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A point explosion in a freely expanding medium

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Abstract. We present an exact invariant solution for a point explosion in an inertially expanding medium. The blast wave dynamics continuously joins the self-similar Sedov asymptotics at early times and asymptotical comoving with the ambient medium at late times. The applications to galactic supershell blowout dynamics and cosmological explosions are discussed.

Key words: Shock waves – ISM: supernova remnants – early Universe

1. Prologue

Perhaps no approach in a theory of explosive phenomena has made such a profound impact on the development of analytical methods of investigation as the classical self-similar Sedov solution for a blast wave generated by a point explosion in a uniform static medium (Sedov 1946, 1959; Taylor 1950). Known in hydrodynamics long before 1946, self-similarity most impressively demonstrated, in this instant, a lucky opportunity to obtain an exact analytical solution to a complicated hydrodynamic problem described by the nonlinear partial differential equations. This marvellous ability to obtain exact solutions owes its existence to the fact that self-similarity advantageously employs some symmetrical properties of the physical system, namely its ability for scaling. Ruling out those degrees of freedom which are invariant under scaling transformations allows to diminish the dimensionality of the problem in hand and to study the reduced problem by simpler and more available means.

Self-similarity in the Sedov problem was dictated by necessity because the number of the dimensional governing parameters was four and only four which allowed to obtain solution through the solely possible power-law combination of independent variables and dimensional parameters with the rational power index. Later it was understood that self-similarity may arise also in case when dimensional analysis fails so that the power-law similarity indices may become irrational (Zel'dovich & Raizer 1966). These two cases are distinguished now as similarities of the first and second kind respectively (Barenblatt 1979).

A remarkable success of similarity technique inspired and continues to inspire the followers to search for the solutions for blast waves dynamics in all possible situations in a self-similar form.

There are many situations in which self-similarity arises naturally and is dictated by the physics of the problem, because the number of essential physical factors is exactly the required minimum. There are much more situations in which the number of parameters exceeds the required minimum number and self-similarity is generally broken but can be allowed under some particular conditions imposed on these extra parameters. In the former case the self-similar solutions reflect the generic, universal behaviour of the physical system manifesting itself either as an intermediate (an adiabatic stage of supernova remnant expansion into an interstellar medium (Sedov 1959), an adiabatic shock wave in an exponential atmosphere (Raizer 1964; Grover & Hardy 1966), an adiabatic interaction of supernova blast wave with a circumstellar envelope (Chevalier 1982; Nadyozhin 1985), a blast wave in an inhomogeneous medium with evaporating clouds (Chieze & Lazareff 1981; White & Long 1991)), or an end-state asymptotics (cosmological blast waves (Ikeuchi et al. 1983; Bertschinger 1983; Kovalenko & Sokolov 1993), cosmological detonation waves (Bertschinger 1985; Kazhdan 1986), etc.). In the latter case the self-similar solutions describe just very particular, non-generic situations that have no chance to be realized in nature except perhaps under specially prepared initial, boundary or some other conditions. An enormous amount of works, however, which assume self-similarity a priori artificially adjusting the governing parameters to a required form in order to obtain an analytical or semianalytical solution clearly indicate at once popularity and scantiness of available analytical tools.

Indeed, similarity method remains nowadays perhaps a unique well-known systematic method to search for (and to find) exact unsteady solutions for blast waves. At the same time the blast wave dynamics in actual situations is affected by various factors such as the radiative energy losses behind the shock wave, nonuniformity of the unperturbed medium, non-zero ambient pressure, magnetic fields, and many others, which destroy the self-similarity of the problem. In this case, as far as we are

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aware, there were no precedents of finding an exact unsteady solution – the usual practice is either to make use of the different analytical approximate methods such as the Kompaneets method (Kompaneets 1960), the sector approximation (Laumbach & Probstein 1969), the thin shell approximation (Chernyi 1957), the shell moment approximation (Ostriker & McKee 1988), composite combinations of these four (Koo & McKee 1990), etc., or to study the problem numerically. Meanwhile there exists a more powerful and general method of analytical study – the symmetry analysis of differential equations which incorporates similarity analysis as one of the particular cases. It shares much the same principal idea of reduction of the system of equations under investigation, but it allows to operate with more various symmetries, not only with the simple scaling. If the similarity analysis partly leans upon the intuitive approach, the symmetry analysis provides direct instructions how to find symmetries and their associated solutions. Successfully used in many fields from the field theory to applied hydrodynamics (Vinogradov 1989; Coggeshall & Axford 1986), it is unfortunately still not widely spread in astrophysical science. At the same time the nonlinear astrophysical problems formulated in terms of partial differential equations offer a fertile field for employment of the symmetry analysis where it performs at its best. We believe that this modern mathematical technique extending analytical tools of analysis and complementing power (though still not outperforming flexibility) of the numerical analysis may soon be in common use not only for pure mathematicians but also for applied researchers especially in connection with the development of software for symbolic computations. Indication of a progress becomes increasingly clear (Ibragimov 1994, 1995). The present paper provides an example of how this formalism allows to obtain exact analytical solutions to the complicated nonlinear problems in astrophysics.

2. Introduction

One of the factors governing the blast wave dynamics is the non-stationarity of the background. This appears in various astrophysical situations. Let us outline some of them.

