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# Some new results on the central overlap problem in astrometry

**M. Rapaport**

Observatoire de Bordeaux, URA 352, CNRS/INSU, BP 89, F-33270 Floirac, France

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**Abstract.** The central overlap problem in astrometry has been revisited in the recent last years by Eichhorn (1988) who explicitly inverted the matrix of a constrained least squares problem. In this paper, the general explicit solution of the unconstrained central overlap problem is given. We also give the explicit solution for an other set of constraints; this result is a confirmation of a conjecture expressed by Eichhorn (1988). We also consider the use of iterative methods to solve the central overlap problem. A surprising result is obtained when the classical Gauss Seidel method is used; the iterations converge immediately to the general solution of the equations; we explain this property writing the central overlap problem in a new set of variables.

**Key words:** astrometry

## 1. Introduction

In a pioneer paper, Eichhorn (1988) revisited the central overlap problem, a classical problem of reduction of astrometric data first studied by Eichhorn & Jefferys (1971). This problem is concerned with the simultaneous estimation of the frame and stars parameters associated with the observations of the same stars on several plates (Schmidt plates, CCD images), taken at different epochs. Eichhorn (1988) emphasized that the normal equations of the central overlap problem are singular; in order to obtain a unique solution, he imposed a constraint to the unknown parameters; thus he had to solve a ”constrained least squares problem”. The solution of this kind of problem is now well known since the works of Rao (1973), Eichhorn & Russel (1976), Jefferys (1979), Lawson & Hanson (1974). Selecting a particular constraint ( the Eqs. (6) of Eichhorn (1988)), Eichhorn succeeded to invert explicitly the matrix of the constrained normal equations, and he wrote in closed form the solution of the normal equations together with the covariance matrix for the estimates of the stars and frame parameters. He then demonstrated some interesting properties of the solution, for example the non correlation between the stars and stellar parameters.

This kind of problem is often present in the reduction of astrometric data. For example Murray & Corben (1979), Murray et al. (1986) determined proper motions and parallaxes from Schmidt plates of the same zone of the sky applying these methods. More recently we used similar techniques for the determination of parallaxes of faint stars using Schmidt and/or CCD frames (Hawkins et al. (1998), Ducourant et al. (1998)).

In Sect. 2 of this paper, the explicit general solution of the (singular) equations of the central overlap problem is given. We then show that the results obtained by Eichhorn (1988) can be deduced from the general solution obtained in this paper.

In Sect. 3, it is assumed that the parameters of r stars (position, proper motion and parallax) are known; this is equivalent to enforce a particular constraint on the stars parameters; the (unique) solution of the normal equations of the central overlap problem subject to this constraint is then determined in explicit closed form; we also write explicitly the covariance matrix of the stars parameters. This result is a confirmation of a conjecture of Eichhorn (1988) asserting that other sets of constraints exist, permitting to obtain closed formulae for the solution of the normal equations.

In Sect. 4, the iterative approach of the resolution of the central overlap problem is considered. The Gauss Seidel method is used to solve the central overlap problem, and we prove that it has a very special behaviour: the algorithm converges after the first iteration to the general solution of the equations which is therefore obtained by a rather unusual way.

In Sect. 5, a change of variables is proposed for the central overlap problem; it is shown that with these variables, the equations are written in a very simple form and the properties of the solutions are very easily obtained. Finally concluding remarks are made in Sect. 6, concerning mainly the problem of missing observations.

## 2. The equations of the central overlap problem

### 2.1. The general solution

Most of the notations used in this paper are identical to those introduced by Eichhorn (1988). We consider n frames and the same m stars are present on each frame. The standard coordi-

*Send offprint requests to*: M. Rapaport nates $\zeta_{\nu,\mu}$ and $\eta_{\nu,\mu}$ of the $\mu$th star on the $\nu$th frame are classically written

$$\zeta_{\nu,\mu} = \zeta_\mu(t_\nu) = \sum_{\lambda=1}^{l_1} \gamma_{\nu,\lambda} X_{\lambda,\mu} \tag{1}$$

and a similar equation in $\eta$. The coefficients $\gamma_{\nu,\lambda}$ are known functions of the epoch $t_\nu$ of the $\nu$th frame. The generality of the results of this work is not restricted if we consider, as Eichhorn (1988) writes, the "most common and in almost situations sufficient formulation"

$$\zeta_{\nu,\mu} = \zeta_{0,\nu} + t_\nu \mu_\mu + P(t_\nu)\varpi_\mu \tag{2}$$

and an analog equation in $\eta_{\nu,\mu}$ , where $\zeta_{0,\mu}$, $\mu_\mu$, $\varpi_\mu$ are the $l_1 = 3$ usual stars parameters (position, proper motion, parallax). $P(t_\nu)$ or $P_\nu$ is the parallax factor in the $\zeta$ coordinate.

