direction in Euclidean space the transformations between two inertial reference systems may involve also a time-independent rotation. In addition to these properties Newtonian theory of gravitation states that in the inertial reference system the gravitational force between two material points is proportional to the product of their masses divided by the square of the distance between them.

However, if a reference system differs from an inertial one by a time-dependent rotation, it is no more inertial. The equations of motion of a free particle relative to this rotating reference system look more complicated than in an inertial one. Namely, there appear Coriolis and centrifugal forces. The inertiality of a reference system is also violated if the origin of this reference system moves with some acceleration relative to an inertial reference system. In that case a free particle shows an acceleration relative to that reference system. In an arbitrary accelerated and rotating cartesian reference system $X^i = P^i_{\ j}(t)\left(x^j - x^j_0(t)\right)$, where x^i are inertial coordinates, $x^i_0(t)$ is the inertial coordinates of the origin of the accelerated reference system X^i and $P^i_{\ j}(t)$ is a time-dependent orthogonal matrix, the equations of motion of a particle read

$$\ddot{X}^i = P^i_{\ j}(t)\left(F^j - \ddot{x}^j_0(t)\right) - 2\varepsilon^i_{\ jk}\,\Omega^j\,\dot{X}^k \quad - \varepsilon^i_{\ jk}\,\dot{\Omega}^j\,X^k - \varepsilon^i_{\ jk}\,\Omega^j\left(\varepsilon^k_{\ mn}\,\Omega^m\,X^n\right), \qquad (1)$$

where $\varepsilon^i_{\ jk}$ is the fully antisymmetric Levi-Civita symbol (so that, e.g., $\varepsilon^i_{\ jk}\,A^j\,B^k$ is the usual vector product of vectors A^i and B^i), $F^i = \ddot{x}^i$ is the force which affects the particle in the inertial reference system, $\ddot{x}^i_0(t)$ and $\Omega^i(t) = \frac{1}{2}\,\varepsilon^i_{\ jk}\,\dot{P}^j_{\ m}(t)\,P^k_{\ m}(t)$ are respectively the acceleration of the origin and the angular velocity of rotation of our reference system relative to an inertial one.

Reference systems being a pure mathematical concept should be "materialized" to be useful in practice. Such a materialization (the corresponding reference frame) consists in ascribing in a self-consistent manner some coordinates (generally speaking, as functions of time) to a set of material objects which can be observed and, therefore, used as material "graduation ticks" of the reference system. Clearly, an inertial reference system of Newtonian mechanics is an idealization which could be materialized only with some accuracy. The required precision depends on the application. Thus, for many kind of experiments (e.g., in particle physics) a reference system at rest relative to an earthbound laboratory can be considered as sufficient approximation to the inertial one. However, another kind of experiments (e.g., Foucault pendulum) allows one to detect the deviation of that reference system from inertiality. Then one could adopt the reference system attached to the center of mass of the Earth or to the barycenter of the solar system as subsequent approximations to an inertial reference system. In each case we neglect the acceleration of the origin of the corresponding reference system caused primarily by the influence of external matter. If we can do so or not depends on the accuracy and the kind of observations we consider.

Here, in this article, we are dealing with the spatial orientation of a reference frame, which is a subtle question indeed. Two ways of fixing such an orientation will be discussed here leading to the concepts of i) dynamically nonrotating reference frames and ii) kinematically nonrotating reference frames.

2.1. *Dynamically nonrotating reference frames in Newtonian physics*

Here we start from a reference system nonrotating relative to an inertial one. The equations of motion in this reference system do not contain Coriolis and centrifugal forces, and, in this sense, the reference system under consideration is *dynamically* nonrotating. Then the motion of a certain dynamical system is studied by means of observational data and analyzed in this dynamically nonrotating reference system. The reference system is then materialized and the corresponding reference frame is established by giving these bodies the coordinate positions and velocities from the calculations. Of course, questions of accuracy enter here in many places (not all forces can be modeled, etc.) and the reference frame is a materialization of the corresponding reference system only at a certain accuracy level. The motion of planets and Moon is used usually for establishing the astronomical dynamically nonrotating barycentric reference frame. Instead of celestial bodies we can also observe the rotational motion of a gyroscope where the torques, acting on the gyro, are known sufficiently well. If the effects of the torques are eliminated, theoretically the resulting spin-axis should be constant both in absolute value and in orientation relative to our dynamically nonrotating reference system. With two appropriately oriented gyroscopes we could fix completely the orientation of our reference frame.

2.2. Kinematically nonrotating reference frames in Newtonian physics

In astronomy "kinematically nonrotating" in a certain sense means nonrotating with respect to remote celestial objects (stars, quasars, etc.). A catalogue of positions of such objects serves as such a kinematically nonrotating reference frame. We can think of a catalogue as giving angular positions on the Euclidean unit-sphere (e.g., usual astronomical equatorial positions (α, δ)).

After having observed remote celestial objects we can *calculate* directions to these objects as would be observed by a fictitious observer situated at rest in the barycenter of the solar system. To this end we have to account for many physical factors: refraction, aberration, parallax, precession and nutation, etc. Usually a Newtonian model of the light consisting of particles, which move uniformly and rectilinear with velocity c, is accepted here. In order to account for aberration, precession and nutation we need also some dynamical models for the translational and rotation motion of the Earth relative to the barycentric inertial reference system of the solar system. Then the time dependence of the barycentric directions to remote objects comes only from their motion with respect to the barycenter of the solar system. Selecting a set of objects which are believed to show either only random proper motions (fundamental stars) or no proper motions at all at the current level of accuracy (e.g., quasars) we can construct a catalogue which establishes a rigid reference frame on the sky. We then make a *fundamental assumption* on the structure of our Universe that such a reference frame shows also no rotation at the same level of accuracy relative to absolute Newtonian space. This assumption should be in agreement with all available kinds of observations. For example, in Newtonian mechanics it should not contradict the results of local dynamical observations: the motion of particles (e.g., planets) relative to such a kinematically nonrotating reference frame should be described by dynamical equations of motion without Coriolis and centrifugal terms.

In the framework of Newtonian mechanics dynamical and kinematical ways to orient our reference frame are merely two practical methods to orient a reference frame to be nonrotating relative to absolute space of Newtonian mechanics. In contrast to this, as we will see below, these two methods lead to principally different definitions of nonrotating reference systems in the relativistic framework.

3. Hierarchy of relativistic astronomical reference systems

General relativity abandons the Newtonian concepts of absolute space and time. Time and space represent now two parts of a single geometrical object: four-dimensional pseudo-Riemannien space-time. Geometrical properties of space-time influence motion of matter and the matter in turn defines the geometrical properties of space-time. In order to analyze any physical phenomena occurring in space-time we need a reference system.

A relativistic reference system is a mathematical procedure which to any given point (event) in physical space-time ascribes four real numbers or coordinates x^α, $\alpha = 0, 1, 2, 3$ of the event with respect to that reference system. One of the four coordinates x^0 is the time coordinate ($t = x^0/c$, c being the light velocity, is usually referred to as coordinate time). Three other coordinates x^i, $i = 1, 2, 3$ are space coordinates. Each reference system (t, x^i) is characterized by its metric tensor $g_{\alpha\beta}(t, x^i)$, which represents a solution of the Einstein field equations of general relativity. By studying the metric tensors of the reference systems and the coordinate transformations between them one can infer all the properties of the reference systems.

Although it is well known that all reference systems covering the space-time region under consideration are mathematically equivalent, one can prefer one reference system or another to perform actual calculations. The basic reasons for such a preference are considerations of convenience, that is, simplicity in solving the problem under study. In fact, the choice of an adequate reference system is an important part of solving any problem in the relativistic framework.