(1) For the first $\sim 10^5$ years the expansion of a Type II supernova remnant occurs inside a rarefied cavity produced by the wind of pre-supernova. The pre-explosion gas expands within a cavity with deceleration ($v \sim 3r/5t$) since the wind loses its energy to the shift of the outer layers of interstellar gas (Weaver et al. 1977). The dynamics of a blast wave inside a preexisting wind-driven cavity was studied numerically by Ciotti and D'Ercole (1989) and Chevalier and Liang (1989).

(2) The earlier stage of supernova explosion is free expansion of matter ejected from the interior of exploded star into a circumstellar and then interstellar space. Gandel'man and Frank-Kamenetskii (1956) and Sakurai (1960) showed that after the blowout of the power-law stellar atmosphere by the blast wave the postshock flow approaches a free homologous expansion $v = r/t$ with the power-law distribution of density which in a spherical case takes the form $\rho \sim (t/r)^n \cdot t^{-3}$ (Chevalier & Soker 1989). For the Type I supernovae it is adopted $n = 5.4 \div 8$ (Colgate & McKee 1969), for the Type II supernovae $n = 9 \div 13$ (Jones et al. 1981). According to the numerical simulations for SN 1987A $n \simeq 9.6$ (Arnett 1989). Free expansion continues until the mass of swept-up ambient matter becomes of the order of the mass of ejecta.

(3) Free expansion of matter with a power-law density distribution behind the blast wave occurs not only after the blowout of a power-law atmosphere but also that of an exponential one (Raizer 1964; Grover & Hardy 1966). In astrophysical environment such a possibility is realized in case of sequential supernova explosions concentrated in OB associations (Bruhweiler et al. 1980). A large-scale cavity with a size comparable with the thickness of the galactic disk is then formed. The cavity expands primarily perpendicular to the disk plane in the direction of the density antigradient. Since the density drops with the distance over the plane of symmetry rapidly, approximately as an exponential or Gaussian law (the best fit is a composite law), the cavity stretches with acceleration and blows out the exponential (Gaussian) atmosphere in finite time throwing out matter to large heights into the gaseous galactic halo (Tomisaka & Ikeuchi 1986; MacLow & McCray 1988; Igumentshchev et al. 1990; Tenorio-Tagle et al. 1990). Such a phenomenon, termed the "galactic fountain", is thought to be one of the basic mechanisms providing matter and heat transport into the halo (Shapiro & Field 1976; Norman & Ikeuchi 1989).

(4) According to the explosive theory of galaxy formation (Ostriker & Cowie 1981; Ikeuchi 1981) the present structure of the universe is the result of sequence of large-scale explosions in the early epoch. Cosmological blast waves are driven into an unperturbed background which expands as $v = 2r/3t$ in case of the flat universe, for which the gravitational energy exactly compensates the kinetic energy, or $v \simeq r/t$ in case of the well-open universe with a negligible gravity. Although the observations from *COBE* do not count in favour of the explosive scenario (Levin et al. 1992) the developed theory (Ikeuchi et al. 1983; Bertschinger 1983) may be relevant to the description of individual explosive phenomena in the intergalactic medium.

Of the contexts listed above the cases of a freely expanding ambient medium into which astrophysical blast waves may propagate constitute a major portion. These are the cases of the post-blowout expansion and the case of a low-density universe with a negligible gravity. The blast wave dynamics in a freely expanding medium has been the subject of numerical (Hausman et al. 1983; Hoffman et al. 1983) and analytical self-similar (Ikeuchi et al. 1983; Bertschinger 1983) and approximate investigations (Ostriker & McKee 1988), mainly in the cosmological context. The distinctive property of the dynamics is that the blast wave gradually decelerates and asymptotically comoves with the expanding background.

In this paper we present an exact analytical solution for a blast wave dynamics in a freely expanding medium which is true in all, initial, intermediate and final state asymptotics. Sect. 3 presents the statement of the problem, the solution is developed in Sect. 4 and its astrophysical applications to interstellar matter physics and cosmology are discussed in Sect. 5.

3. Model

We base our consideration on the model of ideal inviscid fluid with a constant specific heats ratio γ which is described by the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \, , \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla p \, , \tag{2}$$

$$\frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p + \gamma p \nabla \cdot \mathbf{v} = 0 \, , \tag{3}$$

where $\rho, \mathbf{v}, p$ are the density, the velocity and the pressure of the fluid respectively. The unperturbed medium is assumed to be freely expanding according to the law

$$\mathbf{v}_0 = \frac{\mathbf{r}}{t + t_0} \, . \tag{4}$$

The moment $t = -t_0$ corresponds to the beginning of expansion. Suppose at the moment $t = 0$ a point explosion occurs in the origin of coordinates with an instantaneous release of energy E_0. This results in formation of the spherical expanding blast wave. The goal of our paper is to find the dynamics and structure of the resulting flow.

Our consideration focuses on the case of the power-law medium. We also suggest that the mass ejected by a source of explosion is neglible compared to the swept-up mass.