The measured coordinates of the $\mu$th star on the $\nu$th frame are $x_{\nu,\mu}$ and $y_{\nu,\mu}$.The relation between ($x_{\nu,\mu}$, $y_{\nu,\mu}$) and ($\zeta_{\nu,\mu}$, $\eta_{\nu,\mu}$) is established with the introduction of the frame parameters associated to the $\nu$th frame. We again do not restrict the generality of the results of this work if (in order to simplify the notations) the 2.k = 2.3 constants model

$$\begin{aligned} \zeta_{\nu,\mu} = \; & \zeta_{0,\mu} + t_\nu \mu_\mu + P_\nu \varpi_\mu = \\ & x_{\nu,\mu} + A_\nu x_{\nu,\mu} + B_\nu y_{\nu,\mu} + C_\nu \\ \eta_{\nu,\mu} = \; & \eta_{0,\mu} + t_\nu \mu'_\mu + P'_\nu \varpi_\mu = \\ & y_{\nu,\mu} + A'_\nu x_{\nu,\mu} + B'_\nu y_{\nu,\mu} + C'_\nu \end{aligned} \tag{3}$$

is used.

We now retain the hypothesis of Eichhorn (1988,1997) who asserts that the plate parameters can be considered as small quantities; then $x_{\nu,\mu}$ and $y_{\nu,\mu}$ can be replaced in equation (3) by the approximate values $x_\mu$ and $y_\mu$ when they appear multipled by $A_\nu$, $B_\nu$, $A'_\nu$, $B'_{\nu,}$. Following again Eichhorn (1997), the unknowns $\zeta_{0,\mu}$, and $\eta_{0,\mu}$ can be replaced by $\Delta\zeta_{0,\mu}$ and $\Delta\eta_{0,\mu}$, by putting

$$\begin{aligned} \zeta_{0,\mu} &= x_\mu + \Delta\zeta_{0,\mu} \\ \eta_{0,\mu} &= y_\mu + \Delta\eta_{0,\mu} \end{aligned} \tag{4}$$

while $x_{\nu,\mu}$ and $y_{\nu,\mu}$ are replaced by $x_{\nu,\mu} - x_\mu$ and $y_{\nu,\mu} - y_\mu$ which will still be denoted $x_{\nu,\mu}$ and $y_{\nu,\mu}$. All the unknown parameters are now small quantities and the equations associated to the measurements of the $\mu$th star on the $\nu$th frame are written

$$\begin{aligned} \Delta\zeta_{0,\mu} + t_\nu \mu_\mu + P_\nu \varpi_\mu &= x_{\nu,\mu} + A_\nu x_\mu + B_\nu y_\mu + C_\nu \\ \Delta\eta_{0,\mu} + t_\nu \mu'_\mu + P'_\nu \varpi_\mu &= y_{\nu,\mu} + A'_\nu x_\mu + B'_\nu y_\mu + C'_\nu \end{aligned} \tag{5}$$

As in Eichhorn (1988), Eqs. (5) are taken in the order of increasing frame, and within each frame number, in the order of increasing star number.The equations in $\zeta$ only will be written in the remainder of this paper.

The vector $\mathbf{p}$ of the unknown parameters is made up of
- the frame parameters $\mathbf{D_1}, \mathbf{D_2}, \ldots, \mathbf{D_n}$
where $\mathbf{D_i}^t = (A_i, B_i, C_i)$ is the 3-vector of the frame parameters $A_i, B_i, C_i$ of the ith frame
- the star parameters $\boldsymbol{\Delta\zeta_0}$, with $\boldsymbol{\Delta\zeta_0}^t = (\Delta\zeta_{0,1}, \ldots, \Delta\zeta_{0,m})$
$\mu$ with $\mu^t = (\mu_1, \ldots, \mu_m)$
$\varpi$ with $\varpi^t = (\varpi_1, \ldots, \varpi_m)$

$\Delta\zeta_{0,j}, \mu_j, \varpi_j$ are respectively the correction of the position of the jth star for the reference epoch, its proper motion, and parallax. As in Eichhorn (1988), there is no distinction between target and reference stars, so that the stellar parameters of all stars are considered as unknowns.

