Astron. Astrophys. 336, 539–544 (1998)

ASTRONOMY AND ASTROPHYSICS

The asymptotic representation of dynamic tides in close binaries revisited

P. Smeyers and B. Willems[⋆]

Instituut voor Sterrenkunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Heverlee, Belgium

Received 3 October 1997 / Accepted 25 May 1998

Abstract. The asymptotic representation of low-frequency, linear, isentropic dynamic tides in components of close binaries derived by Smeyers (1997) is reconsidered. The constructions of the second-order boundary-layer solutions for the divergence and the radial component of the tidal displacement are made more transparent. Moreover, the determination of the constants involved in the second-order boundary-layer solution for the radial component of the tidal displacement valid near the star's surface is modified. As a result, a corrected expression for the second-order Eulerian perturbation of the gravitational potential at the star's surface caused by the tidal distortion is obtained.

Key words: stars: binaries: close – methods: analytical – celestial mechanics, stellar dynamics

1. Introduction

In a recent investigation, an asymptotic representation of low-frequency, linear, isentropic dynamic tides in components of close binaries has been derived (Smeyers 1997, hereafter referred to as Paper I). The dynamic tide is assumed not to be in resonance with any free oscillation mode of the star. For the sake of simplification, the square of the Brunt-Väisälä frequency is considered to be positive everywhere in the star.

For the construction of the second-order asymptotic solutions for the divergence and the radial component of the tidal displacement, we adopted a method described by Kevorkian and Cole (1981, Sect. 3.3.3; 1996, Sect. 4.3.3). A distinction between various regions in the star is made. At sufficiently large distances from the star's centre and surface, the asymptotic solutions are constructed in terms of the usual radial coordinate, considered as a slow coordinate, and a fast radial coordinate. The regions near the star's centre and the star's surface are treated as boundary layers because of the singularities appearing at these end points.

In the present investigation, we reconsider the constructions of the second-order boundary-layer solutions in order to make them more transparent by the use of a single coordinate in the boundary layer. Moreover, we modify the determination of the constants involved in the boundary-layer solution for the radial component of the tidal displacement that is valid near the surface. As a result of this modification, we arrive at a corrected expression for the second-order Eulerian perturbation of the gravitational potential at the star's surface.

The plan of the paper is as follows. In Sect. 2, we reconsider the construction of the asymptotic expansions that are uniformly valid from the singular point at $r = 0$. Sect. 3 is devoted to the construction of the second-order boundary-layer solutions near the singular point at $r = 1$. In Sect. 4, the continuity of the gravitational potential and its gradient at the star's surface is imposed. The matching of the second-order boundary-layer solutions valid near $r = 1$ is performed in Sect. 5. Sect. 6, finally, is devoted to concluding remarks.

2. Asymptotic expansions uniformly valid to order ε^2 from the singular point at $r = 0$

The construction of the boundary-layer solutions of order ε^2 near the singular point at $r = 0$ starts as before. In this boundary layer, we use the stretched coordinate

$$\tau(r) = \frac{1}{\varepsilon}\int_0^r K_1^{1/2}(r')\,dr' \tag{1}$$

[see Paper I, Definition (42)] as the single independent variable. This coordinate is identical with the fast coordinate used at larger distances from $r = 0$ and $r = 1$. We now write dominant boundary-layer Eq. (81) of Paper I as

$$\frac{d^2 v_2^{(c)}}{d\tau^2} + \left[1 - \frac{\ell(\ell+1)}{\tau^2}\right] v_2^{(c)} = g_2^{(c)}(\tau). \tag{2}$$

The function $g_2^{(c)}(\tau)$ is a rapidly varying function defined as

$$g_2^{(c)}(\tau) = \varepsilon^{-(\ell+1)}\, f[r(\tau)]. \tag{3}$$

It is of order ε^0 and behaves as $\tau^{\ell+1}$ as $\tau \to 0$. For illustration, the function $f(r)$, which is a main function in the construction of the asymptotic representation, is represented in Fig. 1 for a polytropic model with index $n_e = 3$ and for $\ell = 2$.

Send offprint requests to: P. Smeyers

⋆ Research Assistant of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.)