Nowadays it is widely accepted that in order to describe adequately modern astronomical observations one has to use several relativistic reference systems. Thus, the barycentric reference system (BRS) can be used to model the light propagation from distant celestial objects as well as the motion of bodies within the solar system. The geocentric reference system (GRS) is physically adequate to describe processes occurring near the Earth (Earth's rotation, motion of an Earth's satellite). In the vicinity of an observer one can construct a local reference system (we call such systems Observer's reference systems (ORS)) and use it for modeling phenomena in the neighborhood of the observer. One also could construct a galactic reference system (GalRS) to describe the dynamics of our Galaxy, etc.

The necessity to use several reference systems can be understood from the following example. If we were to characterize terrestrial observers by the difference between their BRS coordinates and the BRS coordinates of the geocenter, the positions of the observers relative to the geocenter would change with time also due to purely relativistic coordinate effects (such as Lorentz contraction, etc.) which have nothing to do with Earth rotation or geophysical factors and vanish if one employs the GRS coordinates instead of the BRS ones. On the other hand, the coordinate positions derived with the VLBI observations are used to investigate local geophysical processes on the Earth and the adequate Geocentric RS allows one to simplify procedure of modeling of such processes.

The basic idea is to construct a special reference system for each material subsystem, in which relativistic equations of motion of a test body inside the considered subsystem *by form* resembles as close as possible the Newtonian ones. Each reference system (the GalRS, BRS, GRS and the ORS) is a local reference system whose origin coincides with the barycenter of the corresponding subsystem (the center of Galaxy, solar system barycenter, geocenter and the observer, respectively), and the influence of outer matter in them are described by tidal potentials only (that is, by potentials whose expansions in powers of local spatial coordinates in the vicinity of the origin of the corresponding reference system start with quadratic terms).

Two advanced relativistic formalisms have been elaborated to tackle this problem in the first post-Newtonian approximation of general relativity. One formalism is due to Brumberg and Kopejkin (Brumberg and Kopejkin (1989), Kopejkin (1991), Brumberg (1991b), see also Klioner and Voinov (1993), and references therein) and another one is due to Damour, Soffel and Xu (1991, 1992, 1993). Although the formalisms are rather different at first glance, it is mainly the concept of mass multipole moments at the first post-Newtonian level which differs in these two formalisms. As far as the effects which we consider in this paper are concerned, the formalisms give the same results. We outline below the structure of the metric tensors of the reference systems introduced by both formalisms as well as the coordinate transformations between them to the precision necessary to consider the effects in the spatial orientation of the relativistic reference systems.

The metric tensor in any reference system (t,x^i) has the following generic form

$$
\begin{aligned}
g_{00}&=1-\frac{2}{c^2}W(t,\mathbf{x})+\mathcal{O}(c^{-4}),\\
g_{0i}&=\frac{4}{c^3}W^i(t,\mathbf{x})+\mathcal{O}(c^{-5}),\\
g_{ij}&=-\delta_{ij}\left(1+\frac{2}{c^2}W(t,\mathbf{x})\right)+\mathcal{O}(c^{-4}).
\end{aligned}
\qquad (2)
$$

Here, W and W^i are the Newtonian and vector potentials of gravity respectively. In this paper we do not have to consider the terms of order $\mathcal{O}(c^{-4})$ in g_{00} and, therefore, do not have to fix completely the coordinate gauge. The following statements are true in a wide class of gauges which covers both the harmonic gauge and the standard PN one. The potentials satisfy the equations

$$
\begin{aligned}
W_{,ii}&=-4\pi G\,\frac{1}{c^2}\,T^{00}+\mathcal{O}(c^{-2}),\\
W^i_{,jj}&=-4\pi G\,\frac{1}{c}\,T^{0i}+\mathcal{O}(c^{-2}),
\end{aligned}
\qquad (3)
$$

which follow from the Einstein field equations. Here $T^{\alpha\beta}$ is the energy-momentum tensor of matter. Although the whole theory is valid for arbitrary $T^{\alpha\beta}$, it is worth noting that for a wide class of matter models $c^{-2}T^{00}$ is the usual Newtonian density of the matter up to terms $\mathcal{O}(c^{-2})$ and $c^{-1}T^{0i}=c^{-2}T^{00}\,v^i+\mathcal{O}(c^{-2})$, where v^i is the velocity field of matter.

Each reference system is constructed for a subsystem of the overall matter distribution. We select a world tube defining the subsystem. All the matter inside this world tube belongs to the system and is called internal. All the matter outside is called external. Since the equations (3) are formally linear, the potentials can be split into internal parts caused by the internal matter of the corresponding subsystem, external parts due to the external matter, and inertial contributions linear with respect to the spatial coordinates, and fixing the origin of each reference system and its rotational state of motion

$$
W=W_{\rm int}(t,\mathbf{x})+W_{\rm ext}(t,\mathbf{x})+Q_i(t)\,x^i, \qquad (4)
$$

$$
W^i=W^i_{\rm int}(t,\mathbf{x})+W^i_{\rm ext}(t,\mathbf{x})-\frac{1}{4}\,c^2\,R^i{}_k(t)\,\dot R^j{}_k(t)\,x^j, \qquad (5)
$$

As we noted above external potentials are at least quadratic with respect to the spatial coordinates, i.e. $W_{\rm ext}=\mathcal{O}(|\mathbf{x}|^2)$, $W^i_{\rm ext}=\mathcal{O}(|\mathbf{x}|^2)$.

$Q_i(t)$ represents the acceleration of a massless particle freely falling in the gravitational field of the external matter with respect to our reference system (t,x^i) (that is, non-geodesic acceleration). In case of the geocentric reference system Q_i results from the interaction of the gravitational potential due to the Earth's nonsphericity with the external gravitational field (mainly with that of the Moon) and amounts to about $4\cdot 10^{-11}\,{\rm m/sec}^2$. For the reference system of an Earth-bound observer $|Q_i|\approx 9.8\,{\rm m/sec}^2$ and represents the acceleration of a freely falling particle with respect to the observer.

The term $R^i{}_k(t)\,\dot R^j{}_k(t)\,x^j$, $R^i{}_j(t)$ being an arbitrary time-dependent orthogonal matrix, reflects the freedom of spatial orientation of the reference systems. If $\dot R^i{}_j(t)=0$, the resulting reference system is a *dynamically nonrotating* one (see, Sect. 5 below). The spatial rotation of the reference system due to that term is not necessarily of relativistic origin.

The origin of each reference system coincides with the center of mass of the corresponding subsystem which means that the multipole expansion of the internal potentials $W_{\rm int}$ does not have a dipole term

$$
\begin{aligned}
W_{\rm int}(t,\mathbf{x})&=G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}M_{i_1i_2\dots i_l}(t)\frac{\partial^l}{x^{i_1}x^{i_2}\dots x^{i_l}}\,\frac{1}{|\mathbf{x}|}\\
&\quad+\mathcal{O}(c^{-2}),\\
M_i&\equiv 0.
\end{aligned}
\qquad (6)
$$

The expansion (6) is mathematically equivalent to the standard expansion of the gravitational potential in terms of spherical harmonics.

