

Modelling gravity in N -body simulations of disc galaxies. Optimal types of softening for given dynamical requirements

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Abstract. Modelling gravity is a fundamental problem that must be tackled in N -body simulations of stellar systems, and satisfactory solutions require a deep understanding of the dynamical effects of softening. In a previous paper (Romeo 1997), we have devised a method for exploring such effects, and we have focused on two applications that reveal the dynamical differences between the most representative types of softened gravity. In the present paper we show that our method can be applied in another, more fruitful, way: for developing new ideas about softening. Indeed, it opens a *direct* route to the discovery of optimal types of softened gravity for given dynamical requirements, and thus to the accomplishment of a physically consistent modelling of disc galaxies, even in the presence of a cold interstellar gaseous component and in situations that demand anisotropic resolution.

Key words: methods: analytical – methods: N-body simulations – galaxies: evolution – galaxies: kinematics and dynamics – galaxies: spiral

1. Introduction

N -body simulations of disc galaxies rely on the use of softening. This artifice removes the short-range singularity of the gravitational interaction, which is dynamically unimportant and computationally troublesome, whereas it leaves the long-range behaviour of gravity unchanged. But softening is also a critical factor in simulations. It controls their quality and can affect their result on scales much larger than the softening length. Its dynamical effects are further exacerbated in the presence of a cold interstellar gaseous component and in situations that demand anisotropic resolution. Thus softening poses a dynamical problem of special concern, which should be probed carefully and in detail (e.g., Hernquist & Barnes 1990; Pfenniger & Friedli 1993; Romeo 1994, hereafter Paper I; Romeo 1997, hereafter Paper II¹; and references therein).

In Paper I, we have investigated how faithful simulations are. In particular, we have concluded that the standard way of introducing softening in the presence of stars and cold interstellar

gas is definitely unsatisfactory in several regimes of astrophysical interest. It is so because important small-scale instabilities of the gaseous component, e.g. those peculiar to star-formation processes, are suppressed just as unphysical noise of the stellar component. Faithfulness requires an appropriate introduction of two softening lengths, one for each component, and also a rigorous specification of the star-gas gravitational interaction.

In Paper II, we have devised a method for exploring the dynamical effects of softening. As a major result, we have shown how to choose the softening length for optimizing the faithfulness of simulations to the Newtonian dynamics. Then we have focused on two applications that reveal the dynamical differences between the most representative types of softened gravity. In particular, we have concluded that it is desirable to improve the current way of introducing anisotropic softening. We need a clearer decoupling of the resolution parallel and perpendicular to the plane, and also more natural planar and vertical softening lengths.

In the present paper, which completes our planned research work about softening, we propose an *innovative* solution to the problem. The understanding of galactic and extragalactic astrophysics is at a crucial stage. Unsolved problems are viewed in new perspectives, which promise major revisions of knowledge (see, e.g., Blitz & Teuben 1996; Block & Greenberg 1996). Recent investigations suggest, for instance, a more enigmatic interplay between stellar disc and bulge/halo (e.g., Lequeux et al. 1995), a clearer relation between cold gas and dark matter in spiral galaxies (e.g., Pfenniger et al. 1994; Pfenniger & Combes 1994; Combes & Pfenniger 1997), and a closer connection between the fractal structures of the interstellar medium and of the universe (e.g., de Vega et al. 1996, 1998). The implications are clear: modelling gravity in N -body simulations of disc galaxies should offer a flexible interface with such a progress. Our solution is to optimize the fidelity of simulations to given dynamical requirements. How do we apply this idea in practice?

1. We impose the requirements in the wavenumber space since this is the natural dynamical domain of gravity, as Pfenniger & Friedli (1993) have previously emphasized.
2. We identify the softening length with the characteristic dynamical scale length.

¹ Sections and equations of that paper are denoted by the prefix II.

3. Then we invert part of the method of Paper II, and the result is the optimal type of softened gravity that satisfies those dynamical requirements.

Our application covers both 2-D and 3-D modelling. The basic cases are extended to more complex situations through recipes for implementing star-gas and anisotropic softening, which have already been motivated (cf. discussions of Papers I and II). Last but not least, each description is complemented by an example that leaves room for creativity.

