

Modeling multi-stream flow in collisionless matter: approximations for large-scale structure beyond shell-crossing

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Abstract. The generally held view that a model of large-scale structure, formed by collisionless matter in the Universe, can be based on the matter model “dust” fails in the presence of multi-stream flow, i.e., velocity dispersion. We argue that models for large-scale structure should rather be constructed for a flow which describes the average motion of a multi-stream system. We present a clearcut reasoning how to approach the problem and derive an evolution equation for the mean peculiar-velocity relative to background solutions of Friedmann-Lemaître type. We consider restrictions of the nonlinear problem and show that the effect of velocity dispersion gives rise to an effective viscosity of non-dissipative gravitational origin. We discuss subcases which arise naturally from this approach: the “sticky particle model” and the “adhesion approximation”. We also construct a novel approximation that features adhesive action in the multi-stream regime while conserving momentum, which was considered a drawback of the standard approximation based on Burgers’ equation. We finally argue that the assumptions made to obtain these models should be relaxed and we discuss how this can be achieved.

Key words: gravitation – hydrodynamics – instabilities – methods: analytical – cosmology: theory – large-scale structure of Universe

1. Why model a multi-stream flow?

The problem of how to treat a multi-stream flow and the question of how to follow approximations beyond shell-crossing time are themes that are often repeated in the history of constructing models for large-scale structure. The application of approximation schemes such as, e.g., Lagrangian perturbation solutions (which, to first order, contain the celebrated “Zel’dovich approximation” (Zel’dovich 1970, 1973) as a special case, see Buchert 1989, 1992) breaks down at the epoch of formation of caustics in the density field. Fluid elements are treated as to propagate freely through each other; multi-stream flow arises beyond the caustic, i.e., the velocity field is no longer single-valued. Self-gravitation of the multi-stream system is the source

for holding structures together: a considerable fraction of the particles is trapped within three-stream systems (called ‘pancakes’ in the cosmological context) which, in the course of time, develop a hierarchy of nested structures due to the formation of N-stream systems with increasing N. Doroshkevich et al. (1980) have demonstrated this in a two-dimensional numerical simulation; the number of streams as a function of time was calculated by Kofman et al. (1994). This hierarchy of structures is also predicted by higher-order Lagrangian perturbation schemes (Buchert et al. 1997); compare also the works by Shukurov (1981) and Fillmore & Goldreich (1984a,b). The notion of ‘non-dissipative gravitational turbulence’ has been advanced for this phenomenon by Gurevich & Zybin (1995 and ref. therein). For further discussions in the framework of Lagrangian perturbation theory on details of multiple shell-crossings related to bifurcations of the Lagrangian solutions and the mathematical problems involved see, e.g., Buchert & Ehlers (1993), Buchert (1994, 1996), Ehlers & Buchert (1997); in the relativistic context: Clarke & O’Donnell (1992).

In order to repair this shortcoming, the “adhesion approximation” (Gurbatov et al. 1989) has been proposed. This model proved to be successful, provided the structure formation process is not followed too deeply into the nonlinear regime (Kofman et al. 1990, Weinberg & Gunn 1990a,b; Sahni & Coles 1995, Shandarin 1997). At late nonlinear stages the model predicts a distribution of large-scale structures which is different from that predicted by N-body simulations, especially for much small-scale power. One reason for this was referred to the model itself, i.e. that the gravitational potential may acquire an additional part via nonlinear evolution on smaller scales (Kofman et al. 1992), it may also be traced back to the model’s drawback of not conserving the linear momentum (Kofman & Shandarin 1988). We advocate the explanation that it may also be a consequence of treating the “viscosity coefficient” as spatially constant, as will become clear later. A recent work by Shandarin & Sathyaprakash (1996) proposes a semi-numerical model which is built in the spirit of ‘adhesion’ of particles, but which has the advantage of conserving momentum by construction. In this paper we will construct models that allow following the structure formation process beyond shell-crossing time analytically having both the ‘adhesive’ property of ‘sticking’ fluid elements and the conservation of momentum.

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We shall emphasize that the notion of a ‘self-gravitating gas’ (which we are going to specify) is more adequate when we speak about the regime of multi-stream flow in collisionless systems than the generally adopted simplification to “dust matter”: in the presence of multi-stream flow, one should focus on the average bulk motion rather than on individual particles. We shall argue that the action of the multi-stream system on the average motion can be modeled by means of pressure-like forces. Taking this point of view we will be able to propose an approximation scheme that is valid in the regime of multi-stream flow. The assumption of isotropic velocity dispersion singles out a model which determines a transport coefficient - not unlike the “viscosity” in the standard “adhesion approximation” - which, however, is of *gravitational origin*. The advantage of this point of view is two-fold: firstly, we do not have to invent a “viscosity term” to mimic gravitational adhesion of fluid elements after shell-crossing, but we can *derive* it from the gravitational action of the multi-stream system itself and, secondly, we can learn about the restrictions we have to impose on the general problem in order to obtain models of this type. The latter will strongly support our opinion that these restrictions should be relaxed. We wish to note that, starting from another reasoning, a similar interpretation of the “viscosity” in Burgers’ equation has been advocated by Ron Kates (priv. comm. and unpublished notes).

