Astron. Astrophys. 335, 622–636 (1998)

ASTRONOMY AND ASTROPHYSICS

Resonant dynamic tides and apsidal motion in close binaries

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Received 16 February 1998 / Accepted 27 March 1998

Abstract. The resonance of a dynamic tide with a free oscillation mode in a component of a close binary system of stars is treated by means of a two-time variable expansion procedure. The treatment is developed with respect to a frame of reference corotating with the star. Both the free oscillation mode and the dynamic tide are considered as linear, isentropic perturbations of a spherically symmetric star. At the lowest order of approximation in the small expansion parameter, the resonant dynamic tide corresponds to the tidally excited oscillation mode.

Furthermore, the effect of the resonance on the secular apsidal motion is determined. A resonant dynamic tide can contribute to a much larger secular apsidal motion than the static tide and the non-resonant dynamic tides of the same degree can do. The contribution can be oriented in the sense opposite to the orbital motion as well as in the same sense. From numerical applications to polytropic models, it appears that especially resonances of dynamic tides with an f-mode or a lower-order g^+-mode bring about larger apsidal motions.

In this context, the determination of the contributions to the secular apsidal motion stemming from the static tides and the non-resonant dynamic tides is reviewed. Sterne's theory (1939) for the effect of the tidal distortion on the apsidal motion in close binary stars, which is used as standard theory, is also reconsidered. The theory is shown to rest on the consideration of non-resonant, low-frequency dynamic tides taken in their lowest-order asymptotic representation.

Key words: stars: binaries: close – stars: oscillations – methods: analytical

1. Introduction

In close binary systems of stars, each component is subject to the dynamic tides generated by its companion. From a theoretical point of view, dynamic tides are often treated as linear, isentropic, forced oscillations of a spherically symmetric star. In one type of approach, they have been described by means of linear superpositions of non-radial pressure and gravity modes of the star by Zahn (1970), Press & Teukolsky (1977), and Kumar et al. (1995). In another type of approach, dynamic tides have been determined by integration of the full fourth-order system of equations governing forced oscillations. This type of approach was initiated by Zahn (1975) and later developed more systematically by Polfliet & Smeyers (1990). Along the same type of approach, an asymptotic representation of low-frequency dynamic tides was recently established (Smeyers 1997, Smeyers & Willems submitted). Dynamic tides were considered as non-adiabatic, forced oscillations by Zahn (1975, 1977) and Savonije & Papaloizou (1983, 1984).

In a fundamental paper for the study of linear, isentropic, free non-radial oscillations of stars, Cowling (1941) has drawn the attention to the possibility of resonances of the tidal disturbance with free oscillations when the orbital period is short. In close binary systems with eccentric orbits, resonances with subharmonics of the orbital period are also possible. Resonance coefficients for low-frequency g^+-modes in a non-rotating star consisting of a convective core and a radiative envelope were evaluated by Zahn (1970) by means of an asymptotic representation. The modification of the excitation coefficient of a low-frequency g^+-mode due to a slow rotation of the star was examined by Rocca (1987). In both investigations, the perturbation of the gravitational potential caused by the response of the star to the tidal action was neglected. Resonance coefficients of toroidal modes in a rotating star were also considered by Rocca (1982).

The perturbation of the gravitational potential that arises due to a star's tidal distortion contributes to the apsidal motion. The standard expression for the apsidal motion was derived by Sterne (1939), as stated by the author, in the limiting case where the orbital period is long compared with the free harmonic periods of the component stars. The author also observed that the theory is inadequate in cases of a resonance of a dynamic tide with a free oscillation mode of the star.

The effect of a resonance between the orbital motion and stellar oscillations on the apsidal motion was investigated by Papaloizou & Pringle (1980). More recently, the validity of the classical apsidal motion formula was checked by Quataert et al.

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(1996). These authors concluded that the differences between the classical formula and results of dynamical calculations are largest for resonances of certain multiples of the orbital frequency with low-order modes.

In this paper, we treat the resonance between a dynamic tide and a free oscillation mode of a uniformly rotating component of a close binary system with an eccentric orbit by applying a two-time variable expansion procedure (Kevorkian & Cole 1981, 1996). The treatment is worked out with respect to a frame of reference that is corotating with the star. Important simplifications are that the rotating star is spherically symmetric, that the star's eigenfrequencies with respect to the corotating frame of reference are those of a non-rotating star, and that the tides are unaffected by the Coriolis force. Furthermore, we adopt the linear and isentropic approximation for both the star's oscillation modes and the tides considered as forced oscillations.

We also derive an expression for the effect of the star's distortion on the apsidal motion by means of a classical perturbation procedure of celestial mechanics.

In view of the interpretation of observed apsidal motions, we reconsider the determination of the effects of disturbances caused in stars by static tides and by non-resonant dynamic tides. We also re-examine the validity of the standard expression for the apsidal motion due to the tidal distortion of a component of a close binary system.

The plan of the paper is as follows. In Sect. 2, we present the equations governing linear, isentropic dynamic tides in a rotating spherically symmetric star with respect to a frame of reference that corotates with the star. In Sect. 3, we reconsider the decomposition of the tide-generating potential in Fourier series in terms of the companion's mean anomaly. We show that, even in binary systems with eccentric orbits, static tides, which are not to be assimilated with the equilibrium tide, are always present besides the dynamic tides. The treatment of the resonance of a dynamic tide with a free oscillation mode of the star is developed in Sect. 4. The contribution to the apsidal motion resulting from the tidal disturbances in the star is derived in Sect. 5. Sect. 6 is devoted to an overview of the determinations of the contributions to the apsidal motion stemming from the static tide and the non-resonant dynamic tides in a star. Numerical applications are considered in Sect. 7. In Sect. 8, the standard theory for the effect of the tidal disturbances on the apsidal motion is reconsidered and its validity is re-examined. Concluding remarks are presented in the final section.

2. Basic equations

Consider a close binary system of stars that are orbiting around each other under the influence of their mutual gravitational force. We refer to these stars as the star and its companion. We consider the companion as a point mass and assume that the star rotates uniformly around an axis perpendicular to the orbital plane with an angular velocity Ω in the sense of the orbital motion.

We start from an inertial frame of reference whose origin coincides with the mass centre C of the binary system and whose x^1x^2-plane corresponds to the orbital plane of the two stars. The axes are assumed to be orthogonal to each other and to have fixed directions in space. Initially, the directions of the x^1- and the x^2-axis are unspecified. We let the direction of the x^3-axis correspond to that of the vector of the star's angular velocity.

First, we pass on to an orthogonal frame of reference whose origin coincides with the mass center C_1 of the star and whose axes x'^1, x'^2, x'^3 are parallel to the axes x^1, x^2, x^3. For the direction of the x'^1-axis, we adopt the direction from the star's mass centre to the periastron in the relative orbit of the companion.

Next, we pass on to an orthogonal frame of reference whose origin and x''^3-axis coincide with those of the orthogonal frame of reference $C_1x'^1x'^2x'^3$, but whose x''^1- and x''^2-axis are corotating with the star. In this corotating frame of reference, we introduce the spherical coordinates $\boldsymbol{r} = (r, \theta, \phi)$. Let M_1 and M_2 be the masses of the star and the companion. If u and v are the radial distance and the true anomaly of the companion in its relative orbit, the transformation formulae from the Cartesian coordinates x^1, x^2, x^3 of a mass element of the star to the spherical coordinates r, θ, ϕ of that mass element are given by

$$\left.\begin{aligned} x^1 &= r\sin\theta\,\cos(\phi+\Omega\,t) - \frac{M_2}{M_1+M_2}\,u\,\cos v,\\ x^2 &= r\sin\theta\,\sin(\phi+\Omega\,t) - \frac{M_2}{M_1+M_2}\,u\,\sin v,\\ x^3 &= r\cos\theta. \end{aligned}\right\} \tag{1}$$

We consider the spherical coordinates r, θ, ϕ as generalized coordinates q^1, q^2, q^3. Let T be the kinetic energy with respect to the inertial frame of reference, Φ the potential of self-gravitation generated by the star, and Φ_2 the gravitational potential generated by the companion. These quantities are taken per unit mass. Furthermore, let P be the pressure, and ρ the mass density. If viscosity is neglected, Lagrange's equations of motion take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^j} - \frac{\partial T}{\partial q^j} = -\frac{\partial}{\partial q^j}\left(\Phi+\Phi_2\right) - \frac{1}{\rho}\frac{\partial P}{\partial q^j}, \quad j=1,2,3. \tag{2}$$

The kinetic energy per unit mass can be expressed as

$$T = \frac{1}{2}\,\mu + \nu_i\,\dot q^i + \frac{1}{2}\,g_{ik}\,\dot q^i\,\dot q^k \tag{3}$$

with

$$\left.\begin{aligned} \mu &= \delta_{k\ell}\,\frac{\partial x^k}{\partial t}\,\frac{\partial x^\ell}{\partial t},\\ \nu_i &= \delta_{k\ell}\,\frac{\partial x^k}{\partial t}\,\frac{\partial x^\ell}{\partial q^i},\\ g_{ij} &= \delta_{k\ell}\,\frac{\partial x^k}{\partial q^i}\,\frac{\partial x^\ell}{\partial q^j}. \end{aligned}\right\} \tag{4}$$

In these expressions, the $\delta_{k\ell}$ are Kronecker's delta's, and Einstein's summation convention is used.

The gravitational potential generated by the companion at any point with coordinates r, θ, ϕ can be expanded as

$$\Phi_2\,(\boldsymbol{r},t) = -\frac{G\,M_2}{u}\sum_{\ell=1}^{\infty}\left(\frac{r}{u}\right)^{\ell}\,P_\ell(\cos\chi). \tag{5}$$

Here G is the constant of gravitation, $P_\ell(x)$ the Legendre polynomial of degree ℓ, and χ the angle, viewed from the star's mass centre, between the direction to the companion and the direction to the point considered. The cosine of the angle χ is determined by

$$\cos\chi = \sin\theta\,\cos(\phi + \Omega\,t - v). \tag{6}$$

Under the assumption that the companion describes an unvarying Keplerian orbit around the star, its radial distance u and true anomaly v satisfy the equations of motion

$$\left.\begin{aligned} &\ddot{u} - u\,\dot{v}^2 = -\frac{G\,(M_1+M_2)}{u^2},\\ &\frac{d}{dt}\left(u^2\,\dot{v}\right) = 0. \end{aligned}\right\} \tag{7}$$

Then, in Eqs. (2), the first term in the expansion of the companion's gravitational force is seen to produce a uniform acceleration of the star towards the companion which is rendered by a part of the term $\partial\nu_j/\partial t$. When both terms are removed, the equations describe the tidal action exerted by the companion on the uniformly rotating star.

The tide-generating potential can be written in the form

$$\varepsilon_T\,W\,(\boldsymbol{r},t) = -\varepsilon_T\,\frac{G\,M_1}{R_1} \sum_{\ell=2}^{4}\left(\frac{R_1}{a}\right)^{\ell-2}\left(\frac{r}{R_1}\right)^{\ell}\left(\frac{u}{a}\right)^{-(\ell+1)} P_\ell(\cos\chi), \tag{8}$$

where R_1 is the radius of the equilibrium star, a the semi-major axis of the companion's relative orbit, and

$$\varepsilon_T = \left(\frac{R_1}{a}\right)^3\,\frac{M_2}{M_1}. \tag{9}$$

The summation is restricted to $\ell = 4$, since, from $\ell = 5$ on, the terms are at least of the second order in $(R_1/a)^3$.