The measurements are written as a vector $\mathbf{l}$ with
$\mathbf{l^t} = (\mathbf{l_1}^\mathbf{t}, \mathbf{l_2}^\mathbf{t}, \ldots, \mathbf{l_n}^\mathbf{t})$
$\mathbf{l_i^t} = (x_{i,1}, x_{i,2}, \ldots, x_{i,m})$
$\mathbf{l_i}$ is the vector of the measurements of the m stars on the $ith$ frame. The overdetermined system of equations of the central overlap problem is now written

$$X\,(\mathbf{p}) = -\mathbf{l}$$

where

$$X = \begin{pmatrix} M & & & -I & -t_1 I & -P_1 I \\ & M & & -I & -t_2 I & -P_2 I \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & M\; -I & -t_n I & -P_n I \end{pmatrix}$$

and

$$M = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 1 \end{pmatrix} \tag{6}$$

M is of maximal rank, but it is well known (Eichhorn 1988) that X is singular: its rank deficiency is equal to $k.l_1$, and so an infinity of solutions exist to the system of the normal equations
$X^t X(\mathbf{p}) = -\mathbf{X^t}(\mathbf{l})$

The normal (singular) equations are developed in the form
For i=1,. . .,n

$$M^t M(\mathbf{D_i}) - M^t(\boldsymbol{\Delta\zeta_0}) - t_i M^t(\mu) - P_i M^t(\varpi) = -M^t(\mathbf{l_i})$$

$$\begin{aligned} & -M(\mathbf{D_1} + \ldots + \mathbf{D_n}) + n\boldsymbol{\Delta\zeta_0} + (\textstyle\sum_i t_i)\mu \\ & +(\textstyle\sum_i P_i)\varpi = \textstyle\sum_i \mathbf{l_i} \\ & -M(t_1\mathbf{D_1} + \ldots + t_n\mathbf{D_n}) + (\sum t_i)\boldsymbol{\Delta\zeta_0} + (\sum t_i^2)\mu \\ & +(\textstyle\sum t_i P_i)\varpi = \sum_i t_i \mathbf{l_i} \\ & -M(P_1\mathbf{D_1} + \ldots + P_n\mathbf{D_n}) + (\sum P_i)\boldsymbol{\Delta\zeta_0} + (\sum P_i t_i)(\mu) \\ & +(\textstyle\sum P_i^2)(\varpi) = \sum_i P_i \mathbf{l_i} \end{aligned} \tag{7}$$

The notations

$$M^+ = \left(M^t M\right)^{-1} M^t \text{ and } \Gamma = \begin{pmatrix} 1 & t_1 & P_1 \\ \vdots & \vdots & \vdots \\ 1 & t_n & P_n \end{pmatrix} \tag{8}$$

are introduced in the following calculations.

The general solution of (7) can be obtained expressing $\mathbf{D_1}$ from the first set of equations, and putting the result in the three last equations. The calculations present no difficulties; the classical properties of the matrix $(Id - MM)^+$ are also used and the expression of the general solution is easily obtained after some short calculations:

$$\begin{pmatrix} \mathbf{\Delta}\zeta_{\mathbf{0}} \\ \mu \\ \varpi \end{pmatrix} = (\Gamma^t\Gamma)^{-1} \begin{pmatrix} (I - MM^+)\sum_i \mathbf{l_i} \\ (I - MM^+)\sum_i t_i\mathbf{l_i} \\ (I - MM^+)\sum_i P_i\mathbf{l_i} \end{pmatrix} + \begin{pmatrix} M(\lambda) \\ M(\lambda') \\ M(\lambda'') \end{pmatrix} \tag{9}$$

$$\mathbf{D_i} = -M^+(\mathbf{l}_i) + \lambda + t_i\lambda' + P_i\lambda''$$

The solution (9) for the stellar parameters is written as the direct sum of two components, the first one contained in the range of $(Id - MM^+)$, the second component corresponding to the arbitrary part of the solution and being an element of the kernel of $(Id - MM^+)$; in this expression, $\lambda, \lambda', \lambda''$ are arbitrary 3-vectors. It is easy to prove that the solution for the stellar parameters can be expressed in the more convenient form:

$$\begin{pmatrix} \mathbf{\Delta}\zeta_{\mathbf{0}} \\ \mu \\ \varpi \end{pmatrix} = \left(\Gamma^t\Gamma\right)^{-1}\Gamma^t \begin{pmatrix} \mathbf{l_1} \\ \mathbf{l_2} \\ \vdots \\ \mathbf{l_n} \end{pmatrix} + \begin{pmatrix} M(\lambda_1) \\ M(\lambda_2) \\ M(\lambda_3) \end{pmatrix} \tag{10}$$