We also think of several implications of such a description, although we are not going to develop them in this paper. It is known that “dust” is only recovered as a singular limit: a pure single-stream system maintains a δ -shaped velocity distribution along trajectories and generates a singularity at shell-crossing. An initially small deviation from a δ -distribution allows, however, to follow the distribution function smoothly beyond shell-crossing; caustics occurring in the density field for “dust matter” are smoothed out by velocity dispersion (Zel’dovich & Shandarin 1982). Thus, besides the obvious advantages implied by a model which can be followed through shell-crossing, it also entails the description of how velocity dispersion evolves, which is a key element in the diagnostics of cosmic structure. The “Jeans’ length”, which has not to be as small as the one associated with the usual thermostatic pressure, represents the scale on which structures are diffused (for a discussion of the Jeans’ length in collisionless systems see Seliverstov 1974, for a simulation of neutrino distributions see Melott 1982, 1983, Ma & Bertschinger 1994), for a Fermi gas a lower limit is set by the phase space constraint (Tremaine & Gunn 1979, Ruffini & Song 1987, Kull et al. 1997). Including velocity dispersion will allow us to access smaller spatial scales than with previous analytical models, and gives meaning to the notion of (virial) equilibrium in a collisionless system, which is a well-studied case in the context of stellar systems (Binney & Tremaine 1987). The treatment of multi-stream flow on cosmological scales will thus connect models for large-scale structure with small-scale systems which, apart from recent efforts to simulate galaxy formation embedded into large-scale structure, have been treated as isolated systems so far. Yet another perspective is opened concerning reconstruction models of the density field from observed peculiar-velocity

data (see Dekel 1994 for a review). Present reconstruction techniques suffer from the fact that they employ nonlinear models in regimes when singularities in the flow developed; the quest for singularity-free models is especially obvious in that context (for a discussion of this issue see Susperregi & Buchert 1997).

We proceed as follows: in Sect. 2 we introduce the system of equations that we are going to study, in Sect. 3 we give an equation governing the evolution of the mean peculiar-velocity in the presence of multi-stream flow, discuss some properties of the dynamical evolution prescribed by this equation, and relate it to commonly employed models of large-scale structure. In Sect. 4 we present our criticism and perspectives.

2. Basic equations in the presence of velocity dispersion

In this section we shall first use physical (i.e., non-rotating Eulerian) coordinates. The introduction of so-called ‘comoving coordinates’ (i.e., scaled Eulerian coordinates indexing fundamental observers in a background cosmology) will be employed when comparing with commonly used approximation schemes in cosmology in the next section.

Let us consider a gas of N particles of mass m which interact only gravitationally. We may not only think of stars (which is the common point of view - see the textbooks by, e.g., Binney & Tremaine 1987, Ogorodnikov 1965 and Saslaw 1985 -), but alternatively of dark matter particles (e.g., non-relativistic massive neutrinos or WIMP’s, e.g., Bertschinger 1993), or corresponding larger mass units (like galaxy halos) or, generally, of any patch of collisionless matter (a fluid element or a Lagrangian particle). The interpretation of our model depends on the context to which we apply it. In any case we have to specify the nature of the particles to see whether the assumptions apply - at least in a phenomenological sense - (e.g., for a gas of galaxy halos we neglect merger events, or dynamical friction; this also violates the particle number conservation; for any patch of matter we implicitly neglect additional terms that would arise by coarse-graining the distribution function, etc.).

We define the one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ such that $f(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v}$ is the probability density that a particle be within the volume element $d\mathbf{x}$ centered at the physical position \mathbf{x} with a physical velocity within the volume element $d\mathbf{v}$ centered at \mathbf{v} at time t . In the mean field approximation, it obeys Vlasov’s equation (e.g., Binney & Tremaine 1987) (summation over repeated indices is understood):

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + g_i \frac{\partial f}{\partial v_i} = 0 \quad , \quad (1)$$

where the mean-field gravitational field strength \mathbf{g} is determined self-consistently by the equations (the comma denotes spatial derivative, i.e., $_{,i} \equiv \frac{\partial}{\partial x_i}$):

$$g_{i,i}(\mathbf{x}, t) = \Lambda - 4\pi GmN \int d\mathbf{v} f(\mathbf{x}, \mathbf{v}, t) \quad , \quad (2a)$$

$$g_{i,j}(\mathbf{x}, t) = g_{j,i}(\mathbf{x}, t) \quad ; \quad (2b)$$

G denotes the gravitational constant and Λ the cosmological constant.

We now define the averaging operation $\langle \dots \rangle$: If $\phi(\mathbf{v})$ is any function of the velocity \mathbf{v} , then

$$\langle \phi \rangle := \frac{mN}{\varrho(\mathbf{x}, t)} \int d\mathbf{v} \phi(\mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \quad , \quad (3)$$

where the mass density is $\varrho = mN \int d\mathbf{v} f$. We define the average velocity $\bar{\mathbf{v}} := \langle \mathbf{v} \rangle$, and the *stress tensor* $\Pi = \varrho \langle (\mathbf{v}\mathbf{v}) - \bar{\mathbf{v}}\bar{\mathbf{v}} \rangle$, which takes velocity dispersion into account, i.e., the fact that at a given position there are several particles with different velocities. (It should be noticed that Π is a positive definite, symmetric tensor).

From Vlasov's equation (1) the equations obeyed by the density ϱ , the average velocity $\bar{\mathbf{v}}$ and the stress tensor Π are found by integrating out (1), $v_j \cdot (1)$, and $v_j v_k \cdot (1)$ over velocity space:

$$\partial_t \varrho + (\varrho \bar{v}_i)_{,i} = 0 \quad , \quad (4a)$$

$$\partial_t \bar{v}_i + \bar{v}_j \bar{v}_{i,j} = g_i - \frac{1}{\varrho} \Pi_{ij,j} \quad , \quad (4b)$$

$$\partial_t \Pi_{ij} + \bar{v}_k \Pi_{ij,k} + \bar{v}_{k,k} \Pi_{ij} = -\Pi_{ik} \bar{v}_{j,k} - \Pi_{jk} \bar{v}_{i,k} - L_{ijk,k} \quad , \quad (4c)$$

where $L_{ijk} = \varrho \langle (v_i - \bar{v}_i)(v_j - \bar{v}_j)(v_k - \bar{v}_k) \rangle$. Eq. (4b) is known as Jeans' equation in stellar systems theory. This is not a closed set of equations for the density ϱ , the average velocity $\bar{\mathbf{v}}$ and the stress tensor Π and (together with (2)) the gravitational field strength \mathbf{g} , because of L_{ijk} . We may in fact find the equation obeyed by this third-rank tensor, but it turns out to depend on the fourth moment of the velocity and so on. We thus get an infinite hierarchy of equations for all moments of the velocity.