Eqs. (2) are solved in two steps. As a first solution of the equations, we consider a uniformly rotating star in the absence of any tidal action of a companion. We neglect the distortion caused by the centrifugal force, so that the rotating star is spherically symmetric. Next, we introduce the tidal force of the companion as a small perturbing force and consider the tidal disturbances in the star as linear, isentropic, forced oscillations. Here we neglect the effects of the Coriolis force. Hence, the tides correspond to those of a non-rotating equilibrium star with spherical symmetry, which is static with respect to the corotating frame of reference.

Let $\left(\delta q^k\right)_T$, with $k = 1,2,3$, be the components of the tidal displacement with respect to the local coordinate basis $\partial/\partial q^1$, $\partial/\partial q^2$, $\partial/\partial q^3$. These components are then governed by the equations

$$g_{ij}\,\frac{\partial^2\left(\delta q^j\right)_T}{\partial t^2} + \mathcal{U}_{ij}\left(\delta q^j\right)_T = -\varepsilon_T\,\frac{\partial W}{\partial q^i},\quad i = 1,2,3, \tag{10}$$

where $\mathcal{U}_{ij}$ is the tensorial operator applying to free, linear, isentropic oscillations of a spherically symmetric star and is determined as

$$\mathcal{U}_{ij}\left(\delta q^j\right)_T = \frac{\partial \Phi'_T}{\partial q^i} - \frac{\rho'_T}{\rho^2}\,\frac{\partial P}{\partial q^i} + \frac{1}{\rho}\,\frac{\partial P'_T}{\partial q^i}. \tag{11}$$

A prime on a quantity denotes the Eulerian perturbation of that quantity.

3. Decomposition of the tide-generating potential

By means of the addition theorem for Legendre polynomials, Expansion (8) of the tide-generating potential can be developed as

$$\varepsilon_T\,W\,(\boldsymbol{r},t) = -\varepsilon_T\,\frac{G\,M_1}{R_1} \sum_{\ell=2}^{4}\sum_{m=-\ell}^{\ell} f_{\ell,m}(t)\left(\frac{r}{R_1}\right)^{\ell} Y_\ell^m(\theta,\phi)\,\exp(\mathrm{i}\,m\,\Omega\,t), \tag{12}$$

where

$$f_{\ell,m}(t) = \frac{(\ell-|m|)!}{(\ell+|m|)!}\,P_\ell^{|m|}(0)\left(\frac{R_1}{a}\right)^{\ell-2} \left(\frac{u}{a}\right)^{-(\ell+1)}\exp(-\mathrm{i}\,m\,v). \tag{13}$$

Here $P_\ell^{|m|}(x)$ is an associated Legendre polynomial of the first kind.

Since they are periodic functions of time with a period equal to the orbital period of the companion, the functions $f_{\ell,m}(t)$ can be expanded in Fourier series in terms of the multiples k of the mean anomaly M as

$$f_{\ell,m}(t) = \sum_{k=-\infty}^{\infty} c_{\ell,m,k}\,\exp(\mathrm{i}\,k\,M), \tag{14}$$

where the coefficients $c_{\ell,m,k}$ are determined as

$$c_{\ell,m,k} = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f_{\ell,m}(t)\,\exp(-\mathrm{i}\,k\,M)\,dM \tag{15}$$

(Polfliet & Smeyers 1990).

Substitution of Fourier Series (14) into Expansion (12) leads to the expansion of the tide-generating potential, at the point with spherical coordinates r,θ,ϕ and at the instant t considered,

$$\varepsilon_T\,W\,(\boldsymbol{r},t) = \varepsilon_T \sum_{\ell=2}^{4}\sum_{m=-\ell}^{\ell}\sum_{k=-\infty}^{\infty} W_{\ell,m,k}\,(\boldsymbol{r}) \exp[\mathrm{i}\,(m\,\Omega + k\,n)\,t - \mathrm{i}\,k\,n\,\tau]. \tag{16}$$

Here n is the mean motion, τ a time of periastron passage, and

$$\varepsilon_T\,W_{\ell,m,k}\,(\boldsymbol{r}) = -\varepsilon_T\,\frac{G\,M_1}{R_1}\,c_{\ell,m,k}\left(\frac{r}{R_1}\right)^{\ell} Y_\ell^m(\theta,\phi) \tag{17}$$

[see, for comparison, Zahn 1970, the expression above his Series (3)]. An individual term in Expansion (16) has the frequency $m\,\Omega + k\,n$ relative to the corotating frame of reference. For $m = 0$ and $k = 0$, the frequency is equal to zero, and the associated terms of the tide-generating potential give rise to static tides. Thus, even in a component of a binary star with an eccentric orbit, static tides are always present besides the various dynamic tides. These static tides should not be assimilated to the equilibrium tide, the definition of which is recalled below. In particular cases, the frequency $m\,\Omega + k\,n$ can be equal to zero even for values of m and k different from zero.

The coefficients $c_{\ell,m,k}$ are real and can be expressed as

$$c_{\ell,m,k} = \frac{(\ell-|m|)!}{(\ell+|m|)!}\, P_\ell^{|m|}(0) \left(\frac{R_1}{a}\right)^{\ell-2} X_k^{-(\ell+1),-m}, \qquad (18)$$

where $X_k^{-(\ell+1),-m}$ is a Hansen coefficient given by

$$X_k^{-(\ell+1),-m} = \frac{1}{\pi}\int_0^\pi \left[\frac{u(M)}{a}\right]^{-(\ell+1)} \cos[k\,M + m\,v(M)]dM \qquad (19)$$

(Ruymaekers & Smeyers 1994). The Hansen coefficients depend only on the eccentricity of the orbit. By means of the relations

$$dM = \left(\frac{u}{a}\right)^2 \frac{dv}{(1-e^2)^{1/2}}, \qquad (20)$$

$$\frac{u}{a} = \frac{1-e^2}{1+e\,\cos v} \qquad (21)$$

[see Smeyers et al. 1991, Eqs. (42) and (35)], they can be transformed into

$$X_k^{-(\ell+1),-m} = \frac{1}{(1-e^2)^{\ell-1/2}} \frac{1}{\pi}\int_0^\pi (1+e\,\cos v)^{\ell-1} \cos(k\,M + m\,v)\,dv. \qquad (22)$$

Hence, a Hansen coefficient $X_k^{-(\ell+1),-m}$ is proportional to $\left(1-e^2\right)^{-(\ell-1/2)}$ and becomes indefinitely large as $e \to 1$.

The coefficients $c_{\ell,m,k}$ also obey the property of symmetry

$$c_{\ell,-m,-k} = c_{\ell,m,k}. \qquad (23)$$

Of particular interest in investigations on dynamic tides are the second-degree coefficients $c_{2,m,k}$. It may be noted that they do not depend on the ratio R_1/a. The coefficient $c_{2,0,0}$ is determined by the simple expression

$$c_{2,0,0} = -\frac{1}{2}\left(1-e^2\right)^{-3/2}. \qquad (24)$$

Furthermore, the coefficients $c_{2,m,k}$ associated with $m=-1$ and $m=1$ are equal to zero, since $P_\ell^{|m|}(0)=0$ for odd values of $\ell+|m|$. The coefficients $c_{2,2,0}$ and $c_{2,-2,0}$ are also equal to zero for all values of the eccentricity.

The variations of the coefficients $c_{2,m,k}$ different from zero are represented in Fig. 1 for values of the orbital eccentricity ranging from $e=0$ to $e=0.9$, and for $k=0,1,\ldots,6$.

Apart from the coefficients $c_{2,-2,1}$ and $c_{2,-2,2}$, the various coefficients belonging to $m=-2$ increase from zero, reach a maximum value at a certain eccentricity, decrease, and become negative. The eccentricity at which the maximum value is reached, increases as k increases.

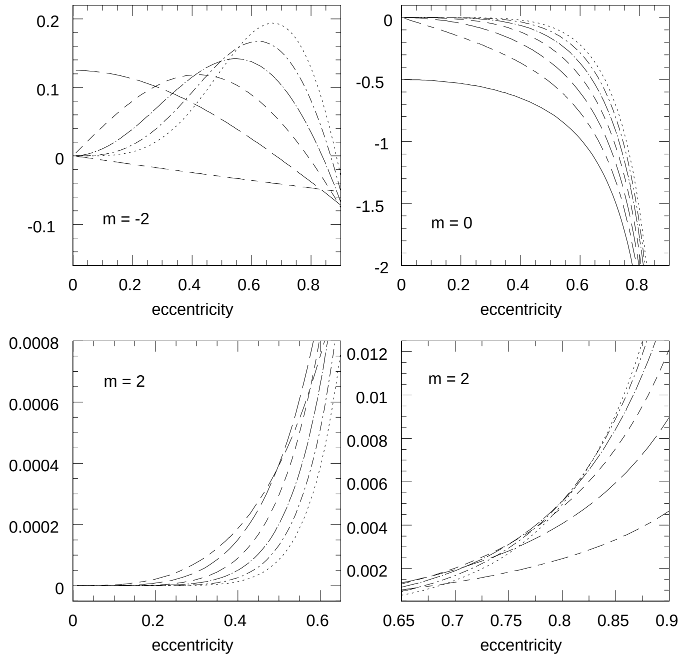

Fig. 1. Variations of the coefficients $c_{2,m,k}$ as functions of the orbital eccentricity e, for k = 0 (——), k = 1 (— — —), k = 2 (— —), k = 3 (– – –), k = 4 (— · —), k = 5 (– · –), k = 6 (· · · · · ·).

The coefficients $c_{2,0,k}$ are all negative and, except for the coefficient $c_{2,0,0}$, take the value 0 at $e=0$. For a given eccentricity, the coefficients $c_{2,0,k}$ decrease in absolute value as k increases.

The coefficients associated with $m=2$ are all positive and smaller in absolute value than the corresponding coefficients $c_{2,0,k}$. For eccentricities $e \lesssim 0.5$, it still holds true that the coefficients decrease as k increases. From $e \simeq 0.5$ on, the curve representing the variation of the coefficient $c_{2,2,1}$ crosses successively the curves representing the variations of the coefficients associated with larger values of k. As the eccentricity increases further, the curves representing the variations of the coefficients $c_{2,2,2}$, $c_{2,2,3}$, ..., one after the other start crossing the curves of the coefficients $c_{2,2,k}$ associated with larger values of k.

Taylor series of the coefficients $c_{2,m,k}$ to order e^3 are given by Smeyers et al. (1991).

In the particular case of a *circular* orbit, the constants $c_{\ell,m,k}$ are different from zero only for $k=-m$. Expansion (16) of the tide-generating potential then reduces to

$$\varepsilon_T\,W(\boldsymbol{r},t) = -\varepsilon_T\,\frac{G\,M_1}{R_1}\sum_{\ell=2}^{4}\sum_{m=-\ell}^{\ell} c_{\ell,m,-m} \left(\frac{r}{R_1}\right)^\ell P_\ell^{|m|}(\cos\theta)\,\exp[\mathrm{i}m(\phi+\Omega\,t-M)]. \qquad (25)$$

The second-degree coefficients involved are $c_{2,0,0}$, $c_{2,-2,2}$, and $c_{2,2,-2}$, which are just the three second-degree coefficients different from zero at $e=0$.