In the motionless cold unperturbed medium with the power-law radial density distribution

$$\rho_0 = \rho_0(r) \propto r^{-\omega}$$

and negligible pressure

$$p_0 = 0 \, ,$$

the strong blast wave dynamics is described by the self-similar Sedov solution (Sedov 1959). The similarity arises naturally because the dynamics in this case is unambigiously determined by four dimensional parameters: t, r, ρ_0 and E_0. Particularly, the shock front position $R_{\rm s}$ is related with the other three variables through the power-law combination:

$$R_{\rm s} = {\rm const}(E_0/\rho_0)^{1/5} t^{2/5} \propto t^{2/(5-\omega)} \, . \tag{5}$$

In case of the freely expanding medium the power-law density distribution evolves in time as

$$\rho_0 = \rho_0(r,t) = \left(\frac{t + t_0}{r}\right)^{\omega} \frac{A}{(t + t_0)^3} \, , \tag{6}$$

where A is a constant, and similarity dies away because an additional essential dimensional parameter t_0 appears. Now $R_{\rm s}$ and t cannot be unambigiously related from the dimensional arguments. Nevertheless, an exact solution can be obtained by using the symmetry analysis.

4. Solution

To develop the solution we have to make a short excursus into the Lie group analysis of differential equations. A reader not familiar with this theory is referred to the textbooks (Ovsiannikov 1982; Ibragimov 1983; Olver 1986).

It is known that the system of n-dimensional hydrodynamic equations for a perfect fluid (1)-(3) with the arbitrary adiabatic index γ admits the $4 + n(n+3)/2$-parametric group of the point symmetry and can be represented by the following set of the infinitesimal operators of symmetry (Ovsiannikov 1982)

$$X_0 = \frac{\partial}{\partial t} \, , \tag{7}$$

$$X_i = \frac{\partial}{\partial x^i} \, , \qquad i = 1, ..., n \, , \tag{8}$$

$$X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j} , \; i,j = 1, ..., n, \tag{9}$$

$$Y_i = t \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i} \, , \qquad i = 1, ..., n \, , \tag{10}$$

$$Z_1 = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} \, , \tag{11}$$

$$Z_2 = t \frac{\partial}{\partial t} - v^i \frac{\partial}{\partial v^i} + 2\rho \frac{\partial}{\partial \rho} \, , \tag{12}$$

$$Z_3 = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p} \, . \tag{13}$$

These operators constitute the linear space with the definite properties (commutator of any pair of operators can be expressed as a linear combination of the entire set) which is called the Lie algebra.

The existence of a group of symmetry of the system (1)-(3) in other words means that the system is invariant under the action of infinitesimal transformations given by (7)-(13). It is worthwile to note that a group generally defines the global transformations given by finite algebraic equations. In case of the finite transformations, finding a group of symmetry is not an easy problem since this procedure involves solving the nonlinear equations. It happened to be more convenient to deal with the infinitesimal transformations because they lead to linear determining equations which are always solvable. In this stage the appearance of algebra which determines the structure of a group is natural. A Lie group and its associated Lie algebra are intimately related so that any one of these allows to recover the other one and vice versa. In what follows we make no difference between them unless it is specially emphasized.

Each transformation we are dealing here with can be assigned a conventional physical meaning. In particular, the operators (7)-(8) describe the invariance of Eqs. (1)-(3) under translations in time and in space, the operators (9) describe invariance under rotations, the operators (10) present the Galilean transformations. The transformations allow to breed solutions once a particular one is known and, what is more important, to construct new classes of particular, so-called invariant, solutions reflecting the corresponding particular symmetrical properties of equations. The invariant solution meeting to X_0 is, say, just a steady state solution, the solution invariant under rotations is a spherically symmetric one and so on. *The self-similar solutions always arise as invariant solutions corresponding to the operators of stretching* (Ovsiannikov 1982). In our case there are three operators of stretching, Z_1, Z_2 and Z_3, which reflect the possibility of scaling the physical independent and dependent variables. It is remarkable that due to the linear nature of algebra any particular linear combination of operators (7)-(13) belongs to this algebra too. This means that in the most general case one can construct the superposition of all $4+n(n+3)/2$ operators with $4+n(n+3)/2-1$ arbitrary constants and thus to obtain the $4+n(n+3)/2-1$-parametric class of particular invariant solutions.

If $\gamma=(n+2)/n$, the group is extended by an additional projective symmetry first found by Ovsiannikov (1958)

$$X_+ \;=\; t^2\frac{\partial}{\partial t}+tx^i\frac{\partial}{\partial x^i}+(x^i-tv^i)\frac{\partial}{\partial v^i}-nt\rho\frac{\partial}{\partial\rho} - \;(n+2)tp\frac{\partial}{\partial p}\;. \quad (14)$$

The invariant solution corresponding to X_+ describes the asymptotic free expansion of fluid into vacuum (Ibragimov 1983).

Obviously, the solution we are interested in must coincide with the standard Sedov solution at early stages ($t\ll t_0$) when the overall expansion of background is not yet pronounced. The Sedov solution can be constructed as invariant solution meeting a certain superposition of Z_1, Z_2 and Z_3. On the other hand, at late times ($t\gg t_0$) the background density ρ_0 drops rapidly with time so that the blast wave dynamics asymptotically resembles expansion into a vacuum and thus should transit to the invariant solution generated by X_+. Both asymptotics must be continuously joined at intermediate times. It would then appear natural to search for a solution satisfying the described condition in the form

$$Z=X_++aZ_1+bZ_2+cZ_3 \quad (15)$$

with some constants a, b and c to be determined later. Close inspection of the structure of Z shows that our suggestion may really justify itself because the coefficients of X_+ are proportional to t or x^i and at $t\approx 0$ and $x^i\approx 0$ the contribution of X_+ to Z is negligible while at large times, conversely, X_+ dominates. Let us recall that the operator X_+ exists only for $\gamma=(n+2)/n$. Henceforward we assume $n=3$, $\gamma=5/3$ which corresponds to the monatomic ideal gas.