The task is now to find a way to close this hierarchy. In hydrodynamics the common practice is to assume that collisions lead to local equilibrium on a short time-scale, so that the hydrodynamical system of equations is reduced to balance equations on a much larger time-scale for the five collisional invariants ϱ , \bar{v}_i and the internal energy U . In dilute gases, the Chapman-Enskog expansion is then applied to find Navier-Stokes' equation and Fourier's law (e.g., Balescu 1991). This method cannot be employed in our case, because collisions are neglected and local equilibrium is thus undefined.

The simplest way to close the hierarchy without resorting to local equilibrium considerations is to neglect velocity dispersion altogether, i.e., $f \propto \delta(\mathbf{v} - \bar{\mathbf{v}})$ which is the definition of "dust matter" (e.g., Gurevich & Zybin 1995). Then $\Pi_{ij} = 0$, $L_{ijk} = 0$ and Eqs. (4) reduce to

$$\partial_t \varrho + (\varrho \bar{v}_i)_{,i} = 0 \quad , \quad (5a)$$

$$\partial_t \bar{v}_i + \bar{v}_j \bar{v}_{i,j} = g_i \quad , \quad (5b)$$

which is the well-known set of equations (together with (2)) employed to model gravitational structure formation, being equivalent to following the motion of individual particles.

As remarked in the introduction, these equations have the problem that their solutions develop caustics, precisely because of neglecting velocity dispersion. We go one step further by explicitly including velocity dispersion from the beginning. Let us therefore assume that there is a small velocity dispersion of order $\varepsilon \ll 1$. This means that the typical relative deviations of particle velocities from the average velocity $\bar{\mathbf{v}}$ are of order ε (i.e., $|\mathbf{v}_{typ} - \bar{\mathbf{v}}| \sim \varepsilon |\bar{\mathbf{v}}|$). This assumption implies that $\Pi_{ij} \sim \varrho \varepsilon^2 |\bar{\mathbf{v}}|^2$ and $L_{ijk} \sim \varrho \varepsilon^3 |\bar{\mathbf{v}}|^3$. To study the effect of velocity dispersion, we shall retain only the leading order terms due to it, i.e. we shall work only up to order ε^2 .¹ Then $L_{ijk,k}$ can be dropped in Eq. (4c) provided L_{ijk} changes spatially on the same scale as \bar{v}_i and we get (together with (2)) a closed set of equations for ϱ , $\bar{\mathbf{v}}$, Π and \mathbf{g} :

$$\partial_t \varrho + (\varrho \bar{v}_i)_{,i} = 0 \quad , \quad (6a)$$

$$\partial_t \bar{v}_i + \bar{v}_j \bar{v}_{i,j} = g_i - \frac{1}{\varrho} \Pi_{ij,j} \quad , \quad (6b)$$

$$\partial_t \Pi_{ij} + \bar{v}_k \Pi_{ij,k} + \bar{v}_{k,k} \Pi_{ij} = -\Pi_{ik} \bar{v}_{j,k} - \Pi_{jk} \bar{v}_{i,k} \quad . \quad (6c)$$

Using this set of equations we go one step further down the hierarchy of equations for the velocity moments. It is, however, still difficult to cope with these equations and, in the present paper, we introduce the strong simplification that the velocity dispersion is approximately isotropic, i.e. the stress tensor Π_{ij} is diagonal and has only one independent component (a pressure-like term):

$$\Pi_{ij} \approx p \delta_{ij} \quad , \quad p > 0 \quad . \quad (7)$$

This assumption is certainly sensible at early stages of the structure formation process: we can think of an initial condition with isotropic velocity dispersion (it has to be so in the case of a homogeneous-isotropic matter distribution). Then, at earlier stages of the evolution, it will remain approximately isotropic. At later stages we have to consider the isotropy of the stress tensor as an *idealization* of the generally anisotropic velocity ellipsoid. This assumption is to be considered a weak point in our analysis, since there is nothing to prevent the velocity distribution from becoming largely anisotropic in contrast to the situation in a collisional system; it may work for single galaxy halos, but the bulk of the medium is not isotropic². Isotropy may, however, provide a good model if violent relaxation is taken into account (see Lynden-Bell 1967, Henriksen & Widrow 1997, White 1996 and Kull et al. 1997).

With this assumption, Eq. (6c) is greatly simplified:

$$\left(\dot{p} + \frac{5}{3} \bar{v}_{i,i} p \right) \delta_{ij} + 2p \sigma_{ij} = 0 \quad , \quad (8)$$

¹ It must be remarked that this is *not* equivalent to assuming that the velocities be Gaussian distributed. Gaussianity imposes an additional constraint on the set of Eqs. (6).

² From the observational point of view velocity dispersion can only be inferred from the line-of-sight information, the isotropy assumption is then employed when concluding on the 3D dispersion value.

where we have introduced the total or Lagrangian derivative along integral curves of the *mean* velocity field, $\dot{\cdot} := \partial_t + \bar{v}_i \partial_i$, and the shear tensor σ_{ij} by splitting the symmetric part of the mean-velocity gradient: $\bar{v}_{(i,j)} = \frac{1}{2}(\bar{v}_{i,j} + \bar{v}_{j,i}) = \frac{1}{3}\bar{v}_{k,k}\delta_{ij} + \sigma_{ij}$.