The present paper is organized as follows. The application is shown in Sects. 2 and 3 (see also Appendix A), and proceeds as in the previous discussion. Comments on related works concerning softening are made in Sect. 4. The conclusions and perspectives are drawn in Sect. 5, where we present our three papers about softening in a more unified view and emphasize their potentially strong impact on galactic dynamics.

2. 2-D modelling

2.1. Inverting part of the method of Paper II

The method of Paper II allows determining the dynamical response of the model to a given type of softened gravity. The basic quantity that describes this response is the reduction factor $\mathcal{S}(|k|s)$ defined in Eq. (II-4), where k has the meaning of a radial wavenumber and s is the softening length. In a few words, this is the factor by which softening reduces the dynamical contribution of self-gravity. The behaviour of $\mathcal{S}(|k|s)$ provides the following information:

- At small $|k|s$, it shows how significantly the stability and collective relaxation properties are affected on large scales. Specifically, a comparison with the reduction factor of 3-D discs with Newtonian gravity reveals how well softening mimics the effects of thickness, as far as density waves are concerned.
- At large $|k|s$, it shows how effectively noise is suppressed on small scales.
- It also contains less direct information about how significantly the equilibrium properties are affected.

The reduction factor $\mathcal{S}(|k|s)$ is related to the point-mass potential $-Gm\varphi_s(R)$ and force $-Gm^2f_s(R)$ through integral transforms. These transforms can easily be inverted, and the inversion formulae that relate $\varphi_s(R)$ and $f_s(R)$ to $\mathcal{S}(|k|s)$ are:

$$\varphi_s(R) = \mathcal{H}_0 \left[\frac{1}{k} \mathcal{S}(|k|s) \right] (R), \quad (1)$$

$$f_s(R) = \mathcal{H}_1[\mathcal{S}(|k|s)](R), \quad (2)$$

where \mathcal{H}_ν denotes the Hankel transform of order ν :

$$\mathcal{H}_\nu[g(k)](R) = \int_0^\infty g(k) J_\nu(kR) k dk, \quad (3)$$

and J_ν denotes the Bessel function of the first kind and order ν (for mathematical and numerical references see Sect. II-2.1.1; in particular, for a swift numerical computation of Hankel

transforms use, e.g., the NAG library). Inverting this part of the method has a direct practical importance and, indeed, is the trick behind the present application: it allows imposing *arbitrary* dynamical requirements in the form of a reduction factor and identifying the optimal type of softened gravity that satisfies such requirements. In addition, the rest of the method allows testing the precise dynamical performance of the modelling.

Here is an example: we want to find a type of softened gravity that mimics the effects of thickness very well and, at the same time, suppresses noise very effectively. This is illustrated in Fig. 1a and in the following discussion. A reduction factor that conforms with the dynamical requirements specified above is:

$$\mathcal{S}(|k|s) = \frac{1}{1 + |k|s} \cdot e^{-(|k|s/\pi)^5}. \quad (4)$$

The first function is the reduction factor of 3-D discs with Newtonian gravity and characteristic scale height s (Shu 1968; Vandervoort 1970; Romeo 1992). The second function acts as a low-pass filter with rather sharp cut-off at radial wavelengths $\lambda \approx 2s$. The benefits of filtering in spectral domain are well known in the context of digital image processing (see, e.g., Jain 1989; Press et al. 1992; see also Vetterling et al. 1992). In our case the dynamical resolution, i.e. the faithfulness in simulating the Newtonian dynamics, is substantially higher than in the standard Plummer softening. Indeed, it can be further improved by choosing sharper filters, at the cost of noticeable oscillations in the gravitational interaction. To some extent, their preference is a matter of taste. On the other hand, too sharp filters make the typical radial wavelength and possibly other dynamical properties hypersensitive to the location of the cut-off, which is unphysical. Thus, they should not be used.