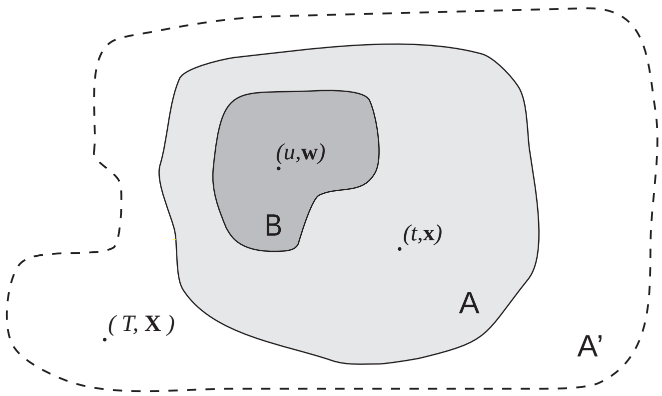

Fig. 1. Hierarchy of material subsystems: the material system A (which may itself be a subsystem of a larger material system A′) is covered with a reference system $(t, \mathbf{x})$; the material subsystem B of the system A is covered with its local reference system $(u, \mathbf{w})$.

In order to construct a reasonable hierarchy of relativistic reference systems we should start from some "mother" or "root" reference system which is constructed under the assumption of isolateness of the corresponding material system. Usually the barycentric reference system of the solar system is used as this "root" reference system, the influence of the gravitational field of our Galaxy and external universe being neglected herewith. It is again a question of accuracy if we can neglect this influence or not. For the "root" reference system both the external parts of the potentials $W_{\rm ext}$ and $W^i_{\rm ext}$ and the acceleration $Q^i(t)$ vanish. Since we consider the material system to be isolated and usually neglect all cosmological effects, there exists a distance at which gravitational field and, therefore, deviation of the space-time from the flat Minkowskian one are arbitrary small. One says that the space-time is asymptotically flat (or asymptotically Minkowskian). The spatial orientation of the "root" reference system could be chosen arbitrarily, but one usually prefers to set $\dot R^{ij} = 0$ and to work with the reference system whose spatial orientation is fixed relative to the asymptotically Minkowskian space. On the contrary, the Observer's reference system is constructed for an observer whose gravitational field could be neglected within the required level of accuracy (e.g., an Earth's satellite, an earthbound telescope, etc.) and the metric tensor of the ORS does not contain internal potentials $W_{\rm int}$ and $W^i_{\rm int}$. All the intermediary reference systems constructed for a massive subsystem of the matter contain all three kinds of terms in their metric tensors. For example, the metric tensor of the geocentric reference system contains internal potentials induced by the Earth's itself, the external potentials due to other bodies of the solar system, and the inertial contributions.

Let us consider some material system A and some subsystem B (see, Fig. 1), and two local reference systems covering them, (t, x^i) and (u, w^i), respectively. System A may itself be a subsystem of a larger material system A′. The coordinate transformations between the two reference systems (t, x^i) and (u, w^i) in both the Brumberg-Kopeikin formalism and the Damour-Soffel-Xu one can be written in the following generic form in notation

$$u = t - \frac{1}{c^2}\left(A(t) + \dot x_2^i r^i\right) + \mathcal{O}(c^{-4}), \tag{7}$$

$$w^i = \underset{(2)}{R}{}^i_{\ j}\,\underset{(21)}{R}{}^j_{\ k}\left(r^k + \frac{1}{c^2}\left(\left(\frac{1}{2}\dot x_2^k \dot x_2^l + D^{kl}\right)r^l + D^{klm} r^l r^m\right)\right) + \mathcal{O}(c^{-4}), \tag{8}$$

$$\frac{d}{dt}A = \frac{1}{2}|\dot{\mathbf{x}}_2|^2 + \underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t)), \tag{9}$$

$$D^{ij} = \delta^{ij}\,\underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t)), \tag{10}$$

$$c^2\,\underset{(21)}{R}{}^i_{\ k}\,\frac{d}{dt}\,\underset{(21)}{R}{}^j_{\ k} =$$
$$+2\left(\frac{\partial}{\partial x^j}\underset{(1)}{W}{}^i_{\rm bg}(t, \mathbf{x}_2(t)) - \frac{\partial}{\partial x^i}\underset{(1)}{W}{}^j_{\rm bg}(t, \mathbf{x}_2(t))\right)$$
$$-\frac{1}{2}\left(\dot x_2^i\,\underset{(2)}{R}{}^k_{\ m}\,\underset{(21)}{R}{}^m_{\ j}\,\underset{(2)}{Q}_k - \dot x_2^j\,\underset{(2)}{R}{}^k_{\ m}\,\underset{(21)}{R}{}^m_{\ i}\,\underset{(2)}{Q}_k\right)$$
$$-\frac{3}{2}\left(\dot x_2^i \frac{\partial}{\partial x^j}\underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t)) - \dot x_2^j \frac{\partial}{\partial x^i}\underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t))\right), \tag{11}$$

$$D^{ijk} = \frac{1}{2}\left(\delta^{ij}\ddot x_2^k + \delta^{ik}\ddot x_2^j - \delta^{jk}\ddot x_2^i\right), \tag{12}$$

where the functions $A(t)$, $\underset{(2)}{R}{}^i_{\ j}(t)$, $\underset{(21)}{R}{}^i_{\ j}(t)$, $D^{ij}(t) = D^{ji}(t)$, $D^{ijk}(t) = D^{ikj}(t)$ depend only on the coordinate time t of the first reference system, $x_2^i(t)$ is the position of the origin of the second reference system relative to the first one, $\dot x_2^i = dx_2^i/dt$, $\ddot x_2^i = d^2x_2^i/dt^2$, $r^i = x^i - x_2^i(t)$, and $\delta^{ij} = \mathrm{diag}(1,1,1)$ is the Kronecker symbol (the identity matrix).

Corresponding transformations in DSX notation read

$$c\,t = z^0(u) + e^0_a(u) w^a + \mathcal{O}(c^{-3}), \tag{13}$$

$$x^i = z^i(u) + e^i_a(u) w^a + \frac{1}{c^2} e^i_a(u)\left(\frac{1}{2}A^a w^2 - w^a A^k w^k\right) + \mathcal{O}(c^{-4}), \tag{14}$$

$$\frac{1}{c}\frac{d}{du}z^0 = 1 + \frac{1}{c^2}\left(\frac{1}{2}|\dot{\mathbf{x}}_2|^2 + \underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t))\right) + \mathcal{O}(c^{-4}), \tag{15}$$

$$e^0_a = \frac{1}{c} e^i_a \frac{d}{du} z^i + \mathcal{O}(c^{-3}), \tag{16}$$

$$e^i_a = \left(1 - \frac{1}{c^2}\underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t))\right)\left\{\delta^{ij} + \frac{1}{2c^2}\dot x_2^i \dot x_2^j\right\} \times \underset{(2)}{R}{}^a_{\ k}(t)\,\underset{(21)}{R}{}^k_{\ j}(t) + \mathcal{O}(c^{-4}), \tag{17}$$

$$A^a = e^i_a \frac{d^2}{du^2} z^i(u) + \mathcal{O}(c^{-2}), \tag{18}$$

$$z^i(u) = x^i_2(t(u)). \qquad (19)$$

The gravitational potentials $\underset{(1)}{W}_{\rm bg}(t, \mathbf{x}_2(t))$ and $\underset{(1)}{W}{}^i_{\rm bg}(t, \mathbf{x}_2(t))$ represent the "background gravitational field" in the first reference system, that is, the gravitational potentials in the metric of the first reference system and generated by the matter external with respect to the subsystem for which the second reference system is constructed. Note that $\underset{(1)}{W}_{\rm bg}$ and $\underset{(1)}{W}{}^i_{\rm bg}$ contain all three kinds of the terms appearing in the gravitational potentials $\underset{(1)}{W}$ and $\underset{(1)}{W}{}^i$ in the first reference system: [a part of] internal potentials $\underset{(1)}{W}_{\rm int}$ and $\underset{(1)}{W}{}^i_{\rm int}$, the external potentials $\underset{(1)}{W}_{\rm ext}$ and $\underset{(1)}{W}{}^i_{\rm ext}$ as well as the inertial contributions $\underset{(1)}{Q}_i x^i$ and $-\frac{1}{4}\,c^2\,\underset{(1)}{R}{}^i{}_k\,\underset{(1)}{\dot R}{}^j{}_k(t)\,x^j$, as appeared in the metric tensor of the first reference system (t, x^i). $\underset{(2)}{Q}_i$ is the acceleration of a massless particle freely falling in the background gravitational field $\underset{(1)}{W}_{\rm bg}$ with respect to the reference system (u, w^i). It appears in (4) written for the second reference system (u, w^i). The matrix $\underset{(2)}{R}{}^i{}_j(t)$ is an arbitrary time-dependent orthogonal matrix, appearing in the metric tensor of the second reference system (u, w^i) (e.g., in (5) written for the second reference system). The matrix $\underset{(21)}{R}{}^i{}_j(t)$ is also an orthogonal matrix, whose time dependence is defined by (11). This matrix represents an additional relativistic rotation and will be discussed in Sect. 5.1.2.