Notice that Eq. (8) imposes a strong kinematical restriction, namely, $\sigma_{ij} = 0$. Although we generalize the basic system of equations by allowing for a nonvanishing p , the interpretation of p in terms of a strictly isotropic velocity dispersion will accordingly restrict the kinematics of the mean motion. This makes clear that, unless we do not consider p as a phenomenological generalization of the ‘‘dust matter model’’, the assumption (7) is far too restrictive. It is, however, one of the assumptions which *has to* be imposed in order to recover the ‘‘adhesion approximation’’.

Keeping this in mind we go on by evaluating the trace of Eq. (8); it yields a relation between the density and the ‘‘dynamical pressure’’:

$$\dot{p} = \partial_t p + \bar{v}_i p_{,i} = -\frac{5}{3}p\bar{v}_{i,i} \quad . \quad (9)$$

Combining this equation with the continuity equation (6a) we find

$$\dot{p} = \frac{5}{3}p\dot{\varrho} \quad . \quad (10)$$

For this equation it is straightforward to find its general (Lagrangian) integral (which we may interpret as an ‘equation of state’ for a single volume element):

$$p(\mathbf{X}, t) = \kappa(\mathbf{X})\varrho(\mathbf{X}, t)^{5/3} \quad , \quad (11)$$

where κ is positive and of order ε^2 , and \mathbf{X} are Lagrangian coordinates which label fluid elements and coincide with the Eulerian coordinates \mathbf{x} at the initial time $t = t_0$. (Note that the integration of (10) implies that $\kappa \neq 0$; $\kappa = 0$ is a singular limit of the equations under consideration.)

If the physical system were a true fluid, (11) would be interpreted as describing the adiabatic evolution of each volume element in an ideal gas (consistently with the fact that we have assumed $L_{ijk,k} = 0$, which would be interpreted as the absence of heat flux between neighbouring volume elements). But unlike in a fluid, there are no collisions in the system we are studying that prevent velocity dispersion from growing and/or becoming highly anisotropic.

In the sequel we restrict the general integral (11) to the case where the initial data $p(\mathbf{X}, t_0)$ and $\varrho(\mathbf{X}, t_0)$ are chosen such that κ is independent of \mathbf{X} . Therefore, we drop the freedom of giving $p(\mathbf{X}, t_0)$ independently of $\varrho(\mathbf{X}, t_0)$. Thus, along any trajectory, we have the relationship $p = \kappa\varrho^{5/3}$ with the same constant κ . Consequently, we have the same relation also at any point in Eulerian space. The integral then attains the status of a (global) ‘equation of state’ for the medium occupying Eulerian space. However, here, it is not imposed as an equation of state

in thermodynamics, but follows from the dynamical equations under consideration and the assumptions employed.

Taking this result into Eq. (6b) we get our final set of equations:

$$\partial_t \varrho + (\varrho \bar{v}_i)_{,i} = 0 \quad , \quad (12a)$$

$$\partial_t \bar{v}_i + \bar{v}_j \bar{v}_{i,j} = g_i - \frac{5}{3}\kappa\varrho^{-1/3}\varrho_{,i} \quad , \quad (12b)$$

$$g_{i,i} = \Lambda - 4\pi G\varrho \quad , \quad g_{i,j} = g_{j,i} \quad . \quad (12c)$$

The difference between the closed set of Eqs. (12) and the one usually employed in cosmology (2,5) is the pressure-like term $\propto \varrho^{-1/3}\varrho_{,i}$, which takes into account (under the approximations stated above) velocity dispersion (i.e., the dynamical reaction due to multi-stream regions).

3. Equation for the mean peculiar-velocity relative to a background Hubble flow

We shall now show what novel features arise in the dynamical evolution prescribed by (12) due to the inclusion of the pressure-like term in contrast to the evolution prescribed by (2,5).

In order to compare our model with the commonly studied approximation schemes for large-scale structure we shall perform a change of variables. We consider the homogeneous-isotropic solutions of the basic equations (Friedmann-Lemaître backgrounds) characterized by the expansion factor $a(t)$ (and Hubble’s function $H = \dot{a}/a$), and define ‘comoving coordinates’ \mathbf{q} , an *average* peculiar-velocity field $\bar{\mathbf{u}}$ and a (mean-field) gravitational peculiar-acceleration \mathbf{w} as follows:

$$\mathbf{q} := \frac{1}{a}\mathbf{x} \quad , \quad \bar{\mathbf{u}} := \bar{\mathbf{v}} - H\mathbf{x} \quad , \quad \mathbf{w} := \mathbf{g} + \frac{4\pi G\varrho_H - \Lambda}{3}\mathbf{x} \quad , \quad (13)$$

where $\varrho_H(t)$ is the homogeneous background density (satisfying $\dot{\varrho}_H + 3H\varrho_H = 0$), which coincides with the spatially averaged density, if we impose periodic boundary conditions on $\bar{\mathbf{u}}$ and \mathbf{w} ; in this case the Hubble-flow exists and is uniquely defined (see Buchert & Ehlers 1997)³.