Correspondence to: Paul.Smeyers@ster.kuleuven.ac.be

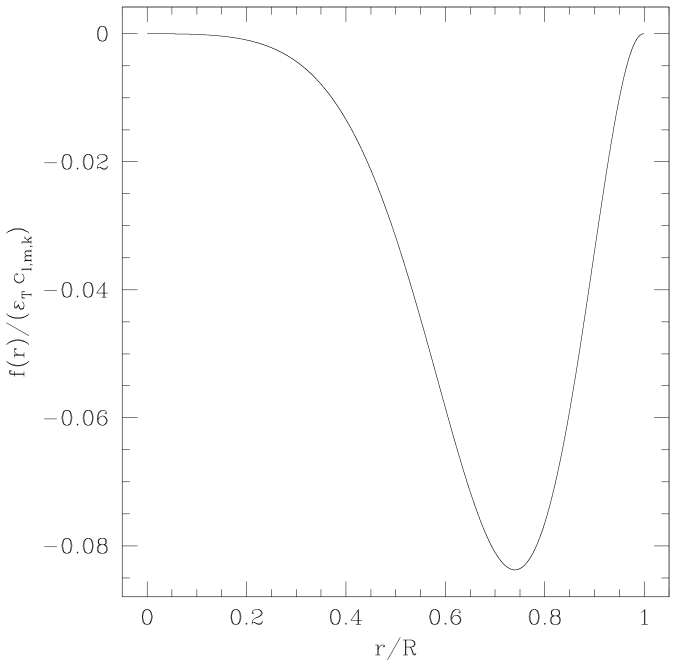

Fig. 1. Variation of the function $f(r)/(\varepsilon_T\, c_{\ell,m,k})$ for a polytropic model with index $n_e = 3$ and for $\ell = 2$.

The solution of the equation satisfying the requirement that the divergence of the tidal displacement be finite at $r = 0$ now takes a more convenient form than Solution (82) in Paper I:

$$v_2^{(c)}(\tau) = A'_{2,c}\,\tau^{1/2}\,J_{\ell+1/2}(\tau) - \left(\frac{\pi}{2}\right)^{1/2}\tau^{1/2} \left[I_Y^{(c)}(\tau)\,J_{\ell+1/2}(\tau) - I_J^{(c)}(\tau)\,Y_{\ell+1/2}(\tau)\right], \qquad (4)$$

where the functions $I_J^{(c)}(\tau)$ and $I_Y^{(c)}(\tau)$ are redefined as

$$\left.\begin{aligned} I_J^{(c)}(\tau) &= \left(\frac{\pi}{2}\right)^{1/2}\int_0^{\tau} g_2^{(c)}(\tau')\,\tau'^{1/2} J_{\ell+1/2}(\tau')\,d\tau', \\ I_Y^{(c)}(\tau) &= \left(\frac{\pi}{2}\right)^{1/2}\int_0^{\tau} g_2^{(c)}(\tau')\,\tau'^{1/2} Y_{\ell+1/2}(\tau')\,d\tau'. \end{aligned}\right\} \qquad (5)$$

As may be noted, we pass over Equality (84) of Paper I and directly introduce the appropriate constant $A'_{2,c}$.

Similarly, dominant boundary-layer Eq. (87) of Paper I is now written as

$$\frac{d^2 w_2^{(c)}}{d\tau^2} - \frac{\ell(\ell+1)}{\tau^2}\,w_2^{(c)} = g_2^{(c)}(\tau) - v_2^{(c)}(\tau). \qquad (6)$$

The solution of the equation corresponds to Solution (89) of Paper I.

Asymptotic Expansions (90) determined in Paper I for the divergence and the radial component of the tidal displacement in the boundary layer near the singular point at $r = 0$ remain valid.

For $\tau \geq \tau_M$, the Bessel functions appearing in the function $v_2^{(c)}(\tau)$ are determined by Hankel's first asymptotic approximations. It follows that

$$\begin{aligned} v_2^{(c)}(\tau) = {} & A'_{2,c}\left(\frac{2}{\pi}\right)^{1/2}\sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ & - I_Y^{(c)}(\tau_M)\,\sin\left(\tau - \ell\,\frac{\pi}{2}\right) - I_J^{(c)}(\tau_M)\,\cos\left(\tau - \ell\,\frac{\pi}{2}\right) \\ & + \left[\int_{\tau_M}^{\tau} g_2^{(c)}(\tau')\,\cos\left(\tau' - \ell\,\frac{\pi}{2}\right) d\tau'\right]\sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ & - \left[\int_{\tau_M}^{\tau} g_2^{(c)}(\tau')\,\sin\left(\tau' - \ell\,\frac{\pi}{2}\right) d\tau'\right]\cos\left(\tau - \ell\,\frac{\pi}{2}\right). \end{aligned} \qquad (7)$$