We are looking for the spherically symmetric solution. In spherical coordinates Z takes the form

$$Z \;=\; \left(t^2+(a+b)t\right)\frac{\partial}{\partial t}+r(a+t)\frac{\partial}{\partial r} + \;(r-tv-bv)\frac{\partial}{\partial v}+\rho(2b+c-3t)\frac{\partial}{\partial\rho} + \;p(c-5t)\frac{\partial}{\partial p}\;. \quad (16)$$

The invariant solution is derived as follows. First we find the invariants J of the group solving the equation

$$ZJ=0\,. \quad (17)$$

Eq. (17) is a linear first order partial differential equation that can be solved with the help of the method of characteristics. It has four functionally independent solutions

$$J_1=rt^{-a/(a+b)}(t+a+b)^{-b/(a+b)}\;, \quad (18)$$

$$J_2=(\frac{vt}{r}-1)(t+a+b)\;, \quad (19)$$

$$J_3=\rho r^{-(2b+c)/a}(t+a+b)^{(3a+2b+c)/a}\;, \quad (20)$$

$$J_4=p\rho^{-1}r^{-2}t^2(t+a+b)^2\;. \quad (21)$$

To draw a parallel between the invariant and self-similar solutions it is worthwile to turn from the invariants J_i to the more habitual non-dimensional invariant variables λ, R, V and Z in the following way:

$$J_1 \;=\; \frac{\alpha\lambda}{a+b}\;, \qquad J_2=(a+b)(V-1)\;,$$
$$J_3 \;=\; AR\;, \qquad J_4=\frac{3}{5}(a+b)^2Z\;. \quad (22)$$

Here α is the dimensional constant to be determined later.

Any function of invariant is also invariant, so we can construct invariant solution suggesting that R, V and Z are functions of λ. Expressing the physical variables in an explicit form we get

$$r=\frac{\alpha\lambda}{a+b}t^{a/(a+b)}(t+a+b)^{b/(a+b)}\;, \quad (23)$$

$$\rho=(t+a+b)^{-(3a+2b+c)/a}r^{(2b+c)/a}AR(\lambda)\;, \quad (24)$$

$$v=\frac{r}{t}\left[1+\frac{a+b}{t+a+b}(V(\lambda)-1)\right]\;, \quad (25)$$

$$c_{\rm s}^2=\frac{5}{3}\frac{p}{\rho}=r^2t^{-2}(t+a+b)^{-2}(a+b)^2Z(\lambda)\;, \quad (26)$$

where we have introduced the sound velocity $c_{\rm s}$. In order that invariant solution (23)-(26) coincide with the self-similar Sedov solution at early times $0<t\ll t_0$, we have to set

$$a=\frac{2}{5-\omega}t_0\;, \quad (27)$$

$$b=\frac{3-\omega}{5-\omega}t_0\;, \quad (28)$$

$$c=-\frac{6}{5-\omega}t_0\;, \quad (29)$$

after which Eqs. (23)-(26) finally transform to

$$r=\alpha\frac{\lambda}{t_0}t^{2/(5-\omega)}(t+t_0)^{(3-\omega)/(5-\omega)}\;, \quad (30)$$

$$\rho = \left(\frac{t+t_0}{r}\right)^{\omega} \frac{A}{(t+t_0)^3} R(\lambda) \ . \tag{31}$$

$$v = \frac{r}{t}\left[1 + \frac{t_0}{t+t_0}(V(\lambda)-1)\right] \ . \tag{32}$$

$$c_{\rm s}^2 = \frac{r^2}{t^2}\left(\frac{t_0}{t+t_0}\right)^2 Z(\lambda) \ . \tag{33}$$

The variable λ in (30)-(33) plays the role of the independent invariant variable. The constant α is traditionally defined in such a way that $\lambda = 1$ at the shock front; its rigorous definition is deferred until Eq. (60). Hence the range of the invariant variable is $\lambda_0 \leq \lambda \leq 1$, where λ_0 is either a positive constant in case of a shell-like solution or zero if a solution extends up to the center of symmetry.

After the substitution of Eqs. (30)-(33) into the system (1)-(3) the latter one is reduced to the system of ordinary differential equations

$$\lambda\left[-V' + \left(\frac{2}{5-\omega} - V\right)\frac{R'}{R}\right] = (3-\omega)V \ , \tag{34}$$

$$\lambda\left[\left(\tfrac{2}{5-\omega} - V\right)V' - \tfrac{3Z}{5}\left(\tfrac{R'}{R} + \tfrac{Z'}{Z}\right)\right] = V^2 - V + (2-\omega)\tfrac{3Z}{5} \ , \tag{35}$$

$$\lambda\left(\frac{2}{5-\omega} - V\right)\left[\frac{Z'}{Z} - \frac{2R'}{3R}\right] = \left(2 + \frac{2}{3}\omega\right)V - 2 \ , \tag{36}$$

which completely coincide with the system derived by Sedov for the self-similar solution (1959). This is not surprising because at small times $t \approx 0$ expressions (30)-(33) are consistent with the self-similar solution, therefore the reduced systems must coincide as well. But since the reduced system does not depend on time, it preserves its form for all moments $t > 0$.