In terms of these variables, Eqs. (12) become (spatial derivatives refer now to \mathbf{q} and time derivatives are taken at constant \mathbf{q} ; hereafter, we drop the bar above \mathbf{u} for notational simplicity):

$$\partial_t \varrho + 3H\varrho + \frac{1}{a}(\varrho u_i)_{,i} = 0 \quad , \quad (14a)$$

$$\partial_t u_i + \frac{1}{a}u_j u_{i,j} + H u_i = w_i - \frac{5}{3}\frac{\kappa}{a}\varrho^{-1/3}\varrho_{,i} \quad , \quad (14b)$$

$$w_{i,i} = -4\pi G a(\varrho - \varrho_H) \quad , \quad w_{i,j} = w_{j,i} \quad . \quad (14c)$$

³ It is straightforward to show that this result carries over to the case where velocity dispersion is present, since the Hubble flow cancels in the expression for Π , $\Pi = \varrho(\langle \mathbf{v}\mathbf{v} \rangle - \bar{\mathbf{v}}\bar{\mathbf{v}}) \equiv \varrho(\langle \mathbf{u}\mathbf{u} \rangle - \bar{\mathbf{u}}\bar{\mathbf{u}})$, i.e., Π is a tensor field on the torus, and the divergence of the tensor appears in the basic equations, i.e. the corresponding flux vanishes by integrating over the torus.

Combining (14b) and (14c) we find the following equation, using the Lagrangian derivative operator $\dot{\cdot} := \partial_t|_x + \bar{v}_i \partial_i|_x = \partial_t|_q + \frac{u_i}{a} \partial_i|_q$ (written in vector form):

$$\dot{\mathbf{u}} + H\mathbf{u} = \mathbf{w} + \zeta \Delta \mathbf{w} \quad , \quad (15)$$

with the coefficient $\zeta = \frac{5}{3} \frac{\kappa}{4\pi G a^2} \varrho^{-1/3} > 0$, which depends on density and explicitly on time. The difference between Eq. (15) and the one generally used to model large-scale structure formation (which is found by setting $\zeta = 0$) is the $\Delta \mathbf{w}$ term.

Recall first the case of “dust” (i.e., no velocity dispersion, $\zeta = 0$). In the weakly nonlinear regime Zel’dovich’s approximation (Zel’dovich 1970, 1973) is a successful model until shell-crossing singularities develop. The trajectories in that approximation obey the parallelism of peculiar-gravitational acceleration and peculiar-velocity (see, e.g. Bildhauer & Buchert 1991, Kofman 1991, Buchert 1992),

$$\mathbf{w} = F(t)\mathbf{u} \quad , \quad F(t) = 4\pi G \varrho_H \frac{b(t)}{\dot{b}(t)} \quad , \quad (16)$$

where $b(t)$ is the growing mode solution of the linear theory of gravitational instability for “dust” (i.e., it solves the equation $\ddot{b} + 2H\dot{b} - 4\pi G \varrho_H b = 0$).

Since Eqs. (14) were obtained under the condition of small velocity dispersion, we can try to extrapolate Zel’dovich’s approximation (16) into this regime (which is equivalent to solving Eq. (15) by iteration) and thus find from Eq. (15):

$$\dot{\mathbf{u}} + (H - F)\mathbf{u} = \zeta F(t) \Delta \mathbf{u} \quad . \quad (17)$$

In order to construct the model we have to derive from the solution of Eq. (17) the trajectory field of the *average flow* $\mathbf{q} = \mathbf{F}(\mathbf{X}, t)$ by quadrature: $\mathbf{u} = a\dot{\mathbf{F}}$. Changing the temporal variable from t to b (this is possible since $b(t)$ is a monotonically increasing function of time) and defining a rescaled velocity field $\tilde{\mathbf{u}} = \mathbf{u}/a\dot{b}$, Eq. (17) becomes

$$\frac{d\tilde{\mathbf{u}}}{db} = \mu \Delta \tilde{\mathbf{u}} \quad , \quad \left(\frac{d}{db} := \frac{\partial}{\partial b} + \tilde{\mathbf{u}} \cdot \nabla \right) \quad , \quad (18)$$

where $\mu = \zeta F(t)/\dot{b} > 0$. If μ were independent of density, this equation would become 3D Burgers equation, whose solution is analytically known. The fact that μ depends on density presents an obstacle for finding an analytical solution. The principal advantage of Eq. (18) over the same equation with $\mu = 0$ is that it does not lead to caustic formation, since velocity dispersion smoothes out the singularity (Zel’dovich & Shandarin 1982, Shandarin & Zel’dovich 1989; see Ginanneschi 1998 and ref. therein for a thorough analysis). Therefore, this equation could be used as it stands to follow the dynamical evolution beyond the time when singularities in the “dust” continuum would arise. To be more precise, “shell-crossing” would still happen, but this doesn’t manifest itself as a singularity in the average

flow, rather as a (smooth) peak in the density field. Morphologically distinct patterns, classified by the Lagrange-singularity theory in the case of “dust” (see Arnold et al. 1982), will also emerge in the sense of smoothed-out images of critical sets on the Lagrangian manifold of the “dust” medium. This picture might not be true for large velocity dispersion.

As time goes by and the system becomes more and more virialized, Eqs. (14) cease to be a good description of the dynamical evolution, because velocity dispersion generically both grows and becomes anisotropic.

Eq. (18) is formally equivalent to the key equation of the “adhesion model” in which the coefficient μ is positive and constant in space and time (Gurbatov et al. 1989), and we see that the $\Delta \mathbf{u}$ term behaves as a viscous force. In fact, in the singular limit $\mu \rightarrow 0$ (the so-called inviscid limit), we approach a singular case of the “adhesion model” (Gurbatov et al. 1989; see also Gurbatov et al. 1983, 1985), the so-called “sticky particle model” for which geometrical construction techniques have been advanced (Pogosyan 1989, Kofman et al. 1990, Sahni & Coles 1995). For further details on models based on Burgers’ equation the reader may also consult the book by Gurbatov et al. (1991) and the interesting article by Vergassola et al. (1994). Further insight into possible applications is provided by a model of Jones (1996) for a two-component collisionless-baryonic system, which is related to the “adhesion model”.