A partial integration in the last two terms yields, at the same degree of approximation in the powers of τ,

$$\begin{aligned} v_2^{(c)}(\tau) = {} & A'_{2,c}\left(\frac{2}{\pi}\right)^{1/2}\sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ & - I_Y^{(c)}(\tau_M)\,\sin\left(\tau - \ell\,\frac{\pi}{2}\right) - I_J^{(c)}(\tau_M)\,\cos\left(\tau - \ell\,\frac{\pi}{2}\right) \\ & - g_2^{(c)}(\tau_M)\cos(\tau - \tau_M) + g_2^{(c)}(\tau). \end{aligned} \qquad (8)$$

Matching Condition (96) of Paper I is then satisfied to order $\gamma_1^{(c)}(\varepsilon) = \varepsilon^2$ by the relations

$$\left.\begin{aligned} A_2^* = {} & -\varepsilon^{\ell+1}\left\{\left[A'_{2,c}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(c)}(\tau_M)\right]\sin\left(\ell\,\frac{\pi}{2}\right)\right. \\ & \left. + I_J^{(c)}(\tau_M)\,\cos\left(\ell\,\frac{\pi}{2}\right) + g_2^{(c)}(\tau_M)\,\cos\tau_M\right\}, \\ B_2^* = {} & \varepsilon^{\ell+1}\left\{\left[A'_{2,c}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(c)}(\tau_M)\right]\cos\left(\ell\,\frac{\pi}{2}\right)\right. \\ & \left. - I_J^{(c)}(\tau_M)\,\sin\left(\ell\,\frac{\pi}{2}\right) - g_2^{(c)}(\tau_M)\,\sin\tau_M\right\}. \end{aligned}\right\} \qquad (9)$$

These relations replace Relations (97) of Paper I.

Furthermore, matching Condition (98) of Paper I is still satisfied to order $\gamma_2^{(c)}(\varepsilon) = \varepsilon^2$ by Relations (103) and (104) presented there.

Consequently, the asymptotic expansions of the divergence and the radial component of the tidal displacement that are uniformly valid to order ε^2, from $r = 0$ to a sufficiently large distance from $r = 1$, remain given by Expansions (105) of Paper I:

$$\left.\begin{aligned} \alpha^{(c,u)}(r;\varepsilon) &= \varepsilon^{\ell+3}\,K_5(r)\,v_2^{(c)}(\tau), \\ \xi^{(c,u)}(r;\varepsilon) &= \xi_0(r) + \varepsilon^2\,K_6(r) \\ & \qquad \left[\varepsilon^{\ell+1}\,v_2^{(c)}(\tau) + \frac{G_2^{(o)}(r)}{K_6(r)} - f(r)\right]. \end{aligned}\right\} \qquad (10)$$

3. Boundary-layer solutions of order ε^2 near the singular point at $r = 1$

Proceeding as for the boundary layer near $r = 0$, we use the stretched coordinate

$$\tau_s(r) = \frac{1}{\varepsilon}\int_r^1 K_1^{1/2}(r')\,dr' \qquad (11)$$

as the single independent coordinate and write dominant boundary-layer Eq. (122) of Paper I as

$$\frac{d^2 v_2^{(s)}}{d\tau_s^2} + \left[1 - \frac{(n_e+1)^2 - 1/4}{\tau_s^2}\right] v_2^{(s)} = g_2^{(s)}(\tau_s)\,. \tag{12}$$

The function $g_2^{(s)}(\tau_s)$ is a rapidly varying function defined as

$$g_2^{(s)}(\tau_s) = \varepsilon^{-(n_e+3/2)}\, f\left[r(\tau_s)\right]. \tag{13}$$

It is of order ε^0 and behaves as $\tau_s^{n_e+3/2}$ as $\tau_s \to 0$.

The solution of the equation satisfying the requirement that the divergence of the tidal displacement be finite at $r = 1$ takes the form

$$v_2^{(s)}(\tau_s) = A'_{2,s}\, \tau_s^{1/2}\, J_{n_e+1}(\tau_s) - \left(\frac{\pi}{2}\right)^{1/2} \tau_s^{1/2} \left[I_Y^{(s)}(\tau_s)\, J_{n_e+1}(\tau_s) - I_J^{(s)}(\tau_s)\, Y_{n_e+1}(\tau_s)\right], \tag{14}$$

where the functions $I_J^{(s)}(\tau_s)$ and $I_Y^{(s)}(\tau_s)$ are redefined as

$$\left.\begin{aligned} I_J^{(s)}(\tau_s) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau_s} g_2^{(s)}(\tau'_s)\; \tau_s'^{1/2}\, J_{n_e+1}(\tau'_s)\; d\tau'_s, \\ I_Y^{(s)}(\tau_s) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau_s} g_2^{(s)}(\tau'_s)\; \tau_s'^{1/2}\, Y_{n_e+1}(\tau'_s)\; d\tau'_s. \end{aligned}\right\} \tag{15}$$

Here too, we directly introduce the appropriate constant $A'_{2,s}$.