It is not difficult to prove that the forms (7)–(12) and (13)–(19) of the transformation are equivalent. The transformation is a generalization of the Lorentz transformation of special relativity.

4. Spatial rotation of relativistic reference systems

From now on we consider only the reference systems described in Sect. 3. In order to discuss the rotational state of motion of reference systems as well as relative rotation of their spatial axes, we give the following

Definition 1. A reference system (u, w^i) rotates kinematically relative to another reference system (t, x^i) if in the transformations between spatial coordinates x^i and w^i there is a time-dependent orthogonal matrix $R^i{}_j(t)$ $(\det(R^i{}_j) = +1)$

$$w^i = R^i{}_j(t)\,S^j{}_k(t)\,r^k + \mathcal{O}(r^2). \qquad (20)$$

Here $r^i = x^i - x^i_2$, x^i_2 is the trajectory of the origin of the reference system (u, w^i) relative to the reference system (t, x^i), and $S^j{}_k(t) = S^k{}_j(t)$ is a symmetric matrix representing nonrotational deformations of the coordinate chart.

If the rotation is small, the matrix $R^i{}_j(t)$ can be expanded as

$$R^i{}_j(t) = R^i{}_j(t_0) + F^i{}_j(t) + \mathcal{O}(||F^i{}_j||^2), \qquad (21)$$

where $F^i{}_j = -F^j{}_i$ is an antisymmetric matrix and $R^i{}_j(t_0)$ is a time-independent orthogonal matrix. Dropped in (21) are the terms at least quadratic relative to the angle of rotation between the moments of time t_0 and t, which is of order of $||F^i{}_j||$. If the rotation has a relativistic post-Newtonian origin, $||F^i{}_j|| \sim \mathcal{O}(c^{-2})$ and the higher order terms can be formally neglected for a sufficiently long period of time. Such an approximation was used , e.g., in the original version of the Brumberg-Kopeikin formalism (see, Brumberg (1991b), Kopeikin (1991)). However, keeping the rotational matrix $R^i{}_j$ in closed form enables one to discuss the rotation in more general way allowing also time-independent Newtonian rotations between the reference systems.

Let us note that a matrix product of two orthogonal matrices is also an orthogonal matrix and, for example, according to Definition 1 the rotation of the two reference systems (t, x^i) and (u, w^i) related by (7)–(8) or (13)–(14) is defined by $R^i{}_j = \underset{(2)}{R}{}^i{}_k\,\underset{(21)}{R}{}^k{}_j$.

5. Rotational state of motion of reference systems in the theory of relativity

In principle all relativistic reference systems covering the space-time region under study are mathematically equivalent and we could choose any rotational state of motion of a reference system (that is, any $R^i{}_j$ may appear in (5) and (8)). However, there are two choices of $R^i{}_j$ which are certainly interesting for practical applications. The cases of the global ("root") reference system and the local ones should be distinguished herewith.

5.1. Dynamically nonrotating reference systems

One choice of the rotational state of motion of a relativistic reference system is certainly preferred, which leads to the so-called dynamically nonrotating reference system. It is commonly known that dynamically nonrotating reference systems are a relativistic generalization of nonrotating, though possibly accelerated, reference systems of Newtonian mechanics (often referred to as quasi-inertial ones). An important property of those reference systems is that the equations of motions of test particles relative to them do not contain Coriolis and centrifugal forces ($\Omega^i = 0$ in (1)). The same property is true in the relativistic framework. However, some care must be taken to give a rigorous meaning of that statement. Relativistic equations of motion are much more complicated than the Newtonian ones and one could artificially extract terms resembling the Coriolis and centrifugal accelerations (e.g., by expanding the relativistic equations of motion in power series in the vicinity of the origin of the reference system in use).

5.1.1. Dynamically nonrotating global reference system

Let us first consider the "root" reference system constructed for an isolated material system. Far away from the material sources the gravitational field asymptotically vanishes and the space-time asymptotically becomes flat Minkowskian space-time of special relativity. We can define the "root" reference system in such a way that its spatial axes infinitely far from the material sources do not rotate relative to a cartesian inertial reference system in the asymptotic Minkowskian space. Mathematically it can be expressed as

$$\lim_{|\mathbf{x}|\to\infty} g_{\alpha\beta}(t,\mathbf{x}) = \eta_{\alpha\beta}, \tag{22}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ is the Minkowskian metric. We can say that the asymptotically Minkowskian reference system obeying (22) is dynamically nonrotating in the sense that the equations of motion of a test particle far away from the gravitating masses coincide with those in an inertial reference system of special relativity and contain no Coriolis and centrifugal forces. Eq. (22) implies certain boundary conditions for Eqs. (3) defining the gravitational potentials $W = W_{\rm int}$ and $W^i = W^i_{\rm int}$.

5.1.2. Dynamically nonrotating local reference systems

For a local reference system constructed for a subsystem of an isolated material system the concept of dynamically nonrotating spatial orientation becomes more intricate. For a reference system of a massless observer it can be defined as follows.

Definition 2. A local reference system of a massless observer is called dynamically nonrotating if vectors tangent to its coordinate lines at its origin move according to the Fermi-Walker transport rules.

The Fermi-Walker transport is a well defined mathematical procedure (see, e.g., Synge, 1960 for details) providing a complete system of equations for the components of the tangent vectors in the reference system in which the motion of the observer is considered.

An important property of a dynamically nonrotating local reference system is that the equations of motion of a test particle relative to that system do not contain Coriolis or centrifugal forces. The metric tensor of a local reference system (t, x^i) of a massless observer does not contain internal potentials and therefore, can be expanded in Taylor series in the vicinity of the origin $x^i = 0$. The terms of the second and higher orders with respect to x^i represent the tidal gravitational field of gravitating masses, while the linear terms are inertial contributions. Within our scheme of approximations (2)–(19) one can also see that if the mixed component of the metric, g_{0i}, shows such a linear term, the equations of motion of a test particle in the vicinity of the origin do contain Coriolis and centrifugal forces. The latter circumstance allows one to judge easily whether a given reference system is dynamically nonrotating or not: it is basically sufficient to check the mixed component of its metric.

In case of a local reference system of a massive material subsystem the concept of a dynamically nonrotating orientation requires even more detailed definition. The metric tensor of such a reference system contains not only inertial terms and external tidal gravitational potentials, but also internal contributions. The idea of introducing a dynamically nonrotating local reference system of a massive subsystem comes from the split of the metric tensor into three well recognizable pieces: (1) the gravitational influence of internal matter of the subsystem (at the level of accuracy under consideration the internal gravitational field is the same as if the subsystem were isolated), (2) tidal gravitational terms due to external material sources, (3) inertial contributions which depend on the orientation of the spatial axes of the local reference system, and also on the trajectory of its origin. This split is indicated in (4)–(5). Then we can give the following

Definition 3. A local reference system of a massive material subsystem defined by (2)–(6) is called dynamically nonrotating if in (5) the linear term vanishes: $\dot{R}^i_j(t) \equiv 0$.