2.2. Implementing star-gas softening

How do we model gravity in the presence of two components, such as stars and cold interstellar gas? Let us think in the alternative way of finite-sized particles interacting with Newtonian gravity, as Dyer & Ip (1993) have partly suggested (for an ABC of finite-sized particles see Appendix A). Then the answer is simple. Each component turns out to have its own positive reduction factor $\mathcal{S}(|k|s)$, where now s is the scale length of the particle mass distribution. So the star-star and gas-gas interactions are as in the one-component case, while the star-gas interaction potential $-Gm_s m_g \varphi_{s-g}(R)$ and force $-Gm_s m_g f_{s-g}(R)$ are determined unequivocally by the inversion formulae:

$$\varphi_{s-g}(R) = \mathcal{H}_0 \left[\frac{1}{k} \sqrt{\mathcal{S}_s(|k|s_s) \cdot \mathcal{S}_g(|k|s_g)} \right] (R), \quad (5)$$

$$f_{s-g}(R) = \mathcal{H}_1 \left[\sqrt{\mathcal{S}_s(|k|s_s) \cdot \mathcal{S}_g(|k|s_g)} \right] (R). \quad (6)$$

Our recipe for implementing star-gas softening has strong advantages. Indeed, its characteristics are *fundamental* for modelling the complex roles that such components play in regimes of astrophysical interest, as we have concluded in Paper I.

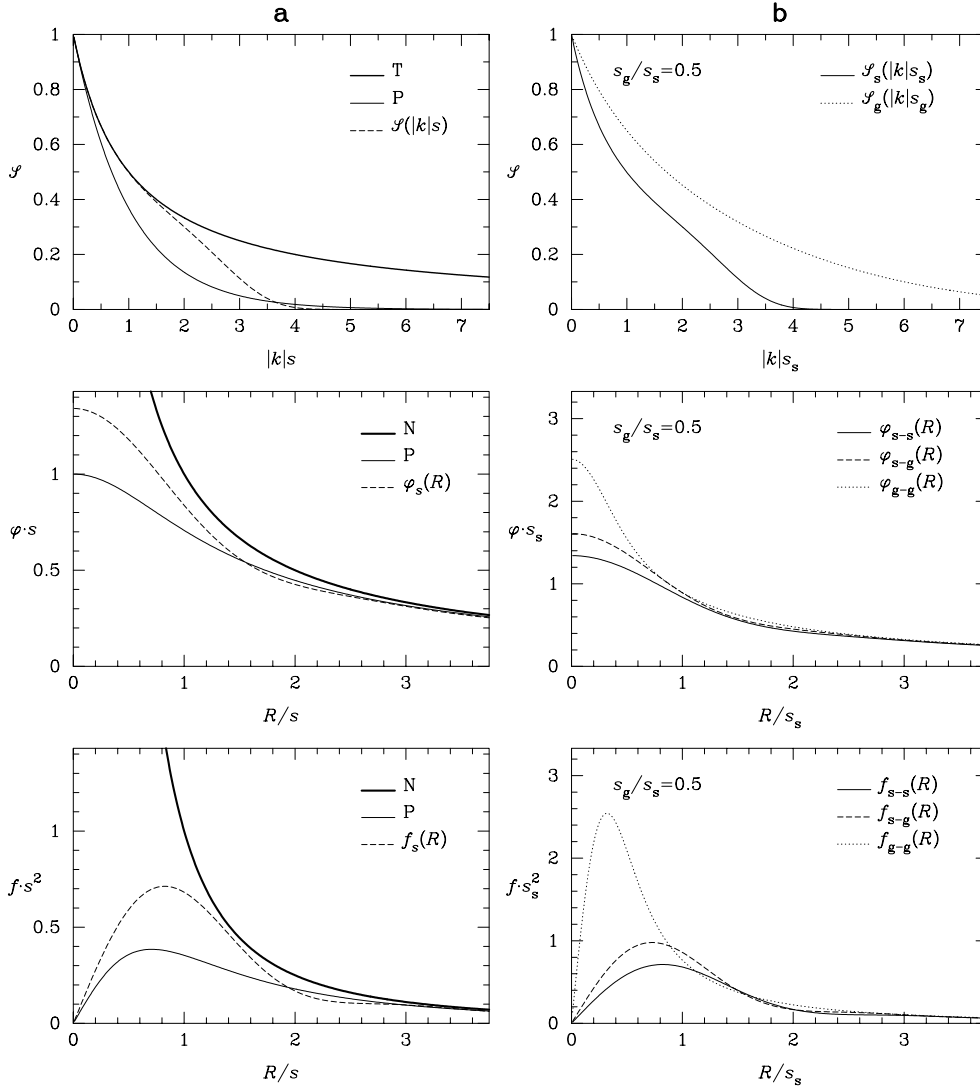


Fig. 1a and b. Examples of 2-D modelling: **a** one-component case (cf. Sect. 2.1), **b** two-component case (cf. Sect. 2.2). The abbreviations N, T and P mean Newtonian gravity, thickness and Plummer softening, respectively