When moreover $\Omega\,t = M$, the tide-generating potential is time-independent relative to the uniformly rotating star and reduces to the potential raising equilibrium tides (Kopal 1978, p. 47). The star's uniform angular velocity Ω is then equal to the mean motion n of the companion and is determined by Kepler's third law as

$$\Omega = \left[\frac{G\left(M_1+M_2\right)}{a^3}\right]^{1/2}. \tag{26}$$

4. Dynamic tides in resonance with a free oscillation mode

We concentrate on the partial dynamic tide generated by a single term of Expansion (16) of the tide-generating potential. Let the term be

$$\varepsilon_T\, W_{\ell,m,k}\left(\boldsymbol{r}\right)\,\exp\left[\mathrm{i}\left(\sigma_T\, t - k\, n\, \tau\right)\right] \tag{27}$$

with the angular frequency

$$\sigma_T = m\,\Omega + k\,n. \tag{28}$$

We consider the frequency σ_T of the tide-generating potential to be close to the eigenfrequency of a free oscillation mode of the star.

Let $\left(\delta q^j\right)_{\lambda,\mu,\nu}(\boldsymbol{r})$, with $j=1,2,3$, be the components of the Lagrangian displacement of a free oscillation mode belonging to the spherical harmonic $Y_\lambda^\mu(\theta,\phi)$. The subscript ν designates the type and the order of the mode, for example, the mode g_5^+. Furthermore, let $\sigma^2_{\lambda,\nu}$ be the square of the eigenfrequency of the mode. The azimuthal number μ does not appear among the subscripts of the eigenfrequency, since the eigenvalue problem of the free oscillation modes of a non-rotating spherically symmetric star is degenerate with respect to that number. A free oscillation mode λ,μ,ν is solution of the wave equations

$$\sigma^2_{\lambda,\nu}\, g_{ij}\left(\delta q^j\right)_{\lambda,\mu,\nu} - \mathcal{U}_{ij}\left(\delta q^j\right)_{\lambda,\mu,\nu} = 0,\quad i=1,2,3. \tag{29}$$

We specifically assume that the partial dynamic tide with frequency σ_T is close to resonance with the free oscillation modes associated with the eigenfrequency $\sigma_{\ell,N}$ and belonging to the spherical harmonics of degree ℓ and the various admissible azimuthal numbers m. For the sake of simplification, we abbreviate the collective indices λ,μ,ν and ℓ,m,N to s and S, respectively.

We apply a two-time variable expansion procedure to Eqs. (10). To this end, we pass on to the dimensionless time variable

$$t^* = \sigma_{\ell,N}\, t. \tag{30}$$

Eqs. (10) then become

$$g_{ij}\,\frac{\partial^2\left(\delta q^j\right)_T}{\partial t^{*2}} + \frac{1}{\sigma^2_{\ell,N}}\,\mathcal{U}_{ij}\left(\delta q^j\right)_T = -\varepsilon_T\,\frac{1}{\sigma^2_{\ell,N}}\,\frac{\partial W_{\ell,m,k}}{\partial q^i}\,\exp\left[\mathrm{i}\left(\frac{\sigma_T}{\sigma_{\ell,N}}\,t^* - k\,n\,\tau\right)\right],\quad i=1,2,3. \tag{31}$$

Next, we define the small expansion parameter

$$\varepsilon = \frac{\sigma_{\ell,N}-\sigma_T}{\sigma_{\ell,N}} \tag{32}$$

and set

$$\varepsilon_T = \varepsilon\, f_T,\quad f_T \in \mathbb{R}. \tag{33}$$

(Kevorkian and Cole 1981, p. 168).

Furthermore, we introduce a fast time variable as

$$t^+ = t^*\left[1 + O\left(\varepsilon^2\right)\right] \tag{34}$$

and a slow time variable as

$$\tilde{t} = \varepsilon\, t^*. \tag{35}$$

We also introduce the following expansions for the components of the tidal displacement in terms of the star's free oscillation modes:

$$\left(\delta q^j\right)_T(\boldsymbol{r},t) = \sum_{s'}\left[F^{(0)}_{s'}\left(t^+,\tilde{t}\right) + \varepsilon\, F^{(1)}_{s'}\left(t^+,\tilde{t}\right) + O\left(\varepsilon^2\right)\right]\left(\delta q^j\right)_{s'}(\boldsymbol{r}),\quad j=1,2,3. \tag{36}$$

The expansions can be restricted to the free spheroidal modes, since the tidal force does not produce any vorticity around the local normal to a spherical equipotential surface of the star (see also Press & Teukolsky 1977).

At order ε^0, it follows from Eqs. (31) that

$$g_{ij}\,\frac{\partial^2}{\partial t^{+2}}\sum_{s'} F^{(0)}_{s'}\left(t^+,\tilde{t}\right)\left(\delta q^j\right)_{s'} + \frac{1}{\sigma^2_{\ell,N}}\,\mathcal{U}_{ij}\sum_{s'} F^{(0)}_{s'}\left(t^+,\tilde{t}\right)\left(\delta q^j\right)_{s'} = 0,\quad i=1,2,3. \tag{37}$$

By using Eqs. (29), multiplying all terms by $\rho\,\overline{\left(\delta q^i\right)_s}$, where the bar denotes the complex conjugate, integrating over the volume V of the equilibrium star, and taking into account the orthogonality property of the spherical harmonics, one derives the equation

$$\frac{\partial^2}{\partial t^{+2}}\,F^{(0)}_s\left(t^+,\tilde{t}\right) + \frac{\sigma^2_{\lambda,\nu}}{\sigma^2_{\ell,N}}\,F^{(0)}_s\left(t^+,\tilde{t}\right) = 0. \tag{38}$$

A general solution of it is given by

$$F^{(0)}_s\left(t^+,\tilde{t}\right) = A^{(0)}_s\left(\tilde{t}\right)\exp\left(\mathrm{i}\frac{\sigma_{\lambda,\nu}}{\sigma_{\ell,N}}\,t^+\right), \tag{39}$$

where $A^{(0)}_s\left(\tilde{t}\right)$ is a yet undetermined complex function. The lowest-order solutions for the components $\left(\delta q^j\right)_T(\boldsymbol{r},t)$ of the tidal displacement then take the form

$$\left(\delta q^j\right)_T(\boldsymbol{r},t) = \sum_{s'} A^{(0)}_{s'}\left(\tilde{t}\right)\left[\exp\left(\mathrm{i}\frac{\sigma_{\lambda',\nu'}}{\sigma_{\ell,N}}\,t^+\right)\right]\left(\delta q^j\right)_{s'}(\boldsymbol{r}),\quad j=1,2,3. \tag{40}$$

Next, at order ε, it follows from Eqs. (31) that

$$g_{ij}\,\frac{\partial^2}{\partial t^{+2}}\sum_{s'} F^{(1)}_{s'}\left(t^+,\tilde{t}\right)\left(\delta q^j\right)_{s'}(\boldsymbol{r})$$

$$+\frac{1}{\sigma_{\ell,N}^2}\,\mathcal{U}_{ij}\sum_{s'}F_{s'}^{(1)}\left(t^+,\tilde{t}\right)\left(\delta q^j\right)_{s'}(\boldsymbol{r})$$
$$=-2\,g_{ij}\,\frac{\partial^2}{\partial t^+\,\partial\tilde{t}}\sum_{s'}A_{s'}^{(0)}\left(\tilde{t}\right)$$
$$\left[\exp\left(\mathrm{i}\frac{\sigma_{\lambda',\nu'}}{\sigma_{\ell,N}}\,t^+\right)\right]\left(\delta q^j\right)_{s'}(\boldsymbol{r})$$
$$-\frac{f_T}{\sigma_{\ell,N}^2}\,\frac{\partial W_{\ell,m,k}}{\partial q^i}\left\{\exp\left[-\mathrm{i}\left(\tilde{t}+k\,n\,\tau\right)\right]\right\}\exp\left(\mathrm{i}t^+\right),$$
$$i=1,2,3. \qquad (41)$$

Again, we use Eqs. (29), multiply all terms by $\rho\,\overline{(\delta q^i)_s}$, integrate over the volume V of the equilibrium star, and take into account the orthogonality property of the spherical harmonics. We are then led to consider the overlap integral with regard to the mode s

$$\int_V \rho\,\frac{\partial W_{\ell,m,k}}{\partial q^i}\,\overline{(\delta q^i)_s}\,dV, \qquad (42)$$

which is proportional to the work done by the tidal force, and thus to the energy deposited in the star, through this mode.

The components of the Lagrangian displacement can be expressed with respect to the local coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$ as

$$\left.\begin{array}{rcl}(\delta r)_s(\boldsymbol{r}) &=& \xi_{\lambda,\nu}(r)\,Y_\lambda^\mu(\theta,\phi),\\ (\delta\theta)_s(\boldsymbol{r}) &=& \dfrac{\eta_{\lambda,\nu}(r)}{r^2}\,\dfrac{\partial Y_\lambda^\mu(\theta,\phi)}{\partial\theta},\\ (\delta\phi)_s(\boldsymbol{r}) &=& \dfrac{\eta_{\lambda,\nu}(r)}{r^2}\,\dfrac{1}{\sin^2\theta}\,\dfrac{\partial Y_\lambda^\mu(\theta,\phi)}{\partial\phi},\end{array}\right\} \qquad (43)$$

with $Y_\ell^m(\theta,\phi)=P_\ell^{|m|}(\cos\theta)\,\exp(\mathrm{i}m\phi)$. Here $P_\ell^{|m|}(\cos\theta)$ is an associated Legendre function.