The boundary conditions for the problem are the standard conservation laws at the strong adiabatic shock jump

$$\rho_0(v_0 - \dot{R}_{\rm s}) = \rho_1(v_1 - \dot{R}_{\rm s}) \ , \tag{37}$$

$$\rho_0(v_0 - \dot{R}_{\rm s})^2 = p_1 + \rho_1(v_1 - \dot{R}_{\rm s})^2 \ , \tag{38}$$

$$\frac{1}{2}(v_0 - \dot{R}_{\rm s})^2 = \frac{5}{2}\frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - \dot{R}_{\rm s})^2 \ , \tag{39}$$

where the subscript 1 denotes the values taken immediately behind the shock front. The shock front velocity $\dot{R}_{\rm s}(t)$ can be found by setting $\lambda = 1$ in Eq. (30) and differentiating this with respect to time:

$$\dot{R}_{\rm s} = \left(\frac{2}{5-\omega}\frac{1}{t} + \frac{3-\omega}{5-\omega}\frac{1}{t+t_0}\right) R_{\rm s}.$$

Substituting $\dot{R}_{\rm s}$, the preshock and postshock values taken from Eqs. (4), (6) and (31)-(33) into (37)-(39) we find that the boundary conditions are compatible with the invariant form and can be eventually written as

$$R(1) = 4 \ , \tag{40}$$

$$V(1) = \frac{3}{2(5-\omega)} \ , \tag{41}$$

$$Z(1) = \frac{5}{4(5-\omega)^2} \ . \tag{42}$$

We again find that the boundary conditions coincide with those of Sedov for $\gamma = 5/3$ (Sedov 1959). Since both the dynamic equations and the boundary conditions are the same as in the Sedov case we conclude that the solution of the problem (34)-(36), (40)-(42) is exactly the Sedov one and hence we may draw on the known solution from (Sedov 1959):

$$\lambda(V) = \left[\frac{2}{3}(5-\omega)V\right]^{\alpha_1}\left[\frac{10(5-\omega)}{3}V - 4\right]^{\alpha_2} \times \left[\frac{\omega-5}{\omega-2}(1-2V)\right]^{\alpha_3} \ , \tag{43}$$

$$R(V) = 4\left[\frac{\omega-5}{\omega-2}(1-2V)\right]^{\beta_1}[4 - 2(5-\omega)V]^{\beta_2} \times \left[\frac{10(5-\omega)}{3}V - 4\right]^{\beta_3} \ , \tag{44}$$

$$Z(V) = -\frac{1}{3}V^2\left[V - \frac{2}{5-\omega}\right]\left[V - \frac{6}{5(5-\omega)}\right]^{-1} \ , \tag{45}$$

$$\alpha_1 = \frac{2}{\omega-5} \ , \tag{46}$$

$$\alpha_2 = \frac{2}{13-5\omega} \ , \tag{47}$$

$$\alpha_3 = -\frac{2(5\omega^2 - 26\omega + 41)}{3(5-\omega)(13-5\omega)} \ , \tag{48}$$

$$\beta_1 = \frac{2(\omega-3)(5\omega^2 - 26\omega + 41)}{3(\omega-1)(13-5\omega)} \ , \tag{49}$$

$$\beta_2 = \frac{2(9-4\omega)}{3(\omega-1)} \ , \tag{50}$$

$$\beta_3 = \frac{3(3-\omega)}{13-5\omega} \ . \tag{51}$$

Here the variable V for $\omega \leq 2$ varies in the range

$$\frac{6}{5(5-\omega)} \leq V \leq \frac{3}{2(5-\omega)} \ . \tag{52}$$

In case $2 < \omega < 3$ the solution bifurcates to a hollow density distribution and then V varies in the range

$$\frac{3}{2(5-\omega)} \leq V \leq \frac{2}{5-\omega} \ . \tag{53}$$

Altogether, the invariant solution is given by the formulas (30)-(33), (43)-(53).

An inspection of expressions (31), (33) shows that the profiles of density and pressure remain similar throughout the whole lifetime of the blast wave, their form coincides with the profiles of the Sedov self-similar solution. At the same time according to Eq. (32) the profile of velocity changes with time (Figs. 1 and 2) and tends to the homologous one, $v \approx r/t$, at $t \to \infty$.

It is important that in the matter reference frame the blast wave decelerates and eventually comes to rest: the relative jump of velocity

$$\frac{v_1 - v_0}{v_0} \sim t^{-1}$$

and the mass flux through the discontinuity

$$j = \rho_0(v_0 - \dot{R}_\mathrm{s}) \sim t^{-4}$$

asymptotically vanish. This means that the blast wave asymptotically encloses the finite mass

$$M = \lim_{t\to\infty} \frac{4\pi}{3-\omega}\rho_0(R_\mathrm{s})R_\mathrm{s}^3 = \frac{4\pi A}{3-\omega}\left(\frac{\alpha}{t_0}\right)^{3-\omega} .$$

However, at any finite moment t the mass flux is non-zero, and since the sound speed of preshock gas is identically zero by suggestion, the discontinuity remains a strong shock wave the whole time of expansion.