One can show that Definition 3 is in agreement with Definition 2 in the limit of a massless subsystem. One can also say that the vectors tangent to the coordinate lines at the origin of the local dynamically nonrotating reference system of a massive subsystem move according to the Fermi-Walker transport in the background gravitational field defined by $W_{\rm bg}$ and $W^i_{\rm bg}$ (see, discussion after (19)). The post-Newtonian equations of motion of a test particle in the vicinity of the origin (i.e., the center of mass of the subsystem) can be shown to contain terms due to internal matter, tidal terms due to external masses and possibly the term Q_i reflecting acceleration of the origin. No Coriolis and centrifugal forces appear there.

However, dynamically nonrotating reference systems moving along different world lines may rotate with respect to each other in the sense of Definition 1. Let us consider again two reference systems (t, x^i) and (u, w^i) whose metric tensors have the form (2)–(6) and the transformations between them are defined by (7)–(12) or (13)–(19). If the orthogonal matrices $\underset{(1)}{R}{}^i{}_j(t)$ and $\underset{(2)}{R}{}^i{}_j(u)$ appearing in the metric tensors of the first and the second reference systems, respectively, vanish, both reference systems are dynamically nonrotating. However, from the transformations (7)–(12) or (13)–(19) one can see that the spatial axes of the reference systems rotate relative to each other, and this rotation is defined by the matrix $\underset{(21)}{R}{}^i{}_j(t)$ defined by (11). The rotation is usually called relativistic precession and consists of three components: (1) Lense-Thirring or gravitomagnetic precession due to the background vector potential $\underset{(1)}{W}{}^i_{\rm bg}$ (the first line of (11)), (2) Thomas precession due to the acceleration Q^i_2 (the second line of (11)), and (3) de Sitter or geodetic precession due to Newtonian background potential $\underset{(1)}{W}_{\rm bg}$ (the last line of (11)). Let us note that the sum of all three terms is also sometimes called geodetic precession.

5.2. Kinematically nonrotating reference systems

Another possible definition of the rotational state of motion of a reference system in the relativistic framework leads to so called kinematically nonrotating reference systems. Again cases of global and local reference systems should be distinguished.

5.2.1. Kinematically nonrotating global reference systems

The concept of a kinematically nonrotating reference system, as it is usually understood, is defined rather operationally and probably comes from the basic Newtonian (or Machian) idea of astrometry: we want to construct a reference system and the corresponding reference frame showing no rotation relative to distant celestial objects (stars or quasars). Such a property is used as a "definition" of the rotational state of motion of the relativistic reference systems to be used in astronomical practice given by the recent recommendations of the IAU (1991) (see, Recommendation II in Bergeron (1991)). However, from the relativistic point of view, such a definition is ambiguous.

Here as in Newtonian physics there are two aspects of the problem: astronomical (observational) and physical (theoretical). It is also clear that the idea itself of a reference system showing no rotation relative to distant celestial objects is applicable only to reference systems covering the space region where those distant objects reside, that is, to global reference systems.

From a theoretical point of view some attempts to provide more mathematical rigor to that notion have been undertaken in Soffel (1989) where spatial reference directions were discussed from the viewpoint of Weyl parallelism and the notion of a stellar compass was introduced to mathematically describe a global kinematically nonrotating reference system covering an isolated material system. Within our approximation scheme the "root" reference system of a hierarchy is constructed under the assumption of isolateness of the material system under study. Independent of the "root" reference system itself the space-time produced by an isolated material system is asymptotically Minkowskian and it is quite natural to call a global reference system kinematically nonrotating if its spatial axes far away from the material sources show no rotation relative to the asymptotically Minkowskian space. Mathematically this again can be expressed by (22). Therefore, the "root" reference system can be constructed to be both dynamically and kinematically nonrotating.

From the observational point of view even for such global asymptotically Minkowskian reference systems the property of a reference system and the corresponding reference frame to show no rotation relative to a set of remote objects is no more than a *conjecture* concerning the trajectories of the remote objects relative to that reference system. We can adopt *by definition* that a set of remote objects do not move with respect to the reference system in question or at least their motion can be neglected (usually, because of large distance to the objects and/or our poor accuracy of observations). At most, we can admit that the sources move relative to our reference system with a constant velocity (it is better to say that the origin of our reference system moves relative to "motionless" distant sources with a constant velocity), which would change positions of all of them (because of secular aberration), but would not give time-dependent variations of the positions. The assumption for a set of sources to be motionless should be verified by observations. In fact, at different levels of observational accuracy one can consider different sets of sources to be at rest. For example, if our accuracy be of order of $1'$ and the period of observation be a few decades, we could believe that all the stars are at rest relative to the barycentric RS. Contemporary accuracy of observations allows us to measure proper motions of many stars and, therefore, reveals them to move relative to each other and to the barycentric RS. Not discussing the internal structure of quasars which can provide in some cases a time-dependence of their effective observable positions, we can say that at the present level of accuracy we cannot measure proper motions of quasars and, therefore, we can believe quasars to be at rest with respect to the barycentric RS.

5.2.2. Kinematically nonrotating local reference systems

In order to define a local kinematically nonrotating reference system, we have to appeal to some global reference system to provide a link from that local reference system to distant objects. In the hierarchy of relativistic reference system there is one "root" reference system which is nonrotating both dynamically and kinematically.

Definition 4. If a local reference system does not rotate with respect to a kinematically nonrotating global reference system (a "root" reference system of a hierarchy) according to Definition 1, then that local reference system is also kinematically nonrotating.

This definition seems to be intuitively clear. However, here we have two important complications. The property of two reference systems to show no rotation relative to each other in the sense of Definition 1 is neither transitive nor symmetric.

Non-transitivity. This question has been technically discussed in Klioner (1993). The problem is that the Lorentz transformation without rotation, as it is well known (see, e.g., Møller (1972)), is not transitive. That is, two subsequent Lorentz transformations, each of which gives no spatial rotation, are equivalent to one Lorentz transformation which may involve such a rotation. As a result, unless special (and unrealistic) physical conditions are met, three relativistic reference systems cannot be nonrotating relative to each other in the sense of Definition 1.

Let us make our statement more rigorous. Let us consider three reference systems (t, x^i) (system 1), (u, w^i) (system 2) and (τ, ξ^i) (system 3) (see, Fig. 2). As it was discussed in Klioner (1993) we have three transformations similar to (7)–(8) or (13)–(14): transformations from (t, x^i) to (u, w^i), those from

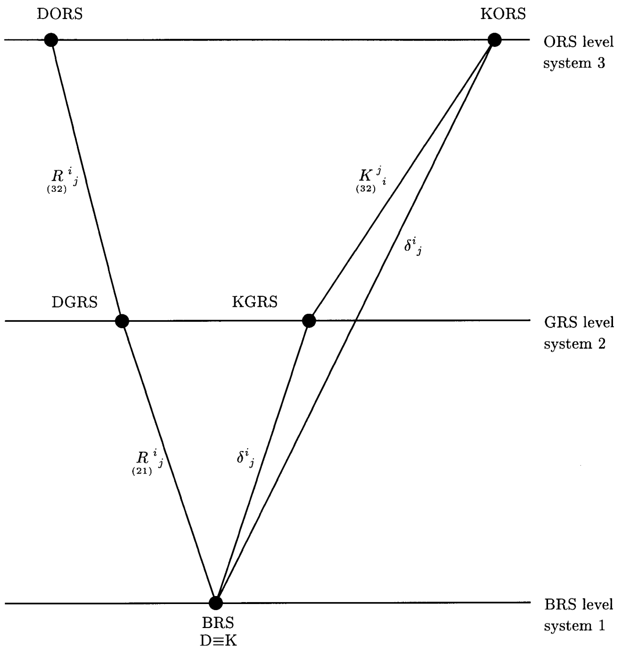

Fig. 2. Hierarchy of relativistic reference systems (see, also Klioner, 1993). Each branch in the graph corresponds to a generalized Lorentz transformation. The value ascribed to a branch ($\underset{(21)}{R}{}^i{}_j$, $\underset{(32)}{R}{}^i{}_j$, $\underset{(32)}{K}{}^j{}_i$ or $\delta^i{}_j$) defines the spatial rotation in the relevant transformations. D and K mean dynamically and kinematically nonrotating reference systems, respectively.