When one uses Expression (17) for $W_{\ell,m,k}$ and properties of the spherical harmonics, one derives that

$$\int_V \rho\,\frac{\partial W_{\ell,m,k}}{\partial q^i}\,\overline{(\delta q^i)_s}\,dV$$
$$=-\delta_{\lambda,\ell}\,\delta_{\mu,m}\,\frac{G\,M_1}{R_1^{\ell+1}}\,\frac{4\,\pi}{2\,\ell+1}\,\frac{(\ell+|m|)!}{(\ell-|m|)!}\,c_{\ell,m,k}$$
$$\int_0^{R_1}\ell\,r^{\ell-1}\left[\xi_{\lambda,\nu}+(\ell+1)\,\eta_{\lambda,\nu}/r\right]\rho\,r^2\,dr. \qquad (44)$$

Two cases must be distinguished. First, when $s=S$, one finds the equation

$$\frac{\partial^2}{\partial t^{+2}}\,F_S^{(1)}\left(t^+,\tilde{t}\right)+F_S^{(1)}\left(t^+,\tilde{t}\right)$$
$$=-2\,\mathrm{i}\,\frac{dA_S^{(0)}}{d\tilde{t}}\,\exp\left(\mathrm{i}t^+\right)$$
$$+f_T\,\mathcal{J}_{k,\ell,m,N}\left\{\exp\left[-\mathrm{i}\left(\tilde{t}+k\,n\,\tau\right)\right]\right\}\exp\left(\mathrm{i}t^+\right), \qquad (45)$$

where

$$\mathcal{J}_{k,\ell,m,N}=-\frac{1}{\sigma_{\ell,N}^2\,I_S}\int_V \rho\,\frac{\partial W_{\ell,m,k}}{\partial q^i}\,\overline{(\delta q^i)_S}\,dV, \qquad (46)$$

and

$$I_S=\int_V \rho\,g_{ij}\left(\delta q^j\right)_S\,\overline{(\delta q^i)_S}\,dV. \qquad (47)$$

The quantity $\mathcal{J}_{k,\ell,m,N}$ is proportional to the ratio of the work done by the tidal force to the kinetic energy, with regard to the star's free oscillation mode S. Since I_S is given explicitly by

$$I_S=\frac{4\,\pi}{2\,\ell+1}\,\frac{(\ell+|m|)!}{(\ell-|m|)!}$$
$$\int_0^{R_1}\left[\xi_{\ell,N}^2+\ell(\ell+1)\,\eta_{\ell,N}^2/r^2\right]\rho\,r^2\,dr, \qquad (48)$$

the quantity $\mathcal{J}_{k,\ell,m,N}$ can be transformed as

$$\mathcal{J}_{k,\ell,m,N}=c_{\ell,m,k}\,Q_{\ell,N} \qquad (49)$$

with

$$Q_{\ell,N}=\frac{G\,M_1}{R_1^{\ell+1}}$$
$$\frac{1}{\sigma_{\ell,N}^2}\,\frac{\int_0^{R_1}\ell\,r^{\ell-1}\left[\xi_{\ell,N}+(\ell+1)\,\eta_{\ell,N}/r\right]\rho\,r^2\,dr}{\int_0^{R_1}\left[\xi_{\ell,N}^2+\ell(\ell+1)\,\eta_{\ell,N}^2/r^2\right]\rho\,r^2\,dr}. \qquad (50)$$

Here $Q_{\ell,N}$ is dimensionless and is determined by the free oscillation mode N of degree ℓ. The number k and the azimuthal number m play a role in the determination of the quantity $\mathcal{J}_{k,\ell,m,N}$ only through the coefficient $c_{\ell,m,k}$.

One removes the resonant terms from the inhomogeneous part of Eq. (45) by setting

$$\frac{dA_S^{(0)}}{d\tilde{t}}=\frac{f_T}{2\,\mathrm{i}}\,\mathcal{J}_{k,\ell,m,N}\,\exp\left[-\mathrm{i}\left(\tilde{t}+k\,n\,\tau\right)\right]. \qquad (51)$$

Integration yields

$$A_S^{(0)}\left(\tilde{t}\right)=\frac{f_T}{2}\,\mathcal{J}_{k,\ell,m,N}\,\exp\left[-\mathrm{i}\left(\tilde{t}+k\,n\,\tau\right)\right]. \qquad (52)$$

Secondly, when $s\neq S$, one removes the resonant terms from the inhomogeneous part of the equation by setting

$$\frac{dA_s^{(0)}}{d\tilde{t}}=0. \qquad (53)$$

We moreover assume that $A_s^{(0)}\left(\tilde{t}\right)$ is equal to zero at some instant $\tilde{t}_0$ so that

$$A_s^{(0)}\left(\tilde{t}\right)=0 \qquad (54)$$

in Solutions (40) for the components $\left(\delta q^j\right)_T(\boldsymbol{r},t)$ of the tidal displacement field.

Consequently, the tide-generating potential determined by Expression (27) gives rise to the tidal displacement field

$$\left(\delta q^j\right)_T(\boldsymbol{r},t)=\frac{1}{\varepsilon}\,\frac{\varepsilon_T}{2}\,\mathcal{J}_{k,\ell,m,N}$$
$$\left(\delta q^j\right)_S(\boldsymbol{r})\exp\left[\mathrm{i}\left(\sigma_T\,t-k\,n\,\tau\right)\right],\quad j=1,2,3. \qquad (55)$$

At the lowest-order of approximation, the tidal displacement field is of order ε^{-1} and corresponds to the resonantly excited oscillation mode N associated with the spherical harmonic of degree ℓ and the azimuthal number m.

We furthermore consider the term belonging to degree ℓ in Expansion (16) of the tide-generating potential which has a frequency equal to $-\sigma_T$ and also leads to a resonance. This term is associated with the azimuthal number $-m$ and the number $-k$. One then arrives at the following global solution for the dynamic tide with frequency σ_T close to the star's eigenfrequency $\sigma_{\ell,N}$:

$$\begin{aligned}\left(\delta q^j\right)_T(\boldsymbol{r},t) &= \frac{1}{\varepsilon}\,\frac{\varepsilon_T}{2}\,\mathcal{J}_{k,\ell,m,N}\\ &\Big\{\left(\delta q^j\right)_{\ell,m,N}(\boldsymbol{r})\,\exp\left[\mathrm{i}\left(\sigma_T\,t-k\,n\,\tau\right)\right]\\ &+\left(\delta q^j\right)_{\ell,-m,N}(\boldsymbol{r})\,\exp\left[-\mathrm{i}\left(\sigma_T\,t-k\,n\,\tau\right)\right]\Big\},\\ &j=1,2,3. \end{aligned}\tag{56}$$

Here we have taken into account that the coefficients $c_{\ell,m,k}$ obey Property (23). The two terms inside the braces are complex conjugate terms, so that their sum is real. Therefore, at the lowest-order of approximation, the components of the resonant dynamic tide can be expressed as

$$\left.\begin{aligned}(\delta r)_T(\boldsymbol{r},t) &= \frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\\ &\quad\xi_{\ell,N}(r)\,P_\ell^{|m|}(\cos\theta)\,\cos\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right),\\ (\delta\theta)_T(\boldsymbol{r},t) &= \frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\\ &\quad\frac{\eta_{\ell,N}(r)}{r^2}\,\frac{\partial P_\ell^{|m|}(\cos\theta)}{\partial\theta}\cos\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right),\\ (\delta\phi)_T(\boldsymbol{r},t) &= -\frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\\ &\quad\frac{\eta_{\ell,N}(r)}{r^2}\,\frac{m}{\sin^2\theta}\,P_\ell^{|m|}(\cos\theta)\\ &\quad\sin\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right).\end{aligned}\right\}\tag{57}$$

The coefficient $\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}/\varepsilon$ is similar to the coefficients c_i introduced by Zahn in his expansion of the tidal displacement in terms of the star's free oscillation modes [Zahn 1970, Definition (4)]. For $m \neq 0$, Solution (57) represents a tidal wave propagating in the rotating star in the azimuthal direction.

Correspondingly, the Eulerian perturbation of the gravitational potential at the star's surface generated by the tidal disturbances is given by

$$\begin{aligned}\Phi'_T(R_1,\theta,\phi;t) &= \frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\,\Phi'_{\ell,N}(R_1)\\ &P_\ell^{|m|}(\cos\theta)\,\cos\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right).\end{aligned}\tag{58}$$

Since the effects of both the dynamic tide with frequency σ_T and the dynamic tide with frequency $-\sigma_T$ are incorporated in $\Phi'_T(R_1,\theta,\phi;t)$, only non-negative values of k must be considered subsequently.

5. Contribution of the resonant dynamic tide to the apsidal motion

The Eulerian perturbation of the external gravitational potential that is generated by the star's tidal distortion is a solution of Laplace's equation of the form

$$\begin{aligned}\Phi'_e(\boldsymbol{r},t) &= C\,\left(\frac{r}{R_1}\right)^{-(\ell+1)}\,P_\ell^{|m|}(\cos\theta)\\ &\cos\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right).\end{aligned}\tag{59}$$

In this solution, C is a constant which is determined by the requirement that the Eulerian perturbation of the gravitational potential be continuous at $r = R_1$. It follows that

$$\begin{aligned}\Phi'_e(\boldsymbol{r},t) &= \frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\,\Phi'_{\ell,N}(R_1)\\ &\left(\frac{r}{R_1}\right)^{-(\ell+1)}\,P_\ell^{|m|}(\cos\theta)\,\cos\left(m\,\phi+\sigma_T\,t-k\,n\,\tau\right).\end{aligned}\tag{60}$$

One obtains the Eulerian perturbation of the external gravitational potential at the companion's position by setting $r = u$, $\theta = \pi/2$, $\phi = v - \Omega\,t$. One then has

$$\begin{aligned}\Phi'_e\left(u,\frac{\pi}{2},v-\Omega\,t;t\right) &= \frac{1}{\varepsilon}\,\varepsilon_T\,\mathcal{J}_{k,\ell,m,N}\,\Phi'_{\ell,N}(R_1)\\ &P_\ell^{|m|}(0)\left(\frac{u}{R_1}\right)^{-(\ell+1)}\,\cos\left(m\,v+k\,M\right).\end{aligned}\tag{61}$$

Taking into consideration Expression (49) for $\mathcal{J}_{k,\ell,m,N}$, we define the dimensionless quantity

$$H_{\ell,N} = -\frac{R_1}{G\,M_1}Q_{\ell,N}\,\Phi'_{\ell,N}(R_1).\tag{62}$$

This quantity is determined by the star's free oscillation mode N of degree ℓ and is independent of the azimuthal number. It is also independent of the normalization adopted for the free oscillation mode.

When one furthermore takes into account Definition (9) of ε_T, the Eulerian perturbation of the gravitational potential at the point where the companion is instantaneously situated can be written as

$$\begin{aligned}\Phi'_e\left(u,\frac{\pi}{2},v-\Omega\,t;t\right) &= -\frac{1}{\varepsilon}\,\frac{G\,M_1}{R_1}\,\left(\frac{R_1}{a}\right)^{\ell+4}\,\frac{M_2}{M_1}\\ &c_{\ell,m,k}\,P_\ell^{|m|}(0)\,H_{\ell,N}\,\left(\frac{u}{a}\right)^{-(\ell+1)}\,\cos\left(m\,v+k\,M\right).\end{aligned}\tag{63}$$

From this expression, the rate of apsidal motion can be derived by means of a classical perturbation procedure of celestial mechanics. By proceeding as in Smeyers et al. (1991), one finds for the disturbing function

$$R(u,v,t) = -\frac{M_1+M_2}{M_1}\,\Phi'_e\left(u,\frac{\pi}{2},v-\Omega\,t;t\right).\tag{64}$$

The rate of change of the longitude ϖ of the periastron in the companion's relative orbit is given by

$$\frac{d\varpi}{dt} = \left[G\left(M_1+M_2\right)a\right]^{-1/2}\,\frac{\left(1-e^2\right)^{1/2}}{e}\,\frac{\partial R}{\partial e}\tag{65}$$

(Sterne 1960, Sect. 4.58; Pascoli 1993, Sect. 6.2).

One may observe that

$$\frac{\partial R}{\partial e} = \frac{\partial R}{\partial u}\,\frac{\partial u}{\partial e} + \frac{\partial R}{\partial v}\,\frac{\partial v}{\partial e} \tag{66}$$

with

$$\left.\begin{aligned} \frac{\partial u}{\partial e} &= -a\,\cos v, \\ \frac{\partial v}{\partial e} &= \sin v\,\frac{2 + e\,\cos v}{1 - e^2} \end{aligned}\right\} \tag{67}$$

[see Smeyers et al. 1991, Eqs. (29) et (30)].