Let us consider how the energy of explosion E_0 is redistributed in the flow with time. The total energy of the flow E_tot occupied by a blast wave of radius R_s at any moment t is equal to a sum of the injected energy E_0 and the kinetic energy $E_\mathrm{tot,0}$ of unperturbed flow:

$$E_\mathrm{tot}(t) = E_\mathrm{tot,0}(t) + E_0 , \tag{54}$$

$$\begin{aligned} E_\mathrm{tot,0}(t) &= \int_0^{R_\mathrm{s}} \rho_0 \frac{v_0^2}{2} 4\pi r^2 \,\mathrm{d}r \\ &= \frac{2\pi A}{5-\omega}\left(\frac{\alpha}{t_0}\right)^{5-\omega}\left(\frac{t}{t+t_0}\right)^2 . \end{aligned} \tag{55}$$

At the same time the total energy is combined as the kinetic, E_kin, plus thermal, E_th, energy:

$$E_\mathrm{tot}(t) = E_\mathrm{kin}(t) + E_\mathrm{th}(t) , \tag{56}$$

$$\begin{aligned} E_\mathrm{kin}(t) &= \int_0^{R_\mathrm{s}} \rho \frac{v^2}{2} 4\pi r^2 \,\mathrm{d}r = 2\pi A \left(\frac{\alpha}{t_0}\right)^{5-\omega}\left(\frac{t_0}{t+t_0}\right)^2 \\ &\times \int_0^1 R(\lambda)\left[V(\lambda) + \frac{t}{t_0}\right]^2 \lambda^{4-\omega}\,\mathrm{d}\lambda , \end{aligned} \tag{57}$$

$$\begin{aligned} E_\mathrm{th}(t) &= \int_0^{R_\mathrm{s}} \frac{3}{2} p \, 4\pi r^2 \,\mathrm{d}r = \frac{18\pi A}{5}\left(\frac{\alpha}{t_0}\right)^{5-\omega}\left(\frac{t_0}{t+t_0}\right)^2 \\ &\times \int_0^1 R(\lambda) Z(\lambda) \lambda^{4-\omega}\,\mathrm{d}\lambda . \end{aligned} \tag{58}$$

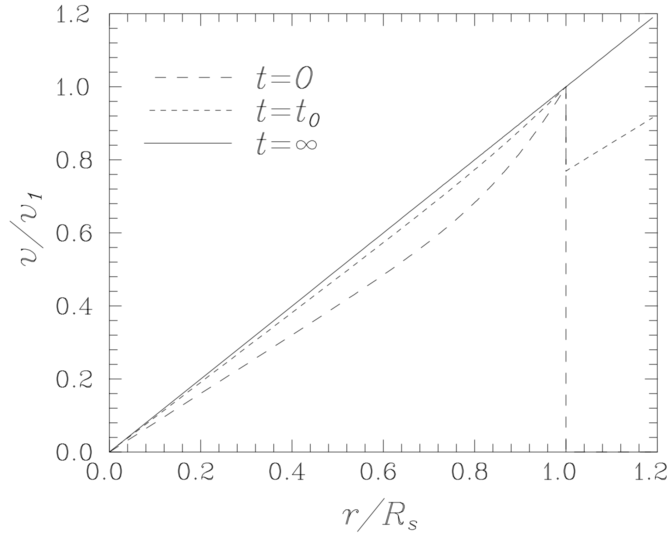

Fig. 1. The normalized velocity profiles at different stages for a filled blast wave in a uniform ambient density

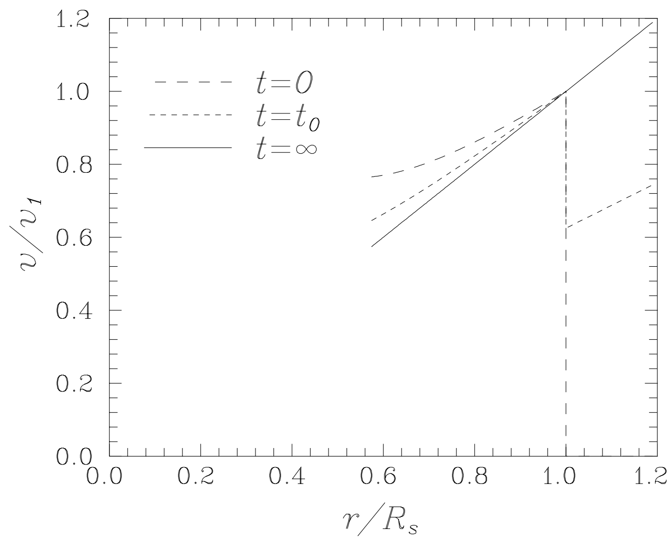

Fig. 2. The same as in Fig. 1 but for a hollow blast wave in a nonuniform ambient density with $\omega = 2.5$. Interior to $r/R_\mathrm{s} = 0.57$ is void

At the initial moment $t = 0$ the injected energy is allocated between the kinetic and thermal energies (in a uniform medium their initial fractions are $E_\mathrm{kin} = 0.28E_0$, $E_\mathrm{th} = 0.72E_0$) while $E_\mathrm{tot,0} = 0$. Thereafter the fraction of the thermal energy monotonically decreases with time down to zero; the energy of explosion E_0 is thus expended for work done on the fluid transfer from the central area outwards:

$$\begin{aligned} E_0 &= \lim_{t\to\infty}\{E_\mathrm{kin}(t) - E_\mathrm{tot,0}(t)\} \\ &= \lim_{t\to\infty}\int_0^{R_\mathrm{s}} (\rho - \rho_0)\frac{v_0^2}{2} 4\pi r^2 \,\mathrm{d}r . \end{aligned} \tag{59}$$