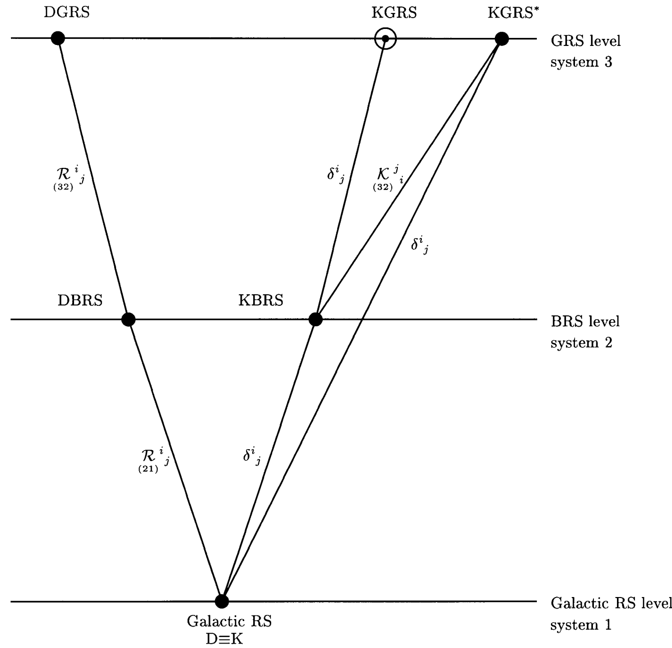

Fig. 3. Hierarchy of relativistic reference systems starting from a Galactical RS. The DBRS and KBRS differ from each other by a negligible secular rotation of order of $\sim 1\mu$as per century. The encircled point at the GRS level of the graph – KGRS – corresponds to the GRS that is used nowadays in astronomical practice. The KGRS* does not rotate relative to the GalRS. The KGRS and KGRS* differ from each other by a time-dependent rotation of order of 8 mas (see text).

(u, w^i) to (τ, ξ^i), and those relating (t, x^i) directly to (τ, ξ^i). All three transformations involve an orthogonal matrix R^i_j in (8).

Let us denote the orthogonal matrix appearing in the transformations from reference system A to reference system B as $\underset{(ba)}{E}{}^i{}_j$. For example, the transformations (7)–(8) or (13)–(14) involve the orthogonal matrix $\underset{(21)}{E}{}^i{}_j = \underset{(2)}{R}{}^i{}_j \, \underset{(21)}{R}{}^j{}_k$.

By considering explicitly two subsequent transformations (7)–(8) or (13)–(14) from (t, x^i) to (u, w^i) and from (u, w^i) to (τ, ξ^i), and the analogous direct transformations from (t, x^i) to (τ, ξ^i) one can show that (see, also Klioner, 1993)

$$\underset{(31)}{E}{}^i{}_j = \underset{(32)}{E}{}^i{}_a \, \underset{(21)}{E}{}^a{}_b \, \underset{(32)}{K}{}^b{}_j + \mathcal{O}(c^{-4}), \tag{23}$$

where

$$c^2 \, \underset{(32)}{K}{}^i{}_k \frac{d}{dt} \underset{(32)}{K}{}^j{}_k =$$

$$-2 w_3^{[i} \frac{d^3}{dt^3} x_2^{j]} - 3 \frac{d}{du} w_3^{[i} \frac{d^2}{dt^2} x_2^{j]} - \frac{d^2}{du^2} w_3^{[i} \frac{d}{dt} x_2^{j]} + \mathcal{O}(c^{-2}). \tag{24}$$

The structure of the matrix $\underset{(32)}{K}{}^i{}_j$ can be seen from the following approximate solution of (24), which is valid for a short period of time

$$\underset{(32)}{K}{}^i{}_j = \delta^i{}_j + \frac{1}{c^2} \left(2 w_3^{[i} \frac{d^2}{dt^2} x_2^{j]} + \frac{d}{du} w_3^{[i} \frac{d}{dt} x_2^{j]} \right) + \mathcal{O}(c^{-4}). \tag{25}$$

Here $A^{[ij]} = (A^{ij} - A^{ji})/2$ is the antisymmetric part of A^{ij}, w_3^i is the position of the origin of the third reference system relative to the second one, and x_2^i is the position of the origin of the second reference system with respect to the first one. Therefore, even if the second reference system do not rotate relative to the first one ($\underset{(21)}{E}{}^i{}_j = \delta^i{}_j$), and the third reference system do not rotate relative to the second one ($\underset{(32)}{E}{}^i{}_j = \delta^i{}_j$), the third reference system may still rotate relative to the first one ($\underset{(31)}{E}{}^i{}_j = \underset{(32)}{K}{}^i{}_j \neq \delta^i{}_j$). Now we come to the conclusion that the property of two RSs being nonrotating relative to each other is *not transitive*.

Let us note that the non-transitivity appears only when we consider three reference systems on three different levels of a hierarchy (that is, when each pair of the three reference systems is related by the generalized Lorentz transformations (7)–(8) or (13)–(14). For example, the KGRS and the DGRS in Fig. 2 are related by a pure spatial rotation

$$\begin{aligned} u_{\rm DGRS} &= u_{\rm KGRS}, \\ w^i_{\rm DGRS} &= \underset{(21)}{R}{}^i{}_j \, w^j_{\rm KGRS}, \end{aligned} \tag{26}$$

and if we consider the DGRS, KGRS and the BRS as the three reference systems, the non-transitivity disappears.

Non-symmetry. It is worth noting that Definition 1 of spatial rotation between two RSs is not symmetric: it defines how reference system (u, w^i) rotates relative to reference system (t, x^i), but not vice verse. In order to judge how reference system (t, x^i) rotates relative to reference system (u, w^i) one should, according to Definition 1, invert the transformations between the reference systems and expand them in Taylor series in the neighborhood of the origin of (t, x^i). Generally speaking, this is not always possible. If (u, w^i) is constructed for a material subsystem of a larger system covered with (t, x^i), the area of definition of (u, w^i) may not even cover the origin of (t, x^i) (e.g., due to acceleration of the origin of (u, w^i)). Nevertheless, for practically important cases in the solar system (for example, for the Barycentric Reference System (BRS) of the solar system and any local planetocentric reference system) the inversion is possible. It can be shown that due to the nonlinear terms in the transformation of the spatial coordinates (8) or (14), the rotational matrix appearing in the inversion of the transformations is not equal to the inversion of the matrix appearing in the direct transformations. Thus, the property of two RSs to show no rotation relative to each other in the sense of Definition 1 turns out to be *nonsymmetric*. Thus, if we consider the BRS and the GRS defined in such a way that coordinate transformations from the barycentric coordinates to the geocentric ones do not involve spatial rotation, the inverse transformations contain a quasi-periodic rotation with maximal amplitude of order of $25\mu\mathrm{as}$ and a typical period of 1 month. This means that an unambiguous definition of kinematically nonrotating GRS must also contain the *direction* of the transformations to the kinematically nonrotating global reference system.