When one also uses Eq. (20), the definition of the mean anomaly M, and Kepler's third law, Eq. (65) becomes

$$\begin{aligned} \frac{d\varpi}{dM} = {} & \frac{1}{\varepsilon}\left(\frac{R_1}{a}\right)^{\ell+3}\,\frac{M_2}{M_1}\,\frac{\left(1 - e^2\right)^{1/2}}{e}\,c_{\ell,m,k}\,P_\ell^{|m|}(0)\,H_{\ell,N} \\ & \left[(\ell + 1)\left(\frac{u}{a}\right)^{-(\ell+2)}\cos(m\,v + k\,M)\,\cos v\right. \\ & -\frac{m}{1 - e^2}\left(\frac{u}{a}\right)^{-(\ell+1)}(2 + e\,\cos v) \\ & \left.\sin(m\,v + k\,M)\,\sin v\right]. \end{aligned} \tag{68}$$

The resulting secular apsidal motion, per revolution of the companion, is given by

$$(\Delta\varpi)_{\rm reson} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\varpi}{dM}\,dM \tag{69}$$

or, explicitly, by

$$(\Delta\varpi)_{\rm reson} = \frac{1}{\varepsilon}\left(\frac{R_1}{a}\right)^{\ell+3}\,\frac{M_2}{M_1}\,H_{\ell,N}\,G_{\ell,m,k} \tag{70}$$

with

$$\begin{aligned} G_{\ell,m,k}(e) = {} & \frac{1}{e\,\left(1 - e^2\right)^{\ell}}\,c_{\ell,m,k}\,P_\ell^{|m|}(0) \\ & \frac{1}{\pi}\left[(\ell + 1)\int_0^{\pi}(1 + e\,\cos v)^{\ell}\,\cos(m\,v + k\,M)\,\cos v\,dv\right. \\ & -m\int_0^{\pi}(1 + e\,\cos v)^{\ell-1}\,(2 + e\,\cos v) \\ & \left.\sin(m\,v + k\,M)\,\sin v\,dv\right], \end{aligned} \tag{71}$$

where we have used Eqs. (20) and (21).

Expression (70) for the contribution of a resonant dynamic tide to the secular apsidal motion, per revolution of the companion, is of the order of ε^{-1}. Apart from the factor $\varepsilon^{-1}\,(R_1/a)^{\ell+3}\;M_2/M_1$, it consists of the product of two factors: the quantity $H_{\ell,N}$ depending on the star's free oscillation mode that is involved in the resonance, and the constant $G_{\ell,m,k}$ depending on the orbital eccentricity.

The sign of Expression (70) is determined by the signs of ε, $H_{\ell,N}$, and $G_{\ell,m,k}$, so that the expression can be positive or negative. Hence, a resonant dynamic tide can contribute to a secular apsidal motion in the sense opposite to the orbital motion as well as in the sense of the orbital motion. This conclusion agrees with the conclusion of Quataert et al. (1996) that a periastron recession instead of a periastron advance may result from a resonance.

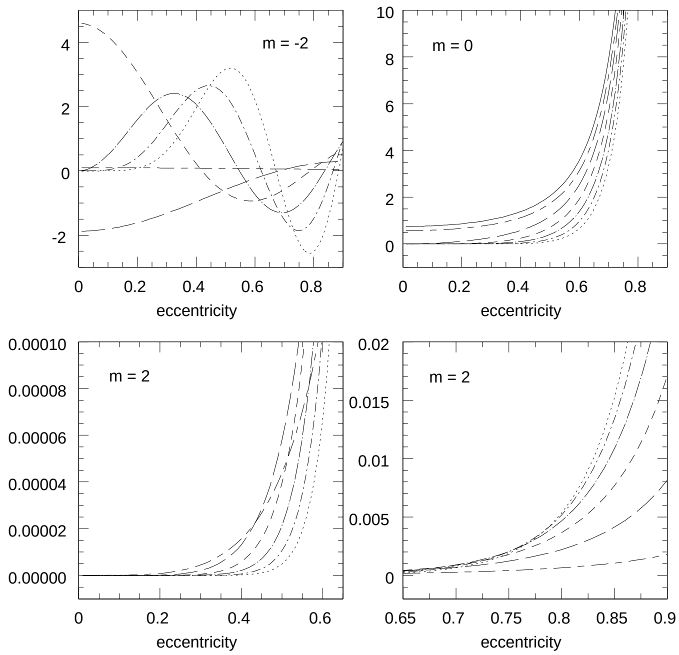

Fig. 2. Variations of the coefficients $G_{2,m,k}$ as functions of the orbital eccentricity e, for k = 0 (——), k = 1 (— — —), k = 2 (— —), k = 3 (– – –), k = 4 (— · —), k = 5 (– · –), k = 6 (· · · · · ·).

The constants $G_{\ell,m,k}$ obey the property of symmetry

$$G_{\ell,-m,-k} = G_{\ell,m,k}. \tag{72}$$

Furthermore, they become indefinitely large as $e \to 1$. In particular, the constant $G_{2,0,0}$ is given by

$$G_{2,0,0} = \frac{3}{4}\left(1 - e^2\right)^{-7/2}. \tag{73}$$

In Fig. 2, the variations of the coefficients $G_{2,m,k}$ different from zero are represented as functions of the orbital eccentricity from $e = 0$ to $e = 0.9$, for $k = 0, 1, \ldots, 6$. The coefficients $G_{2,0,k}$ and $G_{2,2,k}$ are positive and increase monotonically with the eccentricity. The coefficients $G_{2,0,k}$ reach larger values than the coefficients $G_{2,2,k}$ do. The coefficients $G_{2,-2,k}$ have a different behaviour. From $k = 4$ on, they first increase, reach a maximum value, subsequently decrease to a minimum, and again become positive.

Taylor series of the constants $G_{2,m,k}$ exact to order e^2 are given in the Appendix.

6. Global secular apsidal motion due to a star's tidal distortion

For any binary system with an orbital eccentricity different from zero, Decomposition (16) of the tide-generating potential leads to a decomposition of the tides associated with a given degree ℓ

into a *static tide* and an infinite number of *dynamic tides*. In accordance with this decomposition, the contributions stemming from the various tides must be taken into account in the determination of the global apsidal motion due to the star's tidal distortion.

6.1. Contributions of the static tides

The static tide of a degree ℓ is generated by the term in Expansion (16) of the tide-generating potential associated with $m = 0$ and $k = 0$, i.e.

$$- \varepsilon_T \frac{G M_1}{R_1} c_{\ell,0,0} \left(\frac{r}{R_1}\right)^{\ell} P_\ell(\cos\theta). \tag{74}$$

The radial component of the tidal displacement, $\xi_{T;\ell,0,0}(r)$, is determined by the linear, homogeneous, second-order differential equation

$$\frac{d^2\xi_{T;\ell,0,0}}{dr^2} + 2\left[\frac{1}{m(r)}\frac{dm(r)}{dr} - \frac{1}{r}\right]\frac{d\xi_{T;\ell,0,0}}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi_{T;\ell,0,0} = 0, \tag{75}$$

where $m(r)$ is the mass contained inside the sphere with radius r [Polfliet and Smeyers 1990, Eq. (70)]. Eq. (75) is equivalent with Eq. (10) of Sterne (1939) or with Radau's equation. It admits of a particular solution that remains finite near the singular point at $r = 0$ and behaves as $r^{\ell-1}$. Therefore, the admissible solution contains a single undetermined constant.

At the star's surface, the radial component of the tidal displacement must satisfy the inhomogeneous boundary condition

$$\left(\frac{d\xi_{T;\ell,0,0}}{dr}\right)_{R_1} + \frac{\ell-1}{R_1}\left(\xi_{T;\ell,0,0}\right)_{R_1} = \varepsilon_T\,(2\ell+1)c_{\ell,0,0}. \tag{76}$$

By means of this boundary condition, the undetermined constant involved in the admissible solution of Eq. (75) is fixed. The solution for $\xi_{T;\ell,0,0}(r)$ can then be cast in the form

$$\xi_{T;\ell,0,0}(r) = \varepsilon_T\, c_{\ell,0,0}\, \xi_\ell^*(r), \tag{77}$$

where the function $\xi_\ell^*(r)$ depends only on the degree ℓ.

At any radial distance r, the radial component of the tidal displacement is related to the sum of the tide-generating potential and the Eulerian perturbation of the gravitational potential caused by the tidal displacement field as

$$\xi_{T;\ell,0,0}(r) = -\frac{1}{g}\left[\Phi'_{T;\ell,0,0}(r) - \varepsilon_T \frac{G M_1}{R_1} c_{\ell,0,0} \left(\frac{r}{R_1}\right)^{\ell}\right]. \tag{78}$$

From this relation, it follows that the Eulerian perturbation of the gravitational potential at the star's surface can be expressed as

$$\Phi'_{T;\ell,0,0}\left(R_1\right) = -\varepsilon_T \frac{G M_1}{R_1} c_{\ell,0,0}\, 2\, K_{\ell,0} \tag{79}$$

with

$$2\, K_{\ell,0} = \frac{\xi_\ell^*\left(R_1\right)}{R_1} - 1. \tag{80}$$

The constants $K_{\ell,0}$ depend on the star's mass concentration. For illustration, one may consider two extreme cases. First, in the case of the equilibrium configuration with uniform mass density, one has

$$\xi_{T;\ell,0,0}(r) = \varepsilon_T\, c_{\ell,0,0} \frac{2\ell+1}{2\,(\ell-1)} \frac{r^{\ell-1}}{R_1^{\ell-2}}, \tag{81}$$

so that

$$K_{\ell,0} = \frac{3}{4\,(\ell-1)}. \tag{82}$$

Secondly, in the case of a centrally condensed model, one has

$$\xi_{T;\ell,0,0}(r) = \varepsilon_T\, c_{\ell,0,0} \frac{r^{\ell+2}}{R_1^{\ell+1}}, \tag{83}$$

so that

$$K_{\ell,0} = 0. \tag{84}$$

Hence, as the star's central condensation increases, the constants $K_{\ell,0}$ decrease from $3/[4(\ell-1)]$ to 0. The constant $K_{2,0}$ corresponds to the usual apsidal motion constant.

Because of the continuity with Solution (79) at the star's surface, the Eulerian perturbation of the gravitational potential at any external point with spherical coordinates r and θ takes the form

$$\Phi'_e(r,\theta) = -\varepsilon_T \frac{G M_1}{R_1} c_{\ell,0,0}\, 2\, K_{\ell,0} \left(\frac{r}{R_1}\right)^{-(\ell+1)} P_\ell(\cos\theta). \tag{85}$$

From here on, we restrict ourselves to the static tide of $\ell = 2$. Independently of the true anomaly v of the companion, the Eulerian perturbation of the external gravitational potential at $r = u$ and $\theta = \pi/2$ is given by

$$\Phi'_e(u,\pi/2) = -\frac{G M_1}{R_1} \frac{M_2}{M_1} \left(\frac{R_1}{a}\right)^6 c_{2,0,0}\, 2\, K_{2,0}\, P_2(0) \left(\frac{u}{a}\right)^{-3}. \tag{86}$$

Proceeding as in the previous section, one derives the following expression for the contribution of the static tide to the secular apsidal motion, per revolution of the companion:

$$(\Delta\varpi)_{2,0,0} = -\left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1} \frac{\left(1-e^2\right)^{1/2}}{e}\; 3\, c_{2,0,0}\, K_{2,0} \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \left(\frac{u}{a}\right)^{-4} \cos v\, dM. \tag{87}$$

By using Relations (20) and (21), performing the integration, and taking into account Expression (24) for the coefficient $c_{2,0,0}$ and Expression (73) for $G_{2,0,0}$, one derives the contribution of the static tide to the secular apsidal motion, per revolution of the companion,

$$(\Delta\varpi)_{2,0,0} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\, 2\, K_{2,0}\, G_{2,0,0}. \tag{88}$$

The expression consists of a product of positive factors. Hence, the contribution of the static tide of degree $\ell = 2$ to the secular apsidal motion is always positive and is oriented in the sense of the orbital motion. In particular, Expression (88) involves the constant $K_{2,0}$ depending on the mass concentration and the constant $G_{2,0,0}$ depending on the eccentricity of the orbit. It is larger for a star with a lower mass concentration and increases with the orbital eccentricity.