As an example, we want to generalize the type of softened gravity found in the one-component case to the presence of a young disc stellar population and a cold interstellar gaseous component with, say, $s_s : s_g = 2 : 1$ (see, e.g., Mihalas & Binney 1981). This is illustrated in Fig. 1b and in the following discussion. The finite-sized particle implementation of star-gas softening is consistent with the effects of thickness: there are two positive reduction factors, one for each component. Again, the stellar reduction factor is:

$$\mathcal{S}_s(|k|s_s) = \frac{1}{1 + |k|s_s} \cdot e^{-(|k|s_s/\pi)^5}. \quad (7)$$

Concerning cold interstellar gas, the situation is more complex and uncertain (e.g., Combes & Pfenniger 1996; Elmegreen 1996a, b; Ferrara 1996; Lequeux & Guélin 1996; Pfenniger 1996; Pfenniger et al. 1996). Sharp filters should not be used because they over-stress adherence to the effects of thickness, whereas the effects of turbulence and fractality may be more important. Soft filters are safer in that respect, and our preference

goes to the Gaussian member of the family². So the gaseous reduction factor is:

$$\mathcal{S}_g(|k|s_g) = \frac{1}{1 + |k|s_g} \cdot e^{-(|k|s_g/\pi)^2}. \quad (8)$$

Regarding s_s as the characteristic scale height of reference, different values of s_g/s_s have no influence on the stellar functions, modify the star-gas gravitational interaction moderately and change the gaseous functions according to simple scaling laws.

3. 3-D modelling

3.1. Passing from 2D to 3D

In 3-D disc models, $\mathcal{S}(|k|s)$ is no longer the true reduction factor but is still useful for quantifying the dynamical effects of softening parallel to the plane, which now combine with those of vertical random motion. Specifically, a comparison with the

² A simpler, but less instructive, choice would be: $\mathcal{S}_g(|k|s_g) = e^{-|k|s_g}$, i.e. the reduction factor for the standard Plummer softening.