One may wonder how Eqs. (54) and (56) containing the terms with different time dependencies can hold. It would not be difficult to show that this is the case. Substituting expressions (55),

(57) and (58) into Eqs. (54) and (56), equating (54) to (56) and multiplying this by $(t+t_0)^2$ gives us an equation quadratic in time. This equation must be an identity in t, so we obtain three different equations equating terms of equal order in t. The first equation for constant terms determines α

$$\alpha = \left(\frac{E_0}{A\xi}\right)^{1/(5-\omega)} t_0 \ , \tag{60}$$

where

$$\xi = 2\pi \int_0^1 R\left(V^2 + \frac{9}{5}Z\right)\lambda^{4-\omega}\,\mathrm{d}\lambda \ . \tag{61}$$

In case of the uniform medium, $\omega = 0$, Eq. (60) is reduced to the well-known (Sedov 1959)

$$\alpha = 1.15...\left(\frac{E_0}{\rho_\mathrm{i}}\right)^{1/5} t_0^{2/5} \ ;$$

the subscript i hereafter denotes parameters of the ambient fluid taken at the moment of explosion.

The remaining two equations for t and t^2, which with the help of the first one can be reduced to

$$\int_0^1 R\left(V^2 - V + \frac{9}{5}Z\right)\lambda^{4-\omega}\,\mathrm{d}\lambda = 0 \ , \tag{62}$$

$$\int_0^1 R\,(1-V)\,\lambda^{4-\omega}\,\mathrm{d}\lambda = \frac{1}{5-\omega} \ , \tag{63}$$

hold automatically.

5. Discussion

It is convenient to interpret the solution derived above in terms of an effective power-law index

$$\delta = \frac{\mathrm{d}\log R_\mathrm{s}}{\mathrm{d}\log t} \ . \tag{64}$$

From Eq. (30) we see that δ grows monotonically from $2/(5-\omega)$ at the initial moment and asymptotically tends to 1 at late times. Thus the invariant solution presents a continuous joining of two self-similar asymptotics – a really rare event in mathematical physics (Barenblatt 1979). We recall that the solution is true only for $\gamma = 5/3$ but this case is the most interesting one in astrophysical applications. Another merit of the solution is its possible reversibility in time. If we replace $t + t_0$ by $t_0 - t$ and v_0 by $-v_0$ everywhere, the formulas will describe explosion and the subsequent blast wave evolution on a contracting background. Obviously, the moment t_0 will then correspond to the instant of collapse.

The solution found is fundamentally based on two assumptions of adiabaticity and infinite intensity of the blast wave. Let us consider the conditions at which our assumptions become invalid.

First discuss the range of validity of a strong blast approximation. It is clear that as the area occupied by a blast wave expands the strength of the blast should fall off so that our initial assumption may fail with time. The blast wave is considered as strong when the ratio of the postshock p_1 and preshock p_0 pressures is large, that is,

$$k \equiv \frac{p_1}{p_0} \gg 1 \ . \tag{65}$$

The ambient pressure drops adiabatically. In case of the uniform medium it has the form

$$p_0 = \frac{3}{5}\rho_\mathrm{i} c_\mathrm{si}^2 \left(\frac{\rho_0}{\rho_\mathrm{i}}\right)^{5/3} , \tag{66}$$

Then the intensity of the blast wave according to Eqs. (30), (31), (33), (40), (42), (60), (66) is determined by the formula

$$\frac{p_1}{p_0} = \frac{1}{5}\left[\frac{E_0}{\rho_\mathrm{i} t_0^3}\right]^{2/5} \left[\frac{1.15}{c_\mathrm{si}}\right]^2 \left[1 + \frac{t_0}{t}\right]^{6/5} \ . \tag{67}$$

This ratio is a monotonically decreasing function of time asymptotically converging to a non-zero constant at infinity. If

$$E_0 > (5k)^{5/2}\left[\frac{c_\mathrm{si}}{1.15}\right]^5 \rho_\mathrm{i} t_0^3 \tag{68}$$

the blast wave will remain strong the whole time of expansion and our neglect of the ambient pressure will be consistent. Let us make estimates for a supernova exploded inside a galactic fountain. The typical magnitudes of density and sound speed in the rarefied hot gas of the fountain close to the plane of the galactic disk are $\rho \sim 10^{-25} \div 10^{-26}$ g·cm^{-3}, $c_s \sim 10^2$ km·s. The fountain begins to form when several supernovae burst within a short period of time a short distance from each other. For the frequency of supernova explosions it is usually adopted $\nu \sim 5 \cdot 10^{-6}$ yr^{-1} (MacLow & McCray 1988; Tenorio-Tagle et al. 1990) therefore for the well-formed fountain we have $t_0 \gtrsim 10^6$ yr. For the sake of clarity let us assume $k = 5$ which corresponds to the Mach number $Ma \approx 2$. Then from inequality (68) we find $E_0 \gtrsim 4.2 \cdot (10^{52} \div 10^{53})$ erg. Thus, a shock produced by a single Type II supernova with the mean energy release $E_\mathrm{II} = 10^{51}$ erg strongly decays before it reaches the edge of the cavity. This estimate is, however, too uncertain because of the sensitivity of inequality (68) to c_si variation.