The non-transitivity and non-symmetry discussed above should be considered as two illustrations showing that the concept of two reference systems nonrotating relative to each other must be rigorously defined to be useful and unambiguous in the relativistic framework. It is rather meaningless to speak about kinematically nonrotating reference systems in an "absolute" sense. Attempts to use our "Newtonian common sense" in this question often lead to confusions.

6. Reference systems inside the solar system

Let us consider the standard hierarchy of relativistic reference systems consisting of the Observer's RS, Geocentric and the Barycentric ones (Fig. 2). If we neglect the influence of the gravitational field of the Galaxy, the BRS can be considered to be both dynamically and kinematically nonrotating. Here it acts as "root" frame. The metric tensor of the GRS is constructed on the basis of the BRS metric using the generalized Lorentz transformations (7)–(8) or (13)–(14). The orthogonal matrix $\underset{(2)}{R}{}^i{}_j(t)$ entering these transformations can be arbitrary. However, we consider only two choices discussed above: a dynamically nonrotating GRS (DGRS in Fig. 2) corresponding to $\underset{(2)}{\dot{R}}{}^i{}_j(t) \equiv 0$ or kinematically nonrotating one (KGRS in Fig. 2) with $\underset{(2)}{R}{}^i{}_j(t)\, \underset{(21)}{R}{}^j{}_k(t) \equiv \delta^i{}_k$.

The metric tensor of the ORS can be constructed starting from metric tensors of the DGRS, KGRS or directly the BRS, and relevant coordinate transformations. Again, the corresponding orthogonal matrix in the transformations can be chosen to give dynamically or kinematically nonrotating versions of the ORS (DORS and KORS in Fig. 2, respectively). It is very important that both KGRS and KORS do not rotate relative to the BRS, but they do rotate with respect to each other. This rotation can be described by the matrix

$$\underset{(32)}{K}{}^i{}_j = \delta^i{}_j + \frac{1}{c^2}\left(2w_{\rm o}^{[i}a_{\rm E}^{j]} + v_{\rm o}^{[i}\,v_{\rm E}^{j]}\right) + \mathcal{O}(c^{-4}), \qquad (27)$$

where $w_{\rm o}^i$ and $v_{\rm o}^i$ are the position and velocity of the ORS's origin relative to the GRS, $v_{\rm E}^i$ and $a_{\rm E}^i$ are the velocity and acceleration of the geocenter with respect to the BRS.

The rotation $\underset{(21)}{R}{}^i{}_j$ of the DGRS relative to the BRS or of the DGRS relative to the KGRS (see, Eq. (26)) is the well-known geodetic precession (see, e.g., Soffel (1989)), which results mainly from the third (de Sitter) term in (11). This effect can be split into a secular part ~ 1.92 mas per century (it can be called geodetic *precession* in the proper (astronomical) sense) and a periodic part with an amplitude ~ 0.153 mas and a period of one year (geodetic nutation (see, Fukushima (1991)).

As for rotations of the ORS, several cases should be considered. For an Earth-based observer the additional rotation of the DORS relative to the DGRS due to $\underset{(32)}{R}{}^i{}_j$ may amount to $0.66''$ per year plus periodic terms, both the second and third (Tolman and de Sitter) terms in (11) being important. In this case the KORS rotates with respect to the KGRS due to $\underset{(32)}{K}{}^i{}_j$ with a maximal amplitude of 0.02 mas and a diurnal period. For an observer situated on an Earth's satellite $\underset{(32)}{R}{}^i{}_j$ gives the well-known Schiff precession amounting to $8''$ per year for a low orbiting satellite and also contains periodic terms proportional to the eccentricity of the orbit. Here the most important effect comes from the third (de Sitter) term in (11). The effect of $\underset{(32)}{K}{}^i{}_j$ in this case may amount to 0.4 mas with the orbital period.

7. Galactic effects

The effects under investigation come out in an interesting way when considering the influence of the Galaxy upon reference frames constructed on the basis of observations performed from the Earth's surface.

In most cases the construction of the relativistic reference systems to be used for processing modern astronomical observations is based upon the assumption of isolateness of the solar system. That is, the influence of the Galaxy and outer Universe on internal dynamics of the solar system and on the results of astronomical observation of distant celestial sources is supposed to be negligible.

Probably, in the not-so-distant future modern observational techniques (first of all, Very Long Baseline Interferometry) will allow us to measure the time-dependent displacements of extragalactic sources resulted from the acceleration of the solar system relative to the galactic center, and therefore, obtain observational evidence of noninertiality of the BRS (Eubanks, 1993;

Eubanks, *et al*., 1995). The acceleration of the solar system's barycenter changes the velocity of the solar system with respect to extragalactic sources, which in turn results in time dependence of the aberrational shifts of their positions. The effect should be of order of 5 microarcsec per year, provided that the solar system's orbit relative to the Galaxy is roughly circular ($A_{\rm b} = V_{\rm b}^2/X_{\rm b} \sim 2.5\cdot 10^{-10}{\rm m/s}^2$, where $X_{\rm b}$, $V_{\rm b}$ and $A_{\rm b}$ are the position, velocity and the acceleration of the solar system's barycenter relative to the galactic center), and can be larger if the solar system moves in a more complicated way.

It does not mean that we must use a Galactic RS as a "root" reference system for our hierarchy instead of the BRS, but nevertheless some procedure should be adopted to account for the change of the positions of extragalactic sources induced by the noninertiality of the BRS. Therefore, we have to consider one more level of the reference systems in our relativistic hierarchy depicted in Fig. 3. Fig. 3 looks much like Fig. 2, but the root level here represents a reference system bound to the Galaxy as a whole, which we call Galactic RS (GalRS). If we start from the GalRS then we have two versions of the BRS: dynamically nonrotating (DBRS) and kinematically nonrotating (KBRS). The geodetic precession $\underset{(21)}{\mathcal{R}}{}^i{}_j$ of the BRS due to its motion around the Galaxy center is very small, namely of order $\sim 1\mu{\rm as}$ per century (Brumberg, 1991a). This value is far too small to be detectable in the foreseeable future, and, therefore, the difference between DBRS and KBRS is not very important.

In Fig. 3 the GRS is in the same position as the ORS in Fig. 2. The GRS that is in use presently shows no rotation with respect to the BRS (it does not matter with respect to which version of the BRS (KBRS or DBRS), because the difference between them is negligible). However, if we consider the GalRS as our "root" reference system, that is, if we suppose that distant sources do not move with respect to the GalRS (and, therefore, the GalRS is the best possible approximation to a global kinematically nonrotating reference system), we have to say that a kinematically nonrotating GRS should show no rotation with respect to the GalRS. This gives another version of the GRS – KGRS* in Fig. 3. It is again important that the KGRS* rotates relative to the GRS in current use (KGRS), and this rotation, described by analogy with (27) as

$$\underset{(32)}{\mathcal{K}}{}^i{}_j = \delta^i_j + \frac{1}{c^2}\left(2x_{\rm E}^{[i}A_{\rm b}^{j]} + v_{\rm E}^{[i}V_{\rm b}^{j]}\right) + \mathcal{O}(c^{-4}), \tag{28}$$

cannot be neglected. In (28) $x^i_{\rm E}$ and $v^i_{\rm E}$ are the position and velocity of the GRS origin (geocenter) relative to the BRS, $V^i_{\rm b}$ and $A^i_{\rm b}$ are the velocity and acceleration of the solar system barycenter with respect to the GalRS.