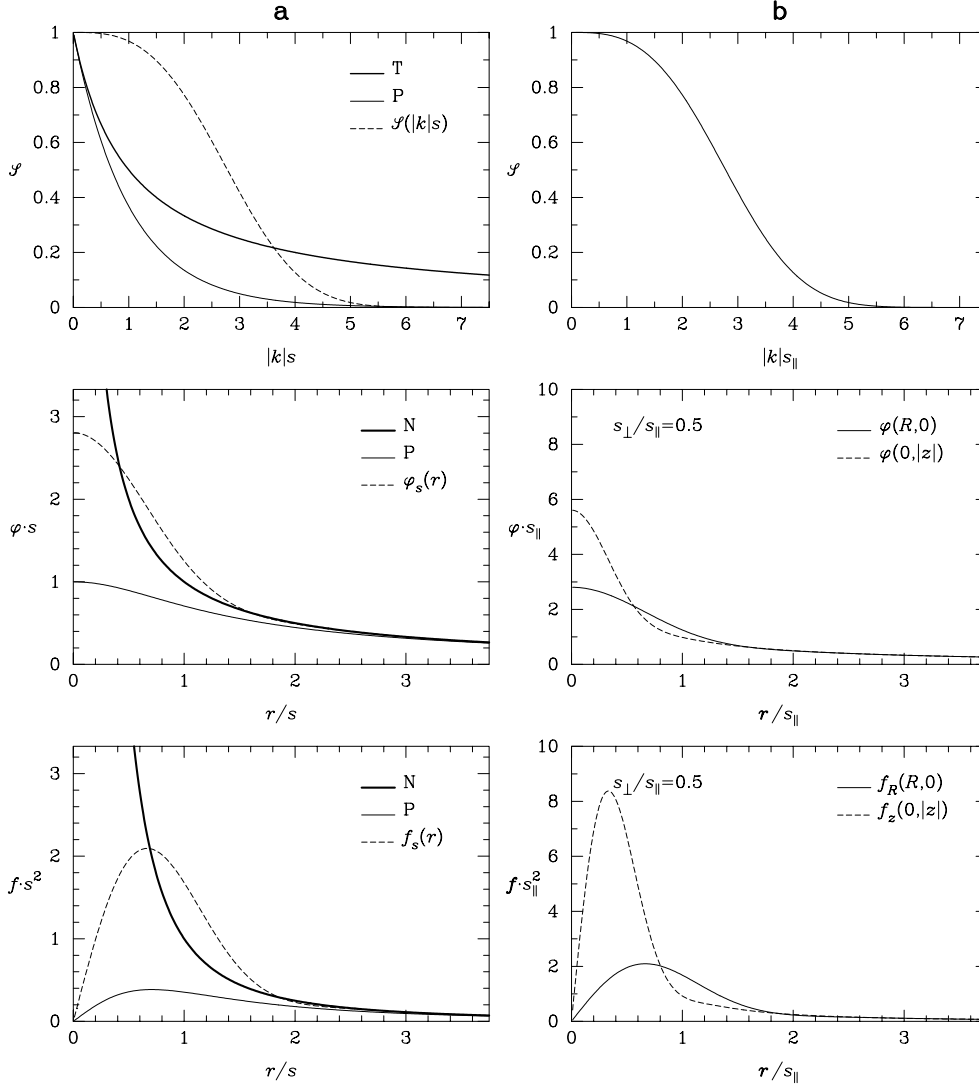


Fig. 2a and b. Examples of 3-D modelling: **a** isotropic case (cf. Sect. 3.1), **b** anisotropic case (cf. Sect. 3.2). The abbreviations N, T and P mean Newtonian gravity, thickness and Plummer softening, respectively

reduction factor of 3-D discs with Newtonian gravity suggests how much softening interferes with the effects of thickness, as far as density waves are concerned. The inversion formulae for $\varphi_s(r)$ and $f_s(r)$ are as in the 2-D case. This is the simplest way of passing from 2-D to 3-D modelling. (The generalization to two components is clear.)

Here is an example: we want to find a type of softened gravity that interferes with the effects of thickness very little and, at the same time, suppresses noise very effectively. This is illustrated in Fig. 2a. The pseudo reduction factor is:

$$\mathcal{S}(|k|s) = e^{-(|k|s/\pi)^3}, \quad (9)$$

and acts as a low-pass filter with rather soft cut-off at $\lambda \sim 2s$. The discussion concerning the dynamical resolution and the action of sharper filters follows the 2-D case closely. On the other hand, only in 3D can we simulate the evolutionary nature of thickness, which arises from the vertical random motion and its subtle coupling with the dynamical properties parallel to the plane (Romeo 1990, 1992; see also Paper I).

3.2. Implementing anisotropic softening

In order to model gravity in situations that demand anisotropic resolution, it is convenient to think in the standard way of point particles interacting with softened gravity, as Zotov & Morozov (1987) have partly suggested. The softening surface is transformed from a sphere of radius s into a spheroid of planar and vertical semi-axes s_{\parallel} and s_{\perp} , respectively. This means that the softening length is the distance from the centre to the surface of the spheroid in the direction of the position vector:

$$s(R, |z|) = \sqrt{\frac{s_{\parallel}^2 R^2 + s_{\perp}^2 z^2}{R^2 + z^2}}. \quad (10)$$

The resolution turns out to be decoupled parallel and perpendicular to the plane, and to be determined by the natural planar and vertical softening lengths. These characteristics of our recipe for implementing anisotropic softening have important advantages, as we have concluded in Paper II. (The generalization to two components is clear.)

As an example, we want to generalize the type of softened gravity found in the isotropic case to situations that demand

moderate anisotropic resolution with, say, $s_{\parallel} : s_{\perp} = 2 : 1$. This is illustrated in Fig. 2b. Regarding s_{\parallel} as the softening length of reference, different values of s_{\perp}/s_{\parallel} have no influence on the planar functions and change the gravitational interaction along the vertical direction according to simple scaling laws.