It may be noted that the contribution of the static tide to the secular apsidal motion differs from Expression (119) established in the standard theory, contrary to an earlier comment of Papaloizou and Pringle (1980, Sect. 2.4).

6.2. Contributions of the non-resonant dynamic tides

Consider a dynamic tide generated by a single term of Expansion (16) of the tide-generating potential. Let the term again be of the form of Expression (27). Here we assume that the angular frequency σ_T is remote from any eigenfrequency of a free oscillation mode of the star and that not both m and k are equal to zero.

6.2.1. General treatment for non-resonant dynamic tides

Non-resonant dynamic tides are governed by the homogeneous fourth-order system of differential equations

$$\frac{d\left(r^2\,\xi\right)}{dr} = \frac{g}{c^2}\, r^2\,\xi + \frac{r^2}{c^2}\left(S_\ell^2 - \sigma_T^2\right)\eta + \frac{r^2}{c^2}\,\Psi, \tag{89}$$

$$\frac{d\eta}{dr} = \left(1 - \frac{N^2}{\sigma_T^2}\right)\xi + \frac{N^2}{g}\,\eta - \frac{1}{\sigma_T^2}\,\frac{N^2}{g}\,\Psi, \tag{90}$$

$$\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\,\frac{d\Psi}{dr}\right) - \frac{\ell(\ell+1)}{r^2}\,\Psi = 4\,\pi\,G\,\rho\left[\frac{N^2}{g}\,\xi + \frac{1}{c^2}\left(\sigma_T^2\,\eta - \Psi\right)\right]. \tag{91}$$

In these equations, $\xi(r)$ and $\eta(r)$ are the radial and the transverse component of the tidal displacement with respect to the coordinate basis $\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi$ [see, for comparison, Expressions (43)], and $\Psi(r)$ is the total perturbation of the gravitational potential defined as

$$\Psi = \Phi' - \varepsilon_T\,\frac{G\,M_1}{R_1}\,c_{\ell,m,k}\left(\frac{r}{R_1}\right)^\ell. \tag{92}$$

Furthermore, g is the gravity, c^2 the square of the isentropic sound velocity, N^2 the square of the Brunt-Väisälä frequency, and

$$S_\ell^2 = \frac{\ell(\ell+1)\,c^2}{r^2}\,. \tag{93}$$

The system of differential Eqs. (89)–(91) depends on the azimuthal number m through σ_T^2. This dependency of the frequency σ_T on the azimuthal number m is a main difference with the frequencies considered in Smeyers et al. (1991). There the frequencies were considered with respect to the non-rotating frame of reference $C_1 x'^1 x'^2 x'^3$.

The solutions of the equations must obey conditions at $r = 0$ and $r = R_1$. At $r = 0$, the tidal displacement must remain finite. At $r = R_1$, the Lagrangian perturbation of the pressure must vanish, and the continuity of the Eulerian perturbation of the gravitational potential and its gradient leads to the inhomogeneous boundary condition

$$\left(\frac{d\Psi}{dr}\right)_{R_1} + \frac{\ell+1}{R_1}\,\Psi_{R_1} + \left(4\,\pi\,G\,\rho\,\xi\right)_{R_1} = -\varepsilon_T(2\ell+1)\,\frac{G\,M_1}{R_1^2}\,c_{\ell,m,k}. \tag{94}$$

Because of the non-homogeneous term, the non-resonant dynamic tide is proportional $\varepsilon_T\,c_{\ell,m,k}$.

The Eulerian perturbation of the gravitational potential at the star's surface due to the star's tidal distortion can be expressed as

$$\Phi'_{T;\ell,m,k}\left(R_1\right) = -\varepsilon_T\,\frac{G\,M_1}{R_1}\,c_{\ell,m,k}\,F_{\ell,m,k}. \tag{95}$$

where $F_{\ell,m,k}$ is a constant given by

$$F_{\ell,m,k} = -\left(\frac{R_1}{G\,M_1}\,\frac{\Psi_{R_1}}{\varepsilon_T\,c_{\ell,m,k}} + 1\right) \tag{96}$$

[Polfliet and Smeyers 1990, Eq. (61)]. It may be observed that the ratio $\Psi_{R_1}/\left(\varepsilon_T\,c_{\ell,m,k}\right)$ is independent of the product $\varepsilon_T\,c_{\ell,m,k}$. The constant $F_{\ell,m,k}$ obeys the property of symmetry

$$F_{\ell,-m,-k} = F_{\ell,m,k}. \tag{97}$$

6.2.2. Asymptotic treatment for lower-frequency, non-resonant dynamic tides

For lower-frequency, non-resonant dynamic tides, an asymptotic representation has been derived by Smeyers (1997) and Smeyers and Willems (submitted). Since the asymptotic representation is established with respect to the non-rotating frame of reference with axes x'^1, x'^2, x'^3, we replace the frequencies $k\,n$ of the dynamic tides appearing in the derivation by $m\,\Omega + k\,n$. The small dimensionless expansion parameter adopted in the asymptotic theory then corresponds to

$$\varepsilon_{\rm dyn} = \frac{R_1^3}{G\,M_1}\,|m\,\Omega + k\,n| = \frac{R_1^3}{G\,M_1}\,|\sigma_T|\,. \tag{98}$$

Furthermore, in the asymptotic theory, the square of the Brunt-Väisälä frequency is assumed to be positive everywhere in the stellar model.

At order $\varepsilon_{\rm dyn}^0$, the asymptotic solution for the radial component of the tidal displacement corresponds to the static solution $\xi_{T;\ell,m,k}(r)$ satisfying homogeneous differential Eq. (75)

and obeying inhomogeneous boundary Condition (76), in which the coefficient $c_{\ell,0,0}$ is now replaced by the coefficient $c_{\ell,m,k}$. By analogy with Solution (77) for $\xi_{T;\ell,0,0}(r)$, the solution for $\xi_{T;\ell,m,k}(r)$ can be expressed as

$$\xi_{T;\ell,m,k}(r) = \varepsilon_T \, c_{\ell,m,k} \, \xi^*_\ell(r). \tag{99}$$

By means of this solution, the total perturbation of the gravitational potential at the star's surface to order $\varepsilon^2_{\rm dyn}$ is determined as

$$\Psi_{R_1} = \varepsilon_T \, \frac{G\,M_1}{R_1} \, c_{\ell,m,k} \left\{ -\frac{(\xi^*_\ell)_{R_1}}{R_1} + \frac{\varepsilon^2_{\rm dyn}}{\ell(\ell+1)} \left[\left(\frac{d\xi^*_\ell}{dr} \right)_{R_1} + 2\,\frac{(\xi^*_\ell)_{R_1}}{R_1} \right] \right\} \tag{100}$$

(Smeyers and Willems submitted). Substitution into Expression (96) for $F_{\ell,m,k}$ and use of Definition (80) of $2\,K_{\ell,0}$ yield for the asymptotic approximation of the constant $F_{\ell,m,k}$:

$$F_{\ell,m,k} = 2\,K_{\ell,0} - \frac{\varepsilon^2_{\rm dyn}}{\ell(\ell+1)} \left[\left(\frac{d\xi^*_\ell}{dr} \right)_{R_1} + 2\,\frac{\xi^*_\ell\,(R_1)}{R_1} \right]. \tag{101}$$

For illustration, the two cases that are extreme with regard to the mass concentration can again be considered. In the case of an equilibrium configuration with uniform mass density, one has

$$F_{\ell,m,k} = \frac{1}{2\,(\ell-1)} \left(3 - \varepsilon^2_{\rm dyn}\,\frac{2\,\ell+1}{\ell} \right), \tag{102}$$

while, in the case of a centrally condensed model, one has

$$F_{\ell,m,k} = -\varepsilon^2_{\rm dyn}\,\frac{\ell+4}{\ell\,(\ell+1)}. \tag{103}$$

Hence, for lower-frequency dynamic tides outside any resonance, the constant $F_{\ell,m,k}$ is positive for an equilibrium configuration with uniform mass density, and is of order $\varepsilon^2_{\rm dyn}$ and negative for a centrally condensed model.

6.2.3. Contribution to the apsidal motion

For any non-resonant dynamic tide of degree ℓ associated with m and k, the time-dependent Eulerian perturbation of the gravitational potential at the star's surface can be expressed as

$$\Phi'_{T;\ell,m,k}\,(R_1,\theta,\phi;t) = -\varepsilon_T\,\frac{G\,M_1}{R_1}\,c_{\ell,m,k}\,F_{\ell,m,k} \, P^{|m|}_\ell(\cos\theta)\,\exp\left[{\rm i}\,(m\,\phi + \sigma_T\,t - k\,n\,\tau)\right]. \tag{104}$$

The non-resonant dynamic tide of the same degree ℓ that is associated with $-m$ and $-k$ has the angular frequency $-\sigma_T$. By using the foregoing expression and the symmetry Properties (23) and (97), one has

$$\Phi'_{T;\ell,m,k}\,(R_1,\theta,\phi;t) + \Phi'_{T;\ell,-m,-k}\,(R_1,\theta,\phi;t) = -\varepsilon_T\,\frac{G\,M_1}{R_1}\,c_{\ell,m,k}\,2\,F_{\ell,m,k} \, P^{|m|}_\ell(\cos\theta)\,\cos\left(m\,\phi + \sigma_T\,t - k\,n\,\tau\right). \tag{105}$$

As for Expression (58), only non-negative values of k must be considered subsequently.

The associated Eulerian perturbation of the external gravitational potential is determined by Solution (59). By imposing the continuity of the Eulerian perturbation of the gravitational potential at the star's surface and using Definition (9) of ε_T, one finds that, at the companion's position $r = u$, $\theta = \pi/2$, $\phi = v - \Omega\,t$,

$$\Phi'_e\left(u, \frac{\pi}{2}, v - \Omega\,t; t\right) = -\frac{G\,M_1}{R_1} \left(\frac{R_1}{a} \right)^{\ell+4} \frac{M_2}{M_1} \, c_{\ell,m,k}\,P^{|m|}_\ell(0)\,2\,F_{\ell,m,k} \left(\frac{u}{a} \right)^{-(\ell+1)} \cos(m\,v + k\,M). \tag{106}$$

This expression is similar to Expression (63). Hence, the contribution to the secular apsidal motion, per revolution of the companion, is similar to that given by Expression (70). It follows that

$$(\Delta\varpi)_{\ell,m,k} = \left(\frac{R_1}{a} \right)^{\ell+3} \frac{M_2}{M_1}\,2\,F_{\ell,m,k}\,G_{\ell,m,k}. \tag{107}$$

6.3. *Total secular apsidal motion resulting from a star's tidal distortion*

We restrict ourselves to the star's tidal distortion resulting from the second-degree terms in Expansion (16) of the tide-generating potential. On the basis of the foregoing analysis, the secular apsidal motion due to a star's second-degree tidal distortion, per revolution of the companion, is given by

$$(\Delta\varpi)_{\rm tidal} = (\Delta\varpi)_{2,0,0} + \sum_{k=1}^{\infty} \left[(\Delta\varpi)_{2,0,k} + (\Delta\varpi)_{2,2,k} + (\Delta\varpi)_{2,-2,k} \right], \tag{108}$$

where $(\Delta\varpi)_{2,0,0}$ is determined by Expression (88), and $(\Delta\varpi)_{2,0,k}$, $(\Delta\varpi)_{2,2,k}$, and $(\Delta\varpi)_{2,-2,k}$ are determined by Expression (107).