It is easy to see that the rotation induced by $\underset{(32)}{\mathcal{K}}{}^i{}_j$ is periodic with a period of one year resulting from the annual motion of the Earth. In order to give its numerical estimate it is necessary to know $V^i_{\rm b}$ and $A^i_{\rm b}$. The term in (28) proportional to $A^i_{\rm b}$ is many orders of magnitude smaller that the other one and can be neglected.

Although the velocity of the solar system relative to the galactic center is known with large uncertainty, for an order-of-magnitude estimate we can consider that the motion of the solar system consists of two parts: 1) the motion of the solar system relative to the local standard of rest (centroid) toward the solar apex whose galactic longitude l and latitude b are ($l = 51^\circ 03', b = 25^\circ 38'$) with a velocity of $V_{\rm apex} = 20$ km/s (Miyamoto, Sôma, 1993); 2) the motion of the centroid toward ($l = 90^\circ$, $b = 0^\circ$) with a velocity $V_{\rm centroid} = 220$ km/s. The total velocity is, therefore, $V_{\rm b} = 235$ km/s and directed to $(l = 87^\circ 13', b = 2^\circ 8') = (\alpha = 20^{\rm h}51^{\rm m}30^{\rm s}, \beta = 47^\circ 39')_{J2000}$.

Therefore, the amplitude of the effect

$$\le \frac{1}{2c^2}|V^i_{\rm b}|\,|v^j_{\rm E}| \sim 0.0080''. \tag{29}$$

We can see that the effect of $\underset{(32)}{\mathcal{K}}{}^i{}_j$ is much larger than modern accuracy of observations. Therefore, once one has to account for the noninertiality of the BRS, one must also consider the rotation described by $\underset{(32)}{\mathcal{K}}{}^i{}_j$. Let us stress that this rotation does *not* represent a new physical effect. It has a purely coordinate nature and can be considered as one more illustration showing that the notion of a kinematically nonrotating reference system is simply a mathematical concept having no profound physical meaning in the theory of relativity.

If we were to adopt the KGRS* instead of the KGRS as a reference system to be used when treating astrometric observations we would get an additional rotation of the celestial reference frame (a set of celestial (BRS) coordinates of distant sources) relative to the terrestrial reference frame (a set of terrestrial (GRS) coordinates of observing sites). Since this rotation is periodic to a high level of accuracy, it can be interpreted as additional terms in the difference between the apparent celestial coordinates and the mean ones

$$\begin{aligned}\Delta\alpha &= K^3 - (K^1\cos\alpha + K^2\sin\alpha)\tan\delta,\\ \Delta\delta &= K^1\sin\alpha - K^2\cos\alpha,\end{aligned} \tag{30}$$

where (α,δ) are right ascension and declination, and $K^i = \frac{1}{2}\varepsilon^i{}_{jk}K^j{}_k$ is a vector defining direction and magnitude of the rotation defined by $K^j{}_k$. For the Galactic effect we have

$$\mathcal{K}^i = \frac{1}{2}\varepsilon^i{}_{jk}\underset{(32)}{\mathcal{K}}{}^j{}_k \approx \begin{pmatrix} -0.0070''\cos l' \\ -0.0015''\cos l' - 0.0059''\sin l' \\ 0.0034''\cos l' - 0.0040''\sin l'\end{pmatrix}, \tag{31}$$

where $l' = 357^\circ.52910918 + 129596581''.0481\,t + \dots$ is the mean anomaly of the Sun, t being measured in Julian Centuries from J2000.0. In (31) we neglected the eccentricity of the Earth's orbit. We know that the luni-solar nutation of the Earth is another source of time variation of apparent celestial coordinates, which is defined by (30) with

$$K^i_{\rm nut} = \begin{pmatrix} \Delta\varepsilon \\ -\sin\varepsilon\,\Delta\psi \\ \cos\varepsilon\,\Delta\psi \end{pmatrix}. \tag{32}$$

However, one can see that the rotation described by (30)–(31) is more general than nutation and cannot be interpreted as additional terms in nutation series. Nutation is a specific rotation since it requires

$$K^2_{\rm nut}\,\cos\varepsilon + K^3_{\rm nut}\,\sin\varepsilon = 0, \tag{33}$$

Table 1. Parallels between the concepts of simultaneity (synchronized clocks) and kinematically nonrotating reference systems.

Concept	Newtonian physics	Relativistic physics
Synchronized clocks	Synchronization relative to Newtonian absolute time	Synchronization relative to coordinate time of a reference system
Kinematically nonrotating reference systems	Rotation relative to Newtonian absolute space	Rotation relative to spatial axes of a global asymptotically Minkowskian reference system

where $\varepsilon \approx 23^{\circ}26'21.412''$ is the obliquity of the ecliptic. The latter condition is not true for the terms proportional to $\sin l'$ in (31).

One could go further and consider higher levels of hierarchy of astronomical reference systems (a reference system tied to Local Group of galaxies, or Cosmic Microwave Background radiation (CMBR)). According to the preliminary results of the COBE mission (Smoot, *et al.*, 1991), the velocity of the solar system with respect to the CMBR field is 365 ± 18 km/s while the solar velocity relative to the Local Group of Galaxy is 308 km/s (Yahil, Tamman & Sandage, 1977). Although it seems to be a problem of very distant future let us note that the order of magnitude of the effects would be the same as in (29)–(31).

8. Concluding remarks

We have demonstrated that the notion itself of a kinematically nonrotating reference system in the framework of relativity, though still quite useful, should be re-formulated more rigorously. It is insufficient to say simply that a local geocentric RS should show no rotation with respect to a set of distant celestial objects, as it was done in the recommendations of the IAU (1991). We have seen that one can find several reference systems satisfying this definition, which would rotate with respect to each other. In fact the kinematical rotation of a local (e.g., geocentric) reference system as defined by Definitions 1 and 4 is a pure mathematical convention which has no profound physical meaning.

In a sense, the basic goal of astrometry, i.e. the construction of an “inertial” celestial reference frame, also should be re-formulated and adapted to the theory of relativity and modern relativistic theory of astronomical reference systems. Probably, one can say that the aim of astrometry is to get a physical realization of a relativistic reference system, that is, to obtain coordinate directions, proper motions, parallaxes, etc. of a set of celestial objects with respect to that reference system. Nowadays the BRS is used as such a reference system and is believed to be the best possible (for the moment) approximation to inertiality. However, in the future one can adopt another reference system, which would be closer to an ”inertial” one. Since in the theory of relativity we can always construct a 4-dimensional coordinate transformations between the BRS and its possible superseder, the BRS position of a source can be easily recomputed into its position relative to that other reference system, and, therefore, the positions of celestial objects relative to one reference system can be used to construct the astrometric materialization of another reference system.

It is interesting that the situation with kinematical rotation looks quite similar to that with the notion of simultaneity and synchronization of remote clocks (see Table 1). As long as the technical accuracy of clocks was low enough one used the Newtonian concept for clock synchronization. In Newtonian physics simultaneity is an absolute notion, and synchronization of clocks is performed relative to the Newtonian absolute time. At some level of accuracy the Newtonian concept goes wrong and the theory of relativity should be used instead. In the theory of relativity both simultaneity and synchronization become “relative” and coordinate-dependent. Synchronization is possible only with respect to a coordinate time of a relativistic reference system (see, e.g., Ashby, Allan, (1979) and Klioner, (1992) for a detailed discussion). Again different reference systems can be used to synchronize clocks depending on physical situation and purposes, and using coordinate transformations one can easily compute clock corrections with respect to coordinate times of different reference systems.

We can speak of another example when a concept, which was believed to be intuitively clear and natural, should be adapted to the framework of relativity, after which the concept becomes rather artificial, less “natural”, but more mathematically rigorous.

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