4. Discussion

Three recent papers concerning softening optimization and conception deserve comment:

- Merritt's (1996) optimization is performed with respect to the Newtonian dynamics in the configuration space, and concerns the softening length. The configuration space does not permit a clear distinction between large-scale dynamical properties and small-scale noise, and also emphasizes the equilibrium state as most representative of the whole dynamics.
- Weinberg's (1996) optimization is comparable to that of Merritt (1996) but concerns orthogonal series force computation, i.e. roughly speaking the softening length and the type of softened gravity.
- Dyer & Ip's (1993) conception is rigid: softened gravity must mimic finite-sized particles. But why? Softening is an artifice: its physical consistency should be scrutinized with respect to basic dynamical requirements, not with respect to the inter-particle force alone. An elastic conception is more useful. Apart from that, Dyer & Ip (1993) have suggested a softening optimization that is not so different from that of Merritt (1996).

5. Conclusions and perspectives

The importance of computer simulations in astrophysics is analogous to that of experiments in other branches of physics. They also serve as a welcome bridge between theories, often restricted to idealized situations, and observations, revealing instead the complexity of nature. Major present objectives are to construct physically consistent N -body models of disc galaxies and to simulate their dynamical evolution, especially in regimes of spiral structure in which a fruitful comparison between theories and simulations can be made (e.g., Pfenniger & Friedli 1991; Junqueira & Combes 1996; Zhang 1996; Bottema & Gerritsen 1997; Fuchs & von Linden 1998; von Linden et al. 1998; Zhang 1998a, b). The construction of such models is indeed a difficult task which has not yet been fully accomplished, and which should eventually provide clues of vital importance to a number of open questions posed by both theories and observations.

Our involvement has been threefold. In Paper I, we have recognized a fundamental problem posed by this research programme (for a concrete use of that analysis and for interesting remarks see, e.g., Junqueira & Combes 1996). In Paper II, we have devised a method for solving this problem. In the present paper, we apply this method and solve the problem, thus laying the foundations of such a plan. The *major result* is that gravity can be modelled so as to optimize the fidelity of simulations, and

the procedure is practicable. The following conclusions point up the whys and wherefores:

1. Optimization is performed with respect to arbitrary dynamical requirements and, in specific examples, with respect to the Newtonian dynamics. This enriches the modelling with an *unprecedented* degree of freedom, which has clear epistemological motivations (cf. Sect. 1, discussion of the present paper).
2. Optimization is performed in the wavenumber space. This is the *appropriate* domain for imposing dynamical requirements on the modelling.
3. Optimization concerns *both* the softening length *and* the type of softened gravity.
4. Softening is conceived as a *double* artifice. The softened gravity and finite-sized particle conceptions are equivalent in the basic cases. Concerning more complex situations, the latter is particularly useful for implementing star-gas softening, whereas the former is particularly useful for implementing anisotropic softening. Thus both conceptions contribute towards the accomplishment of a physically consistent modelling.

Our application is ready for a concrete use. An attractive idea is to employ a particle-particle code together with MD-GRAPE, a highly parallelized special-purpose computer for many-body simulations with an arbitrary central force (Fukushige et al. 1996). We can also employ a classical particle-mesh code. Then the dynamical effects of the grid are known and factorize as those of softening (e.g., Bouchet et al. 1985; Efstathiou et al. 1985; for a review see, e.g., Hockney & Eastwood 1988). So essentially the application proceeds as in the present paper, but it may be useful to act directly on the wavenumber space (e.g., Tormen & Bertschinger 1996). A more complex problem concerns tree codes, which have hierarchical structure and adaptive resolution over multiple scales (e.g., Hernquist 1987; for a review see, e.g., Pfalzner & Gibbon 1996). The solution to that problem would need a more advanced analysis (cf. following discussion). Welcome suggestions about the choice of the code can come from cosmological simulations (e.g., Splinter et al. 1998).

Finally, what about the future? Our approach is connected with the technique of filtering in spectral domain used in the context of digital image processing. This is a rapidly evolving field with growing applications in science and engineering, which can promote further substantial advances in N -body modelling of disc galaxies. For instance, wavelets are ideal for resolving multi-scale problems in space and/or time, such as those concerning turbulence, bifurcations, fractals and many others (see, e.g., Kaiser 1994; Holschneider 1995; Bowman & Newell 1998; for an alternative analysis tool see, e.g., Stutzki et al. 1998). Speculating further, wavelets might be used for speeding up simulations through fast solution of linear systems (cf. Press et al. 1992, pp. 597–599 and 782).