Another constraint imposed on our solution arises from the radiative energy losses. The influence of radiative cooling can be crudely estimated depending on whether the cooling time t_cool exceeds the characteristic time of expansion t_0 or not. If the cooling time is much shorter than t_0, the blast wave enters the cooling phase well before the overall expansion of a background will be dynamically important. In this case our adiabatic solution extended for large times is no longer valid. We do not cite here the estimation of the cooling time in a static power-law medium referring an interested reader to the recent comprehensive discussion of this question in Franco et al. (1994). If t_cool exceeds t_0, the radiative cooling ceases to be a significant factor for dynamics. Indeed, at the moment t_0, the thermal energy is lowered by a factor 4 (see Eq. (58)). Its fraction in the total energy budget decreases much more since the kinetic energy increases $\sim (1 + 2(5 - \omega)/3)^2/4$ times (Eq. (57)). So by the time t_0 the blast wave, being initially for the most part pressure-driven wave, becomes a momentum-driven one. Taking away some of the thermal energy by emerging radiation under these circumstances cannot anywhere affect motion of the blast wave.

Cosmology is a more promising field as far as application of the found solution goes. Cosmological blast waves from a point explosion were studied in connection with the model of the explosive galaxy formation (Ostriker & Cowie 1981; Ikeuchi 1981). Ikeuchi et al. (1983) and Bertschinger (1983) developed a self-similar adiabatic ($\delta = 4/5$) solution (hereafter ITO-B) for a blast wave in an unperturbed flat universe in Friedmann cosmology for which the density parameter $\Omega = \rho/\rho_{\rm crit} = 1$. The most distinctive feature of this solution is that the postshock material is totally confined to a very thin dense shell with a thickness $\sim 0.03 R_{\rm s}$ for $\gamma = 5/3$. Kovalenko and Sokolov (1993) showed that this solution describes in fact a final stage of any spherically symmetric positive energy perturbation in the flat universe. In the flat universe the influence of gravity is never small; the background therefore expands decelerating and our present invariant solution cannot be applied to this case. However it can be applied to a model of an open universe with $\Omega < 1$ for which expansion becomes inertial at late times when $\Omega \to 0$. Ikeuchi et al. (1983) and Ostriker and McKee (1988) considered the final stage of a blast wave against a freely expanding Friedmann background and properly predicted its asymptotical self-similar law $\delta = 1$ and asymptotical comoving with the ambient fluid. At the same time they were mistaken inferring that the postshock material eventually concentrates to even denser shell, in the limit, an infinitely thin shell just behind a shock front. This prediction contradicts the numerical calculations (Hausman et al. 1983; Hoffman et al. 1983). Bertschinger (1983) was more accurate stating that the postshock density distribution may be arbitrarily dependent on the history of a blast wave evolution at the intermediate non-self-similar stages. Based themselves upon the fact that $(\dot{R}_{\rm s} - v_0) \to 0$ all these authors fell victims to a common mistake coming to the conclusion that the shock discontinuity asymptotically degenerates into a contact discontinuity. The true solution, as we can see, shows that in none of the moments discontinuity ceases to be a shock and thus the postshock gas has no tendency to pile up into an infinitely thin shell behind a shock front.

All troubles with the asymptotic self-similar solution arose from the stringent assumption of the exact self-similarity with $\delta = 1$ which eventually led to the condition $\dot{R}_{\rm s} - v_0 \equiv 0$. To correctly resolve the internal structure of the blast wave one has to allow for a small deviation from the exact self-similar law. Luckily, with our new method we are now in a position to follow the asymptotical approach to the final self-similar state and moreover we are able to trace the blast wave evolution from start to finish at least in the limit of negligible gravity. To gain a better understanding of how a blast evolves from the moment of explosion in the universe let us briefly review the general case.

In the $\Omega = 1$ universe the blast wave evolution goes on in the following sequence. Just after an explosion the Sedov profile is established. After several Hubble times the matter gradually escapes the central parts and accumulates to a thin shell behind a shock front; the postshock density distribution eventually acquires a well-pronounced shell-like profile prescribed by a self-similar adiabatic ITO-B cosmological blast wave solution.

In case of the $\Omega < 1$ universe the blast wave behaves similarly but stops the postshock density enhancement earlier either delaying the shell formation or not achieving the final ITO-B state with a thin shell at all. One can discern two limiting cases dependent on the relation between the moment of explosion t_0 measured from the big bang and the Hubble time (we now return to the ususal time scale where the beginning of the background expansion, the big bang, is associated with the moment $t = 0$). Let us define the Hubble time $t_{\rm H}$ as the time at which Ω becomes noticeably small, say $\Omega(t_{\rm H}) = 0.1$. In case of explosion in an early epoch, $t_0 \ll t_{\rm H}$, the blast wave in an open universe experiences three consecutive self-similar episodes of its evolution:

> Sedov stage ($\Delta t = t - t_0 \ll t_0$, $\delta = 2/5$) $\to$ quasi-ITO-B cosmological blast wave stage ($\Delta t \lesssim t_{\rm H}$, $\delta \approx 4/5$) $\to$ "freezing-out" ($\Delta t \gg t_{\rm H}$, $\delta = 1$).

The profile of the blast wave gradually rearranges itself according to the above mentioned scheme.

If explosion occurs at late times $t_0 \gtrsim t_{\rm H}$ when the gravity forces have already fallen off, an intermediate ITO-B stage is lacking. In this case the shell does not form and the blast wave evolution can be described with great accuracy by the present invariant solution.

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