Although we here recover a variant of the “adhesion model”, it should be clear that the new model presented implies an improvement for various reasons. The main one is that the dissipative-like term appears in a natural way together with a clear physical interpretation (i.e., velocity dispersion), unlike in the “adhesion model”, where it is motivated by phenomenological arguments.

This fact also solves the ‘momentum-conservation violation problem’ arising in the “adhesion model” (Kofman & Shandarin 1988; see also Gurbatov et al. 1983, Shandarin & Sathyaprakash 1996). Indeed, if velocity dispersion is neglected, Eq. (5b) describes the evolution of the velocity of *individual* particles, so that appending a $\Delta \mathbf{u}$ force to this equation implies that momentum is not conserved. On the other hand, the velocity field $\mathbf{u}(\mathbf{q}, t)$ is the *mean* velocity of the particles at the (comoving) Eulerian position \mathbf{q} at time t . For the mean velocity, gravity is not the only force, there is also an effective force due to velocity dispersion, so that the momentum balance is not violated. This effective force (pressure) simply provokes a flow of kinetic energy between the bulk average motion and the “random” motion of particles (described by velocity dispersion). The total kinetic energy (the kinetic energy of the mean flow plus internal kinetic energy) changes only due to gravitational work.

The conservation of mass and momentum is already evident from the way we have found Eqs. (14a) and (14b), which are balance equations expressing the conservation of mass and momentum. In fact, since the pressure force appears as a gradient (in general, as the divergence of a tensor for anisotropic velocity dispersion), it describes the transfer of momentum between neighbouring volume elements.

The *reversibility* of the transfer of momentum can also be immediately appreciated from our model equations. The coefficient μ is not related to an irreversible process: if one performs a time reversion $t \rightarrow -t$, $\mathbf{u} \rightarrow -\mathbf{u}$ (\mathbf{w} , ρ remain unchanged), then $F \rightarrow -F$, $\zeta \rightarrow \zeta$, and Eq. (17) is invariant under time reversion. (This is true also for the exact equation (15) as well as for the basic system of equations). Therefore, the coefficient μ should not be considered as a viscosity, a true viscosity coefficient would not change sign under time-reversion. Another difference is that the coefficient μ depends on the initial conditions (through κ), whereas true damping rates are insensitive to initial conditions. We may call the coefficient μ *gravitational multi-stream* or *GM-coefficient*, because it arises from the self-gravitation of multi-stream systems. The phenomenon of “reversible damping” is not unknown in physics (e.g. Landau damping); the phenomenon we are discussing coincides with ‘non-dissipative gravitational turbulence’ as described by Gurevich & Zybin (1995).

Another interesting point is that the GM-coefficient μ in (18) is not arbitrary: it has a dependence on time and density which is determined by the dynamical equations. However, this (very plausible) fact complicates the model which no longer can be solved by the known solution of the 3D Burgers equation. We have not found an immediate variable change to perform Cole-Hopf-type or other transformations in order to reduce the model to the linear diffusion equation. Application of the methods which work in the case of the 3D Burgers equation to more general problems apparently creates obstacles (see, e.g., Nerney et al. 1996). The model must be solved numerically by determining $\mu(\rho, t)$ at each time-step for each particle.

4. Discussion and criticism

We have presented a system of equations that allows studying the structure formation process at stages where the usually employed Euler-Poisson system breaks down. We discussed the possibility of constructing models for large-scale structure which do not suffer from the occurrence of shell-crossing singularities. We also proposed a model which features “adhesion” of fluid elements similar to the commonly used “adhesion approximation”, and allows following the structure formation process beyond stages where a laminar fluid approximation based on “dust” breaks down due to the development of multi-streaming: a “viscous” term appears in the model equation which we derived from the combined action of multi-stream flow and self-gravity.

Besides the obvious advantages that we do not have to *invent* an adhesive term and that momentum conservation is not violated, we also see a further merit of this approach: we have gained some physical insight into the “adhesion model” and could even improve on it.

One may ask why the combined effect of self-gravity and velocity dispersion actually “looks like” a dissipative term. To clarify this, let us add the following illustration: consider a vessel, filled with a non-viscous toy fluid, and placed into the gravitational field of the Earth. Initially, the fluid may be at rest

and its density homogeneous. This state is clearly unstable and the fluid will evolve into a stable, inhomogeneous and stratified state. The overall kinetic energy vanishes in this example, but the gravitational potential energy has decreased. The “missing energy” has not been dissipated away, rather it has been transformed into internal kinetic energy due to compression work; the final volume of the fluid is smaller than the initial volume, and the pressure has always opposed the motion induced by gravity. This is similar to what happens in the cosmological setting: volume elements would have equal tendency to compress or expand due to pressure-like forces, but the presence of gravity favors contraction of volume elements, especially when they are falling onto high-density regions. The effect of transformation of energy into internal velocity dispersion is large, because not only potential energy, but also bulk kinetic energy decreases, when the fluid elements crash onto high-density walls.

We have also shown that pressure-like forces may be relevant even on cosmological scales and an effective Jeans’ length may be larger than the (comoving) globular cluster scale, which is commonly attributed to a hydrodynamical (ideal gas) pressure (compare this to the effect of a gas pressure within pancakes investigated by Sunyaev & Zel’dovich 1972). However, we now understand clearly the limited status of this and similar models. We shall now elaborate on this criticism in more detail.