These are the merits of our contribution. We hope that the trilogy (Papers I–III) and further reflections (Romeo 1998) will encourage N -body experimenters to model gravity so as to optimize the fidelity of their simulations, and that the result will

be a stronger interdisciplinary connection with theories and observations.

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Appendix A: ABC of finite-sized particles

Finite-sized particles interacting with Newtonian gravity are analogous to point particles interacting with softened gravity. The dynamics of one-component 2-D discs containing such particles can be investigated by performing an analysis comparable to that of Sects. II-2.1 and 2.1. In this appendix we report the formulae useful for Sect. 2.2. Let $m\mu_s(R)$ be the particle mass distribution of scale length s . The reduction factor is:

$$\mathcal{S}(|k|s) = \{2\pi \mathcal{H}_0[\mu_s(R)](k)\}^2. \quad (\text{A1})$$

The inversion formula for $\mu_s(R)$ is:

$$\mu_s(R) = \frac{1}{2\pi} \mathcal{H}_0 \left[\sqrt{\mathcal{S}(|k|s)} \right] (R). \quad (\text{A2})$$

Last and most useful, the inversion formulae for the interaction potential $-Gm^2\varphi_s(R)$ and force $-Gm^2f_s(R)$ are:

$$\varphi_s(R) = \mathcal{H}_0 \left[\frac{1}{k} \mathcal{S}(|k|s) \right] (R), \quad (\text{A3})$$

$$f_s(R) = \mathcal{H}_1[\mathcal{S}(|k|s)](R). \quad (\text{A4})$$

(For general references about finite-sized particles see, e.g., Vlasov 1961; Röhrlich 1965; Dawson 1983; Hockney & Eastwood 1988; Birdsall & Langdon 1991; in particular, the last reference contains useful insights in the context of N -body simulations of plasmas.)