So it has been proved in this section that the general solution of the central overlap problem is obtained by means of explicit closed formulae as (9) or (10). It can be noticed that the solution (10) of the central overlap problem is not as in (9) the direct sum of two orthogonal components.

### 2.2. Comparison with Eichhorn's results

Eichhorn (1988) solves the central overlap problem, submitting the stellar (or frame) parameters to a particular constraint, and then solving a constrained least squares problem. The constraint used in his quoted paper is

$$\begin{aligned} \mathbf{D}_1 + \ldots + \mathbf{D}_n &= \mathbf{K_1} \\ t_1\mathbf{D_1} + \ldots + t_n\mathbf{D}_n &= \mathbf{K_2} \\ P_1\mathbf{D_1} + \ldots + P_n\mathbf{D}_n &= \mathbf{K_3} \end{aligned} \tag{11}$$

where $\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3}$ are arbitrary 3-vectors. It is immediately seen on Eqs. (11) and (7) that the solution of the constrained least squares problem of Eichhorn is given by (9) or (10), where $\lambda, \lambda', \lambda''$ or $\lambda_1, \lambda_2, \lambda_3$ are replaced by the constant arbitrary quantities $\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3}$. So it is interesting to notice that the solution obtained by Eichhorn (1988) in a particular case becomes the general solution if the arbitrary constants $\mathbf{K_i}$ of his solution are considered as arbitrary random quantities depending on the observational data.

## 3. Another constraint leading to an explicit solution of the central overlap problem

Eichhorn (1988) conjectures that (11) is not the only set of constraints which allows to find an explicit solution for the central overlap problem. It is the aim of this section to confirm this conjecture. The constraint considered now consists in the assumed knowledge of the parameters of r stars (with $r \geq k$). These stars can always be associated with the indices $\mu = 1, 2, \ldots, r$.

The vector $\mathbf{p}$ of the unknowns is made of
- the frame parameters $\mathbf{D_1}, \ldots, \mathbf{D_n}$ ,with $\mathbf{D_i}^t = (A_i, B_i, C_i)$
- the stars parameters $\mathbf{\Delta}\zeta_0$,$\mu$, $\varpi$ of the m-r target stars.

We will now introduce some notations in order to simplify the next equations of this section. Let us introduce

$$J_m = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \tag{12}$$

M has been partitioned in two submatrices: $M_1$ associated with the measured coordinates of the r known stars, and $M_2$ associated with the m-r target stars. The vector $\mathbf{l_i}$ of the measurements on the ith frame is also partitioned in

$$\mathbf{l_i} = \begin{pmatrix} \mathbf{l_i^{(I)}} \\ \\ \mathbf{l_i^{(II)}} \end{pmatrix} \tag{13}$$

and we notice that $\mathbf{l_i^{(II)}} = \mathbf{J_m^t}(\mathbf{l_i})$
We also introduce the quantities

$$\mathbf{m} = \sum_1^n \mathbf{l}_i \quad \mathbf{m}_\mu = \sum_1^n t_i\mathbf{l}_i \quad \mathbf{m}_\varpi = \sum_1^n P_i\mathbf{l}_i \tag{14}$$

and their partition:

$$\mathbf{m} = \begin{pmatrix} \mathbf{m^{(I)}} \\ \mathbf{m^{(II)}} \end{pmatrix} \text{ resp } (\,\mathbf{m}_\mu, \mathbf{m}_\varpi) \tag{15}$$

The system of equations associated with the central overlap problem where the first r stars are reference stars can be written

$X(\mathbf{p}) = -\mathbf{l}$ with

$$X = \begin{pmatrix} M & & & & -J_m & -t_1J_m & -P_1J_m \\ & M & & & \cdot & \cdot & \cdot \\ & & M & & \cdot & \cdot & \cdot \\ & & & M & -J_m & -t_nJ_m & -P_nJ_m \end{pmatrix} \tag{16}$$