Expression (108) can be written more explicitly as

$$(\Delta\varpi)_{\rm tidal} = \left(\frac{R_1}{a} \right)^5 \frac{M_2}{M_1} \Big[2\,K_{2,0}\,G_{2,0,0} + 2\sum_{k=1}^{\infty} \left(F_{2,0,k}\,G_{2,0,k} + F_{2,2,k}\,G_{2,2,k} + F_{2,-2,k}\,G_{2,-2,k} \right) \Big]. \tag{109}$$

In case of a resonance of a dynamic tide belonging to a given value of m and a given value of k with a free oscillation mode N, the constant $2\,F_{2,m,k}$ must be replaced by $H_{2,N}/\varepsilon$, where the constant $H_{2,N}$ is determined by Definition (62).

7. Numerical applications

For the polytropic models with index $n_e = 2, 3, 4$, we have considered several oscillation modes g^+, the oscillation mode f, and several oscillation modes p belonging to the degree $\ell = 2$. With regard to these modes, we have determined the quantity $H_{2,N}$. It may be recalled that the quantity $H_{2,N}$ is dimensionless and

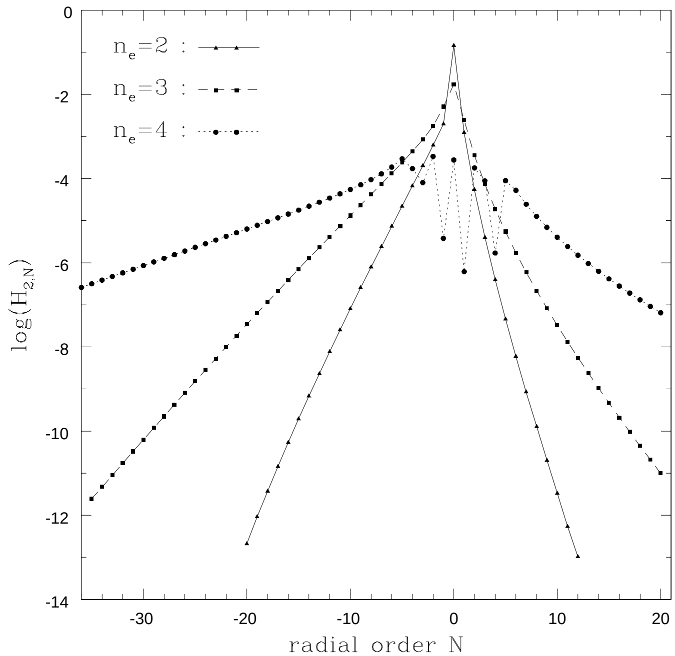

Fig. 3. The logarithms of the quantities $H_{2,N}$ for polytropic models with indices $n_e = 2, 3, 4$ as a function of the radial order N of the mode. g-modes are denoted by negative radial orders.

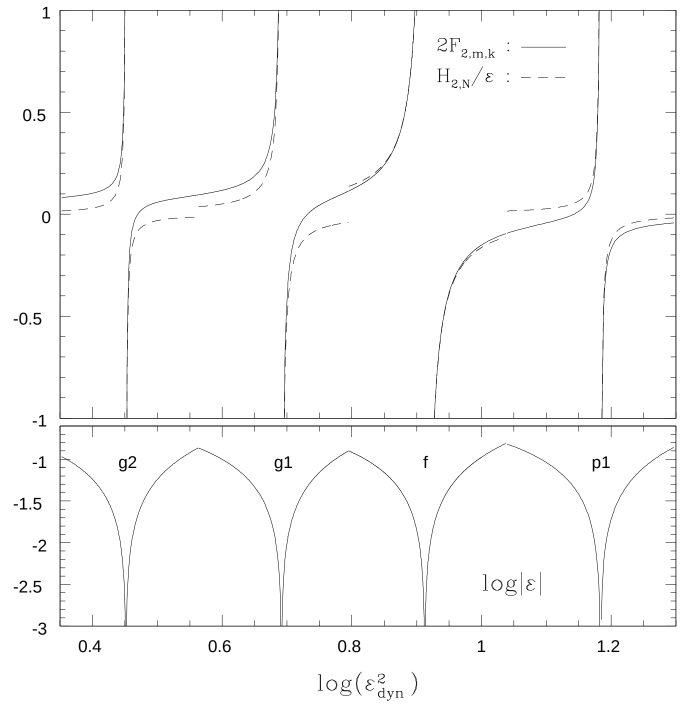

Fig. 4. Variation of the quantities $2\,F_{2,m,k}$ and $H_{2,N}/\varepsilon$ (upper panel), and the logarithm of the small parameter $|\varepsilon|$ (lower panel) as a function of $\log\left(\varepsilon^2_{\rm dyn}\right)$.

independent of the normalization adopted for the determination of the free oscillation modes.

The variation of $\log H_{2,N}$ as a function of the radial order N of the mode is presented in Fig. 3. The order N is negative for the g^+-modes, zero for the f-mode, and positive for the p-modes. In the cases of the polytropes with index $n_e = 2$ and $n_e = 3$, the quantity $H_{2,N}$ is largest for the f-mode and decreases rapidly for both the p- and the g-modes as the radial order of the mode increases. The decrease is less rapid for the polytrope with index $n_e = 3$ than for the polytrope with index $n_e = 2$. Hence, a resonance of a dynamic tide with the f-mode or a lower-order g^+-mode contributes to a larger apsidal motion.

In the case of the polytrope with index $n_e = 4$, the value of the quantity $\log H_{2,N}$ displays an erratic behavior for the lower-order g^+-modes, the f-mode, and the lower-order p-modes. This behaviour can be ascribed to the mixed p- and g^+-character of the modes. For higher-order g^+- and p-modes, the value of $\log H_{2,N}$ decreases as the radial order of the mode increases, but the decrease is sensibly slower than for the polytropes with index $n_e = 3$ and $n_e = 2$.

For the polytrope with index $n_e = 3$, we used Definition (96) in order to consider the variation of the product $2\,F_{2,m,k}$ as a function of the dimensionless orbital frequency $\varepsilon^2_{\rm dyn}$. The variation of the constant is represented in the upper panel of Fig. 4 as a function of the logarithm of $\varepsilon^2_{\rm dyn}$ for the range of the eigenfrequencies of the g_2^+-, the g_1^+-, the f-, and the p_1-mode belonging to $\ell = 2$. Near the resonances with these eigenfrequencies, the variation of $H_{2,N}/\varepsilon$ is also represented. In the lower panel of the figure, the variation of $\log|\varepsilon| = \log|\left(\sigma_T - \sigma_{2,N}\right)/\sigma_{2,N}|$ gives the precise locations of the resonant frequencies of the dynamic tides. One observes that the constant $H_{2,N}/\varepsilon$ tends closely to the constant $2\,F_{2,m,k}$ as $\varepsilon \rightarrow 0$.

8. The standard theory and its validity

The standard theory for the effect of the tidal distortion on the apsidal motion in close binary stars is due to Sterne (1939). The underlying assumption is that the mean motion n of the companion and the angular velocity Ω of the star are so small that the radial distance u and the azimuthal angle $v - \Omega\,t$ of the companion can be considered to be frozen. Under this assumption, the coefficients $c_{\ell,m,k}$ in Expansion (12) for the tide-generating potential are identified as

$$c_{\ell,m,k} = \frac{(\ell-|m|)!}{(\ell+|m|)!}\,P_\ell^{|m|}(0) \left(\frac{R_1}{a}\right)^{\ell-2} \left(\frac{u}{a}\right)^{-(\ell+1)} \exp[{\rm i}\,m\,(\Omega\,t - v)]. \qquad (110)$$

For $\ell = 2$, the coefficients $c_{\ell,m,k}$ different from zero then are

$$\left.\begin{aligned} c_{2,0,0} &= -\frac{1}{2}\left(\frac{u}{a}\right)^{-3}, \\ c_{2,2,0} &= \frac{1}{8}\left(\frac{u}{a}\right)^{-3} \exp[2\,{\rm i}\,(\Omega\,t - v)], \\ c_{2,-2,0} &= \frac{1}{8}\left(\frac{u}{a}\right)^{-3} \exp[-2\,{\rm i}\,(\Omega\,t - v)]. \end{aligned}\right\} \qquad (111)$$

It may be observed that, for an equilibrium tide, the eccentricity of the orbit is assumed to be zero so that

$$u = a, \quad v = M, \qquad (112)$$

and the star is assumed to rotate synchronously with the orbital motion of the companion so that

$$\Omega\, t = M. \tag{113}$$

The coefficients different from zero in the tide-generating potential then reduce to $c_{2,0,0} = -1/2$, $c_{2,2,0} = 1/8$, $c_{2,-2,0} = 1/8$. None of these assumptions is made here, nor one the Equalities (112) and (113) is adopted here.

The functions $\xi_{T;2,0,0}(r), \xi_{T;2,2,0}(r), \xi_{T;2,-2,0}(r)$ are determined in the frame of the theory of static tides described above. They are solutions of Eq. (75) and satisfy boundary Condition (76) at the star's surface. In this boundary condition, the coefficient $c_{\ell,0,0}$ is replaced successively by $c_{2,0,0}$, $c_{2,2,0}$, and $c_{2,-2,0}$ given by Expressions (111).

The Eulerian perturbation of the external gravitational potential at the point with spherical coordinates r, θ, ϕ then takes the form

$$\Phi'_e\,(\boldsymbol{r}) = -\varepsilon_T\, \frac{G\,M_1}{R_1}\, 2\,K_{2,0} \left(\frac{r}{R_1}\right)^{-3} \Big\{c_{2,0,0}\, P_2(\cos\theta) + \Big[c_{2,2,0}\,\exp(2\,\mathrm{i}\,\phi) + c_{2,-2,0}\,\exp(-2\,\mathrm{i}\,\phi)\Big]\, P_2^2(\cos\theta)\Big\}. \tag{114}$$

The perturbing function R at the point $r = u$, $\theta = \pi/2$, $\phi = v - \Omega\, t$, where the companion is situated, is given by

$$R(u,v,t) = \frac{G\,M_1}{R_1}\, \frac{M_1+M_2}{M_1}\, \frac{M_2}{M_1} \left(\frac{R_1}{a}\right)^6 2\,K_{2,0} \left(\frac{u}{a}\right)^{-3} \Big\{c_{2,0,0}\, P_2(0) + \big\{c_{2,2,0}\,\exp[2\,\mathrm{i}\,(v-\Omega\,t)] + c_{2,-2,0}\,\exp[-2\,\mathrm{i}\,(v-\Omega\,t)]\big\}\, P_2^2(0)\Big\}. \tag{115}$$

By partial differentiation and substitution of Expressions (111) for the coefficients $c_{2,0,0}$, $c_{2,2,0}$, $c_{2,-2,0}$, one derives that

$$\frac{\partial R}{\partial u} = -\frac{G\,M_1}{R_1}\, \frac{M_1+M_2}{M_1}\, \frac{M_2}{M_1} \left(\frac{R_1}{a}\right)^6 \frac{1}{a}\, 6\,K_{2,0} \left(\frac{u}{a}\right)^{-7}, \tag{116}$$

$$\frac{\partial R}{\partial v} = 0. \tag{117}$$

It follows that

$$\frac{d\varpi}{dM} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\, \frac{\left(1-e^2\right)^{1/2}}{e}\, 6\,K_{2,0} \left(\frac{u}{a}\right)^{-7} \cos v. \tag{118}$$

This equation is identical to Eq. (37) of Smeyers et al. (1991).