The extrapolation of the relationship (16) into the weakly non-linear regime is well justified for “dust”. If we define the weakly non-linear regime as suggested in (Buchert 1989), namely, as a linearization in Lagrangian space, Zel’dovich’s extrapolation of the Eulerian linear perturbation solution into the nonlinear regime is not only consistent with Lagrangian linearization of the full Euler-Poisson system (Buchert 1989, see also Doroshkevich et al. 1973 for a self-consistency test for part of the system), but arises naturally as a subcase of first-order solutions in a Lagrangian perturbation approach (Buchert 1992). This self-consistency of the “dust model” is mirrored in the fact that the trajectories of Lagrangian first-order perturbations even provide a class of 3D exact solutions to the Euler-Poisson system (Buchert 1989).

If pressure is taken into account, extrapolation of the “dust” model trajectories (as we had to do in order to recover the “adhesion approximation”) is no longer permitted. As we have shown it may be used as a *first* approximation for small velocity dispersion (and this is precisely the method followed in Zel’dovich & Shandarin 1982 and Bharadwaj 1996). A similar remark applies to other assumptions of extrapolation like the Frozen-Potential-Approximation (Brainerd et al. 1993, Bagla & Padmanabhan 1994), which we may employ instead of (16). This approach simply follows the evolution of velocity dispersion along the trajectories dictated by the “dust” model, its main drawback being that the “back-reaction” of velocity dispersion on these trajectories is not taken into account. In fact, the simple version of parallelism (16) is not even justified in the (Eulerian) linear regime (Buchert et al. 1998). The break-down of the assumption (16) after pancake formation is plausible and it is well-known for epochs after shell-crossing to which we want to apply the model (Doroshkevich 1973).

Nevertheless, the “adhesion approximation” works rather well, because the model is taken at face value and not restricted *a posteriori* by the constraints which led us to its derivation. Accordingly, we expect the model we proposed to improve on the “adhesion approximation”, since it specifies the dependence of “adhesion” on the local density. The publication of this work was motivated by this gain of insight and improvement. Let us summarize the constraining assumptions which restrict the general problem to the “adhesion approximation”:

- small velocity dispersion; we keep only the first equation in the hierarchy of velocity moments (A1).
- isotropy of the dispersion tensor which implies isotropy of the mean motion (A2).
- parallelity of peculiar-velocity and -acceleration, both defined relative to a global Hubble-flow (A3).
- a spatially constant relationship between initial density and initial pressure (A4).
- the GM-coefficient is constant in space and time as well as positive (A5) (which could be achieved by *imposing* a relationship $p = \kappa \rho^2$).

This list of assumptions shows that there is enough room for a generalization of the “adhesion approximation”.

Our main concern should be focussed on assumptions (A1) and (A2). Although both of them may be considered a good working hypothesis from a theoretical point of view, we know that the physical situation is not in favor of these assumptions, especially if we follow the average flow further into the nonlinear regime. Velocity dispersion will become quickly large in the multi-stream regime. A hint of this can be seen in observations of velocity dispersion in rich clusters of galaxies which is of the order of 1000 km/s, i.e. certainly not small compared with the bulk speed. However, viewing assumptions (A1) and (A2) together, we would obtain substantially similar models by arguing on phenomenological grounds: assumption (A2) gives an idealized model for the dispersion tensor and it just needs a relationship $p = \beta(\rho)$ to close the hierarchy of velocity moments. In light of a phenomenological approach assumption (A1) or, alternatively, assumption (A5) just specify the function β , but we may as well consider other relationships as discussed in (Buchert et al. 1998). This problem calls for a better model that relates the dispersion tensor to the density and velocity fields, but it does not invalidate the main ideas exemplified with assumption (A1).

Concerning assumption (A3) a first indication of how to proceed in order to investigate a valid extrapolation into the weakly nonlinear regime is furnished by Eq. (15) itself: the adhesive term is proportional to $\Delta \mathbf{w}$ rather than $\Delta \mathbf{u}$. It is possible to derive an evolution equation for the peculiar-gravitational field strength \mathbf{w} . Its specification to the weakly nonlinear regime, however, requires Lagrangian perturbation techniques that lie beyond the scope of the present work. Assumptions (A4), (A5) have their origin in mathematical simplicity and could be easily relaxed in a numerical simulation of the model.