The explicit solution of the normal equations associated to (16) is easily obtained after some calculations. The two following classical matricial identities can be used:

$$\begin{aligned} &\left(I - M_2(M^tM)^{-1}M_2^t\right)^{-1} = I + M_2(M_1^tM_1)^{-1}M_2^t \\ &\left(I + M_2(M_1^tM_1)^{-1}M_2^t\right)M_2(M^tM)^{-1} = M_2\left(M_1^tM_1\right)^{-1} \end{aligned} \tag{17}$$

The solution for the estimates of the stars parameters, and their covariance matrix V is obtained after some easy calculations. This solution is written:

$$\left(\Gamma^t\Gamma\right)\begin{pmatrix} \mathbf{\Delta}\zeta_0 \\ \mu \\ \varpi \end{pmatrix} = \begin{pmatrix} \mathbf{m}^{(II)} - M_2(M_1^tM_1)^{-1}M_1^t(\mathbf{m^{(I)}}) \\ \mathbf{m}_\mu^{(II)} - M_2(M_1^tM_1)^{-1}M_1^t(\mathbf{m}_\mu^{\mathbf{(I)}}) \\ \mathbf{m}_\varpi^{(II)} - M_2(M_1^tM_1)^{-1}M_1^t(\mathbf{m}_\varpi^{\mathbf{(I)}}) \end{pmatrix} \tag{18}$$

and

$$V = \left(\Gamma^t\Gamma\right)^{-1} \otimes \left(I_{m-r} + M_2(M_1^t M_1)^{-1} M_2^t\right) \tag{19}$$

where $\otimes$ represents the Kronecker notation used by Eichhorn (1988). So the Eqs. (18) and (19) correspond to the explicit solution of the central overlap problem, where the constraints consist in the knowledge of the stars parameters of $r$ reference stars, with $r \geq k$. This result is a confirmation of the conjecture by Eichhorn (1988) which asserts that there exists several constraints which allow to write explicitly the solution of the equations of the central overlap problem.

## 4. The iterative solution of the central overlap problem

Eichhorn (1988) emphasizes that most effective resolutions of the central overlap problem were achieved by the use of iterative methods. We will now establish a rather unexpected result concerning the application of the Gauss Seidel iterative procedure to the resolution of the central overlap problem.

The matrix X of the central overlap problem is first partitioned in the form

$$X = (A, B)$$

with

$$A = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix} \quad - B = \begin{pmatrix} I & t_1 I & P_1 I \\ \vdots & \vdots & \vdots \\ I & t_n I & P_n I \end{pmatrix} \tag{20}$$

We now consider the splitting of the matrix $X^t X$ of the normal equations

$$X^t X = M_1 - N_1$$

with

$$M_1 = \begin{pmatrix} A^t A & 0 \\ B^t A & B^t B \end{pmatrix} \; N_1 = \begin{pmatrix} 0 & -A^t B \\ 0 & 0 \end{pmatrix} \tag{21}$$

The Gauss Seidel algorithm for solving the normal equations consists in the iterative solution of

$$\mathbf{p}_{n+1} = M_1^{-1} N_1(\mathbf{p}_n) + M_1^{-1} X^t(-\mathbf{l})$$

Two classical results concerning the convergence of the iterations will be used in the following:
1) If $X^t X$ is definite positive, for every $\mathbf{p_0}$, the sequence $\mathbf{p_n}$ converges to the unique solution of the normal equations.
2 If $X^t X$ is positive semi definite (this is the case for the central overlap problem), the sequence $\mathbf{p_n}$ converges to $\mathbf{a}$ solution of the normal equations. After some easy calculations, we obtain the following results:
a) $M_1^{-1} N_1$ is idempotent.
b) $G M_1^{-1} X^t = 0$ (with $G = M_1^{-1} N_1$)
From a) and b) it is immediate to see that the sequence of the iterations of the Gauss Seidel algorithm is reduced to

$$\mathbf{p}_1 = G(\mathbf{p}_0) - M_1^{-1} X^t(\mathbf{l}) \tag{22}$$

(22) is the general solution of the singular normal equations of the central overlap problem. It can be verified that (22) is equivalent to (9) ou (10). The arbitrary part of the solution is given by $G(\mathbf{p_0})$, where $\mathbf{p_0}$ is an arbitrary vector.