The radial distance u and the true anomaly v are functions of the time, and their slow variations during the revolution of the companion depend on the eccentricity of the orbit. One determines the mean value of the apsidal motion per revolution by using Relations (20) and (21). In this way, one recovers the expression for the secular apsidal motion given by Sterne:

$$(\Delta\varpi)_{\rm standard} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\, K_{2,0}\, 15\, f\left(e^2\right) \tag{119}$$

with

$$f\left(e^2\right) = \left(1-e^2\right)^{-5} \left(1 + \frac{3}{2}\, e^2 + \frac{1}{8}\, e^4\right). \tag{120}$$

In the absence of any resonance, the standard Expression (119) can be compared with the limiting value of Expression (109) for $n \to 0$ and $\Omega \to 0$, i.e. for $\varepsilon_{\rm dyn} \to 0$. From the asymptotic representation of lower-frequency, non-resonant dynamic tides, it follows that the constants $F_{2,m,k}$ are determined by Expression (101). If one neglects the term of order $\varepsilon_{\rm dyn}^2$, the constants are given by

$$F_{2,m,k} = 2\,K_{2,0} \tag{121}$$

for all values of m and k. Expression (109) for the secular apsidal motion, per revolution of the companion, then reduces to

$$(\Delta\varpi)_{\rm tidal} = \left(\frac{R_1}{a}\right)^5 \frac{M_2}{M_1}\, K_{2,0}\, 2\, g(e) \tag{122}$$

with

$$g(e) = G_{2,0,0} + 2 \sum_{k=1}^{\infty} (G_{2,0,k} + G_{2,2,k} + G_{2,-2,k}). \tag{123}$$

We determined the secular apsidal motion, per revolution of the companion, both by means of Expression (119) and by means of Expression (122), for values of the eccentricity ranging from $e = 0$ to $e = 0.9$. The resulting secular apsidal motions agree exactly for any eccentricity in the considered range of values. The agreement is confirmed analytically for small values of the eccentricity. Indeed, by means of the Taylor Series (126) and (127) of the constants $G_{\ell,m,k}$ given in the Appendix, one derives, to order e^2,

$$2\, g(e) = 15 \left(1 + \frac{13}{2}\, e^2\right). \tag{124}$$

This Taylor series corresponds to the Taylor series of the function $15\, f\left(e^2\right)$ considered to the same order of approximation in the eccentricity.

From the agreement, we infer that Expression (119) established by the standard theory is equivalent with Expression (122), which is derived in the frame of the theory of *dynamic* tides in the asymptotic approximation applying to low-frequency dynamic tides.

Expression (122) remains valid as long as the second term in asymptotic Approximation (101) for the constants $F_{2,m,k}$ is small enough in comparison to the first term, i.e. as long as

$$\varepsilon_{\rm dyn}^{-2} >> \frac{1}{12\,K_{2,0}} \left[\left(\frac{d\xi_2^*}{dr}\right)_{R_1} + 2\, \frac{\xi_2^*\,(R_1)}{R_1}\right] \equiv B. \tag{125}$$

Quantities relative to the right-hand member of the inequality are given in Table 1 for the polytropic models with indices $n_e = 2, 3, 4$. It appears that the right-hand member of the inequality increases rapidly as the polytropic index increases. The main reason of this increase is that $K_{2,0}$ tends to zero as the central condensation of the model increases. Hence, the validity of

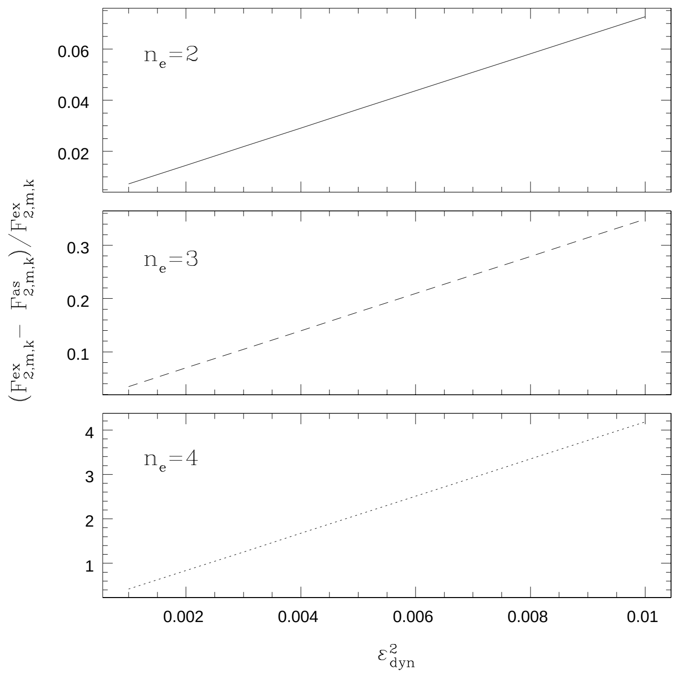

Fig. 5. Relative errors of $F_{2,m,k}$ for the polytropic models with indices $n_e = 2, 3, 4$ as a function of $\varepsilon^2_{\rm dyn}$.

Table 1. Quantities relative to the right-hand member of Inequality (125) for the polytropic models with indices $n_e = 2, 3, 4$

n_e	$\xi_2^*(R_1)/R_1$	$(d\xi_2^*/dr)_{R_1}$	B
2	1.148	3.852	6.923
3	1.029	3.971	34.65
4	1.002	3.998	500.2

Expression (122) may be questioned for models with higher central condensations.

The fact that $K_{2,0}$ tends to zero as the central condensation of the model increases also involves that asymptotic Approximation (101) for the constant $F_{2,m,k}$ rapidly becomes less accurate. The strong decrease of the accuracy is illustrated by Fig. 5, in which the relative error of the constant $F_{2,m,k}$ is represented as a function of the dimensionless orbital frequency $\varepsilon^2_{\rm dyn}$ in the range from 0.001 to 0.01, for the polytropic models with index $n_e = 2, 3, 4$. In the case of the polytropic model with index $n_e = 2$, the relative errors are of the order of only a few percents, while, in the case of the polytropic model with index $n_e = 4$, they amount even to hundreds of percents.

9. Concluding remarks

In this investigation, we have treated the resonance of a dynamic tide with a free oscillation mode in a component of a close binary system of stars by means of a two-time variable expansion procedure. We have developed the treatment with respect to a frame of reference corotating with the star. Both the free oscillation and the dynamic tide are considered as linear, isentropic perturbations of a spherically symmetric star.

The dynamic tide is generated by a single term of the tide-generating potential of the form of Expression (27). This term is characterized by a degree ℓ, an azimuthal number m, and a multiple k of the companion's mean motion. The forced frequency σ_T is determined by Expression (28) and is assumed to be close to the eigenfrequency $\sigma_{\ell,N}$ of a free oscillation mode of the same degree ℓ, and of type and order N.

An appropriate expansion parameter ε is introduced by means of Definition (32). At the lowest order in the expansion parameter, which is the order ε^{-1}, the resonant dynamic tide corresponds to the tidally excited oscillation mode of the star. Its components with respect to the local coordinate basis are given by Expressions (57), in which only non-negative values of the multiple k of the companion's mean motion must be considered.

Next, we have determined the effect of the resonance of the dynamic tide on the secular apsidal motion. The secular apsidal motion, per revolution of the companion, is given by Expression (70). Since the effect is of order ε^{-1}, a resonant dynamic tide can contribute to a much larger apsidal motion than static and non-resonant dynamic tides can do. Apart from the factor $(R_1/a)^{\ell+3}\ M_2/M_1$, Expression (70) consists of the product of the factor $H_{\ell,N}$ depending on the free oscillation mode involved in the resonance and the factor $G_{\ell,m,k}$ depending on the orbital eccentricity. It can be positive or negative. Accordingly, a resonant dynamic tide contributes to an apsidal motion rate either in the sense of the orbital motion or in the opposite sense.

From our numerical applications to polytropic models, it follows that especially resonances of dynamic tides with an f-mode or a lower-order g^+-mode may bring about larger apsidal motions.

In addition, we paid attention to the determination of the contributions to the secular apsidal motion stemming from the static tides and the non-resonant dynamic tides. The contribution stemming from a static tide is given by Expression (88). Since it is positive, the contribution always leads to a periastron advance. Besides, the contribution stemming from a non-resonant dynamic tide is given by Expression (107) and can be positive or negative. Consequently, the total secular apsidal motion is determined by means of Expression (109), in which only positive values of k must be considered.

Finally, we reconsidered the standard theory for the effect of the tidal distortion on the apsidal motion in close binary stars, which is due to Sterne. The assumption underlying the standard theory is that the mean motion n of the companion and the star's angular velocity Ω are so small that the relative position of the companion with respect to the rotating star may be considered to be frozen. We showed that Expression (119) for the secular apsidal motion, which is established by means of the standard theory, corresponds to Expression (122) derived in the frame of the theory of dynamic tides in the asymptotic approximation applying to non-resonant, low-frequency dynamic tides. We also showed that Expression (119) is valid for any orbital eccentricity. Hence, in contrast with a widespread opinion, the standard theory for secular apsidal motion due to the tidal distortion of a star in a close binary system does not simply rest on considerations of equilibrium tides.

Appendix

The following Taylor series of the constants $G_{2,m,k}$ exact to order e^2 hold:

$$\left.\begin{aligned} G_{2,0,0} &= \frac{3}{4}\left(1+\frac{7}{2}\,e^2\right), \\ G_{2,0,1} = G_{2,0,-1} &= \frac{9}{16}\left(1+4\,e^2\right), \\ G_{2,0,2} = G_{2,0,-2} &= \frac{81}{32}\,e^2, \end{aligned}\right\} \quad (126)$$

$$\left.\begin{aligned} G_{2,2,-1} = G_{2,-2,1} &= \frac{3}{32}\left(1-e^2\right), \\ G_{2,2,-2} = G_{2,-2,2} &= -\frac{3}{8}\left(5-\frac{73}{4}\,e^2\right), \\ G_{2,2,-3} = G_{2,-2,3} &= \frac{3}{32}\left(49-455\,e^2\right), \\ G_{2,2-4} = G_{2,-2,4} &= \frac{867}{16}\,e^2. \end{aligned}\right\} \quad (127)$$

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