References

- Birdsall C.K., Langdon A.B., 1991, *Plasma Physics via Computer Simulation*. Institute of Physics Publishing, Bristol
- Blitz L., Teuben P. (Eds.), 1996, *Proc. IAU Symp. 169, Unsolved Problems of the Milky Way*. Kluwer, Dordrecht
- Block D.L., Greenberg J.M. (Eds.), 1996, *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht
- Bottema R., Gerritsen J.P.E., 1997, *MNRAS* 290, 585
- Bouchet F.R., Adam J.-C., Pellat R., 1985, *A&A* 144, 413
- Bowman C., Newell A.C., 1998, *Rev. Mod. Phys.* 70, 289
- Combes F., Pfenniger D., 1996, *Very Cold Gas and Dark Matter*. In: Block D.L., Greenberg J.M. (eds.) *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht, p. 451
- Combes F., Pfenniger D., 1997, *A&A* 327, 453
- Dawson J.M., 1983, *Rev. Mod. Phys.* 55, 403
- de Vega H.J., Sánchez N., Combes F., 1996, *Nat* 383, 56
- de Vega H.J., Sánchez N., Combes F., 1998, *ApJ*, in press (astro-ph/9801224)
- Dyer C.C., Ip P.S.S., 1993, *ApJ* 409, 60
- Efstathiou G., Davis M., Frenk C.S., White S.D.M., 1985, *ApJS* 57, 241
- Elmegreen B.G., 1996a, *What Do We Really Know about Cloud Formation?* In: Blitz L., Teuben P. (eds.) *Proc. IAU Symp. 169, Unsolved Problems of the Milky Way*. Kluwer, Dordrecht, p. 551
- Elmegreen B.G., 1996b, *Constraints on the Surface Density of Gas in Outer Galaxy Disks from Stability Analyses*. In: Block D.L., Greenberg J.M. (eds.) *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht, p. 467
- Ferrara A., 1996, *Why Is the H I Scale Height Constant with Galactocentric Radius?* In: Blitz L., Teuben P. (eds.) *Proc. IAU Symp. 169, Unsolved Problems of the Milky Way*. Kluwer, Dordrecht, p. 479
- Fuchs B., von Linden S., 1998, *MNRAS* 294, 513
- Fukushige T., Taiji M., Makino J., Ebisuzaki T., Sugimoto D., 1996, *ApJ* 468, 51
- Hernquist L., 1987, *ApJS* 64, 715
- Hernquist L., Barnes J.E., 1990, *ApJ* 349, 562
- Hockney R.W., Eastwood J.W., 1988, *Computer Simulation Using Particles*. Institute of Physics Publishing, Bristol
- Holschneider M., 1995, *Wavelets – An Analysis Tool*. Oxford University Press, Oxford
- Jain A.K., 1989, *Fundamentals of Digital Image Processing*. Prentice-Hall, Englewood Cliffs
- Junqueira S., Combes F., 1996, *A&A* 312, 703
- Kaiser G., 1994, *A Friendly Guide to Wavelets*. Birkhäuser, Boston
- Lequeux J., Guélin M., 1996, *Dust and Gas in the Outer Parts of Galaxies*. In: Block D.L., Greenberg J.M. (eds.) *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht, p. 422
- Lequeux J., Dantel-Fort M., Fort B., 1995, *A&A* 296, L13
- Merritt D., 1996, *AJ* 111, 2462
- Mihalas D., Binney J., 1981, *Galactic Astronomy – Structure and Kinematics*. Freeman, San Francisco
- Pfalzner S., Gibbon P., 1996, *Many-Body Tree Methods in Physics*. Cambridge University Press, Cambridge
- Pfenniger D., 1996, *Fundamental Aspects of the ISM Fractality*. In: Block D.L., Greenberg J.M. (eds.) *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht, p. 439
- Pfenniger D., Combes F., 1994, *A&A* 285, 94
- Pfenniger D., Friedli D., 1991, *A&A* 252, 75
- Pfenniger D., Friedli D., 1993, *A&A* 270, 561
- Pfenniger D., Combes F., Martinet L., 1994, *A&A* 285, 79
- Pfenniger D., Martinet L., Combes F., 1996, *Secular Evolution of Galaxy Morphologies*. In: Block D.L., Greenberg J.M. (eds.) *New Extragalactic Perspectives in the New South Africa*. Kluwer, Dordrecht, p. 291
- Press W.H., Teukolsky S.A., Vetterling W.T., Flannery B.P., 1992, *Numerical Recipes in Fortran – The Art of Scientific Computing*. Cambridge University Press, Cambridge
- Röhrlich F., 1965, *Classical Charged Particles – Foundations of Their Theory*. Addison-Wesley, Reading
- Romeo A.B., 1990, *Stability and Secular Heating of Galactic Discs*. PhD thesis, SISSA, Trieste, Italy
- Romeo A.B., 1992, *MNRAS* 256, 307
- Romeo A.B., 1994, *A&A* 286, 799 (Paper I)
- Romeo A.B., 1997, *A&A* 324, 523 (Paper II)

- Romeo A.B., 1998, *N-Body Simulations of Disc Galaxies Can Shed Light on the Dark-Matter Problem*. In: Salucci P. (ed.) *Dark Matter*. Studio Editoriale Fiorentino, Firenze, p. 177
- Shu F.H., 1968, *The Dynamics and Large Scale Structure of Spiral Galaxies*. PhD thesis, Harvard University, Cambridge, Massachusetts, USA
- Splinter R.J., Melott A.L., Shandarin S.F., Suto Y., 1998, *ApJ* 497, 38
- Stutzki J., Bensch F., Heithausen A., Ossenkopf V., Zielinsky M., 1998, *A&A*, in press
- Tormen G., Bertschinger E., 1996, *ApJ* 472, 14
- Vandervoort P.O., 1970, *ApJ* 161, 87
- Vetterling W.T., Teukolsky S.A., Press W.H., Flannery B.P., 1992, *Numerical Recipes Example Book (Fortran)*. Cambridge University Press, Cambridge
- Vlasov A.A., 1961, *Many-Particle Theory and Its Application to Plasma*. Gordon and Breach, New York
- von Linden S., Otmianowska-Mazur K., Lesch H., Skupniewicz G., 1998, *A&A* 333, 79
- Weinberg M.D., 1996, *ApJ* 470, 715
- Zhang X., 1996, *ApJ* 457, 125
- Zhang X., 1998a, *ApJ*, in press
- Zhang X., 1998b, *ApJ*, submitted
- Zotov V.M., Morozov A.G., 1987, *SvAL* 13, 136