Thus in the previous sections, it was proved that the equations of the central overlap problem could be easily solved in explicit form for the more general case ; it was also noticed that the iterative Gauss Seidel algorithm applied to this problem has a very special behaviour. We will now propose an explanation of these results, introducing a change of variables leading to a very simple form of the equations of the central overlap problem.

## 5. The central overlap problem equations in a new set of variables

The change of variables proposed in this section derives from the singular value decomposition of the matrix $M$. This decomposition is classicaly written

$$M = U.\Sigma.V$$

where V is a 3×3 orthogonal matrix, U is a m×m orthogonal matrix, and where

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \tag{23}$$

where $\sigma_i$ are the singular values of M.

The new frame and stars parameters are defined by

$$\begin{aligned} \mathbf{D_i^*} &= V\mathbf{D_i} \\ \mathbf{\Delta}\zeta_0^* &= U^t(\mathbf{\Delta}\zeta_0) \quad resp(\mu, \varpi) \end{aligned} \tag{24}$$

The combination $\mathbf{l_i^*}$ of the observational data defined by $\mathbf{l}_i^* = U^t(\mathbf{l_i})$ are also introduced. With this new set of parameters, the Eqs. (7) associated to the central overlap problem take a very simple form. Suppressing the asteriks in the new variables , the Eqs. (7) become:
For i=1,. . .,n

$$\begin{aligned} \sigma_1 A_i \; -\Delta\zeta_{0,1} \; -t_i\mu_1 \; -P_i\varpi_1 &= -x_{1,i} \\ \sigma_2 B_i \; -\Delta\zeta_{0,2} \; -t_i\mu_2 \; -P_i\varpi_2 &= -x_{2,i} \\ \sigma_3 C_i \; -\Delta\zeta_{0,3} \; -t_i\mu_3 \; -P_i\varpi_3 &= -x_{3,i} \end{aligned} \tag{25}$$

and for j= 4,. . .,m

$$\begin{aligned} \Delta\zeta_{0,j} + t_1\mu_j + P_1\varpi_j &= x_{1,j} \\ \Delta\zeta_{0,j} + t_2\mu_j + P_2\varpi_j &= x_{2,j} \\ &\vdots \\ \Delta\zeta_{0,j} + t_n\mu_j + P_n\varpi_j &= x_{n,j} \end{aligned} \tag{26}$$

The singular character of the equations can be seen on the first set of equations. The second part shows that each group of the new set of star parameters $(\Delta\zeta_{0,j}, \mu_j, \varpi_j)$, for $j \geq 4)$ verifies the same equations. It can be immediately seen that if the constraint considered in Eichhorn (1988) on frame parameters is used, the stars parameters associated with the indices j=1,2,3 verify the same equations than the stars parameters of the other stars. Most of the results of this paper could be easily easily obtained from (25) and (26). The physical meaning of the new parameters introduced above is somewhat less natural than that of the usual frame and stars parameters; nevertheless they are orthogonal combinations of them. With these new variables, the search for the general solution to the equations of the central overlap problem is reduced to a very simple problem, the resolution of the overdetermined system (26) while the arbitrary part of the solution comes from the resolution of the underdetermined system (25).

## 6. Conclusion

The various papers of Eichhorn concerning the central overlap problem suceeded in the achievement of the explicit solution of the problem, when the unknowns are submitted to a particular constraint. Eichhorn (1988) also suggested that it was possible to find other constraints which would permit to obtain an explicit solution to the problem. This conjecture is confirmed in the Sect. 3 of this paper. We furthermore obtained in Sect. 2 the general solution of the ( singular) normal equations of the central overlap problem, in explicit and closed form, and the link between the general solution of this paper and the solution of Eichhorn (1988) has been analysed.

The rather unexpected results obtained in the analysis of the iterative Gauss Seidel methods led us to write the equations of the central overlap problem in a new set of variables using orthogonal transformations on the frame and stars parameters. The equations in these new variables take a very simple form, and we suggest that most of the properties of the solutions of the central overlap problem can be easily obtained from the new equations.

In this paper and in Eichhorn's one (1988), it is supposed that each star is present on each frame, and in fact this hypothesis has an important part in the various calculations. But in practical situations, this hypothesis is not fulfilled ( faint stars, bad images..) and this was for example the case for our CCD images or Schmidt plates taken for the determination of parallaxes; we suggest here that it is again possible to obtain an explicit solution of the central overlap problem in this situation of missing observations.

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