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References

- Arnol'd V.I., Shandarin S.F., Zel'dovich Ya.B., 1982, *Geophys. Astrophys. Fluid Dyn.* 20, 111
- Bagla J.S., Padmanabhan T., 1994, *MNRAS* 266, 227
- Balescu R., 1991, *Equilibrium and Non-equilibrium Statistical Mechanics*, John Wiley & Sons
- Bertschinger E., 1993, in *Statistical Description of Transport in Plasma, Astro-, and Nuclear Physics*, eds.: J. Misguich, G. Pelletier, & P. Schuck, Commack, NY: Nova Science, pp.193-205
- Bharadwaj S., 1996 *ApJ* 472, 1
- Bildhauer S., Buchert T., 1991, *Prog. Theor. Phys.* 86, 653
- Binney J., Tremaine S., 1987, *Galactic Dynamics*, Princeton University Press.
- Bisnovatyi-Kogan G.S., Zel'dovich Ya.B., 1971, *Sov. Astron.* 14, 758
- Brainerd T.G., Scherrer R.J., Villumsen J.V., 1993, *ApJ* 418, 570
- Buchert T., 1989, *A&A* 223, 9
- Buchert T., 1992, *MNRAS* 254, 729
- Buchert T., 1994, *MNRAS* 267, 811
- Buchert T., 1996, in *Proc. IOP 'Enrico Fermi'*, Course CXXXII (Dark Matter in the Universe), Varenna 1995, eds.: S. Bonometto, J. Primack, A. Provenzale, IOS Press Amsterdam, pp. 543-564
- Buchert T., Ehlers J., 1993, *MNRAS* 264, 375
- Buchert T., Ehlers J., 1997, *A&A* 320, 1
- Buchert T., Karakatsanis G., Klaffl R., Schiller P., 1997, *A&A* 318, 1
- Buchert T., Domínguez A., Pérez-Mercader J., 1998 *ApJ*, submitted
- Clarke C.J.S., O'Donnell N., 1992, *Rend. Sem. Math. Univ. Pol. Torino* 50, 39
- Dekel A., 1994, *ARA&A* 32, 371
- Doroshkevich A.G., 1973, *Astrophysics Lett.* 14, 11
- Doroshkevich A.G., Ryabenskii V.S., Shandarin S.F., 1973, *Astrofizika* 9, 257; *Astrophysics* 9, 144 (1975)
- Doroshkevich A.G., Kotok E.V., Novikov I.D., Polyudov A.N., Shandarin S.F., Sigov Yu.S., 1980, *MNRAS* 192, 321
- Ehlers J., Buchert T., 1997, *Gen. Rel. & Grav.* 29, 733
- Fillmore J.A., Goldreich P., 1984a, *ApJ* 281, 1
- Fillmore J.A., Goldreich P., 1984b, *ApJ* 281, 9
- Ginanneschi P.M., 1998, *Physica D*, 115, 341
- Gurbatov S.N., Saichev A.I., Yakushkin I.G., 1983, *Sov. Phys. Usp.* 26, 857
- Gurbatov S.N., Saichev A.I., Shandarin S.F., 1985, *Sov. Phys. Dokl.* 30, 921
- Gurbatov S.N., Saichev A.I., Shandarin S.F., 1989, *MNRAS* 236, 385
- Gurbatov S.N., Malakhov A.N., Saichev A.I., 1991, *Nonlinear Random Waves and Turbulence in Nondispersive Media*, Manchester Univ. Press
- Gurevich A.V., Zybin K.P., 1995, *Sov. Phys. Usp.* 38(7), 687
- Henriksen R., Widrow L.M., 1997, *Phys. Rev. Lett.* 78, 3426
- Jones B.J.T., 1996, in *Proc. IOP 'Enrico Fermi'*, Course CXXXII (Dark Matter in the Universe), Varenna 1995, eds.: S. Bonometto, J. Primack, A. Provenzale, IOS Press Amsterdam, pp. 661-676
- Kofman L.A., 1991, in *IUAP Proceedings on Nucleosynthesis in the Universe*, ed. by K. Sato, Kluwer: Dordrecht, p.495

- Kofman L.A., Shandarin S.F., 1988, in *Friedmann: Centenary Volume*, World Scientific, p.120
- Kofman L.A., Pogosyan D.Yu., Shandarin S.F., 1990, MNRAS 242, 200
- Kofman L.A., Pogosyan D.Yu., Shandarin S.F., Melott A.L., 1992, ApJ 393, 437
- Kofman L.A., Bertschinger E., Dekel A., Gelb J.M., Nusser A., 1994, ApJ 420, 44
- Kull A., Treumann R.A., Böhringer H., 1997, ApJ 484, 58
- Lynden-Bell D., 1967, MNRAS, 136, 101
- Ma C.-P., Bertschinger E., 1994, ApJ 434, L5
- Melott A.L., 1982, Phys. Rev. Lett. 48, 894
- Melott A.L., 1983, ApJ 264, 59
- Nerney S., Schmahl E.J., Musielak Z.E., 1996, Quarterly of Appl. Math. LIV-1, 63
- Ogorodnikov K.F., 1965, *Dynamics of Stellar Systems*, Pergamon Press
- Pogosyan D.Yu., 1989, *Tartu Preprint*, Estonian Acad. Sci.
- Ruffini R., Song D.J., 1987, A&A 179, 3
- Sahni V., Coles P., 1995, Physics Reports 262, 1
- Saslaw W.C., 1985, *Gravitational Physics of Stellar and Galactic Systems*, Cambridge University Press
- Seliverstov V.V., 1974, Sov. Astron. 18, 170
- Shandarin S.F., 1997, in *Proceedings Stochastic Models in Geosystems*, Minneapolis 1994, eds.: S.A. Molchanov and W.A. Woyczynski, IMA Volumes, Springer, pp. 401-413
- Shandarin S.F., Sathyaprakash B.S., 1996, ApJ 467, L25
- Shandarin S.F., Zel'dovich Ya.B., 1989, Rev. Mod. Phys. 61, 185
- Shukurov A.M., 1981, Astrofizika 17, 469; Astrophysics 17, 263 (1982)
- Sunyaev R.A., Zel'dovich Ya.B., 1972, A&A 20, 189
- Susperregi M., Buchert T., 1997, A&A 323, 295
- Tremaine S., Gunn J.E., 1979, Phys. Rev. Lett. 42(6), 407
- Vergassola M., Dubrulle B., Frisch U., Noullez A., 1994, A&A 289, 325
- Weinberg D.H., Gunn J., 1990a, ApJ 352, L25
- Weinberg D.H., Gunn J., 1990b, MNRAS 247, 260
- White S.D.M., 1996, in *Proceedings of the 36th Herstmonceux Conference, "Gravitational Dynamics"*, eds.: O. Lahav, E. Terlevich, R. Terlevich, Cambridge Univ. Press
- Zel'dovich Ya.B., 1970, A&A 5, 84
- Zel'dovich Ya.B., 1973, Astrophysics 6, 164
- Zel'dovich Ya.B., Shandarin S.F., 1982, Sov. Astron. Lett. 8(3), 139