Similarly, dominant boundary-layer Eq. (129) of Paper I takes the form

$$\frac{d^2 w_2^{(s)}}{d\tau_s^2} - \frac{2\,n_e}{\tau_s}\frac{dw_2^{(s)}}{d\tau_s} + \frac{(n_e+1/2)^{1/2} - 1}{\tau_s^2}\, w_2^{(s)} = g_2^{(s)}(\tau_s) - \left[v_2^{(s)}(\tau_s) + 2\,\left(c_s^2\right)^{-1}\, h_2^{(s)}(\tau_s)\right]. \tag{16}$$

The function

$$w_2^*(\tau_s) \equiv w_2^{(s)}(\tau_s) - v_2^{(s)}(\tau_s) \tag{17}$$

is still determined by Solution (132) of Paper I. This solution is transformed into Solution (133) by integration by parts. Subsequently, by application of recurrence Relation (134) for Bessel functions, integration by parts, and use of Equality (135), one brings the solution to the form

$$\begin{aligned} w_2^*(\tau_s) &= C_{2,s}\, \tau_s^{n_e+3/2} + D_{2,s}\, \tau_s^{n_e-1/2} \\ &+2\,\mathcal{N}_s^2 \Big\{-A'_{2,s}\, \tau_s^{-1/2}\, J_{n_e}(\tau_s) \\ &+ \left(\frac{\pi}{2}\right)^{1/2} \tau_s^{-1/2} \left[I_Y^{(s)}(\tau_s)\, J_{n_e}(\tau_s) - I_J^{(s)}(\tau_s)\, Y_{n_e}(\tau_s)\right] \\ &+\tau_s^{n_e-1/2} \int_0^{\tau_s} \tau_s'^{-(n_e+1/2)}\, g_2^{(s)}(\tau'_s)\, d\tau'_s\Big\}. \end{aligned} \tag{18}$$

A main difference with the procedure followed in Paper I is that we no longer rescale the constant $D_{2,s}$.

The asymptotic expansions of the divergence and the radial component of the tidal displacement that are valid to order ε^2 in the boundary layer near the singular point at $r = 1$ are still given by Expansions (139) of Paper I:

$$\left.\begin{aligned} \alpha^{(s)}(r;\varepsilon) &= \varepsilon^{n_e+7/2}\, K_5(r)\, v_2^{(s)}(\tau_s)\,, \\ \xi^{(s)}(r;\varepsilon) &= \xi_0(r) + \varepsilon^{n_e+7/2}\, K_6(r) \left[v_2^{(s)}(\tau_s) + w_2^*(\tau_s)\right]. \end{aligned}\right\} \tag{19}$$

4. Continuity of the gravitational potential and its gradient at $r = 1$

In contrast to the procedure followed in Paper I, we impose the continuity of the gravitational potential and its gradient at $r = 1$ before performing the matching of the second-order asymptotic solutions valid in the boundary layer near $r = 1$.

From asymptotic Expansions (19), it follows that

$$\alpha_{R_1} = \varepsilon^2\, \frac{A'_{2,s}}{2^{n_e+1}\,\Gamma(n_e+2)}\, \left(c_s^2\right)^{-1} \left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2} K_{6,s}, \tag{20}$$

$$\xi_{R_1} = (\xi_0)_{R_1} - \varepsilon^4 \left[A'_{2,s}\, \frac{\mathcal{N}_s^2}{2^{n_e-1}\,\Gamma(n_e+1)} - D_{2,s}\right] \left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2} K_{6,s}, \tag{21}$$

$$\begin{aligned} \left(\frac{d\xi}{dr}\right)_{R_1} &= \left(\frac{d\xi_0}{dr}\right)_{R_1} \\ &-\varepsilon^2 \left[A'_{2,s}\, \frac{1+\mathcal{N}_s^2}{2^{n_e+1}\Gamma(n_e+2)} + C_{2,s}\right] \left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2} K_{6,s} \\ &+\varepsilon^4 \left[A'_{2,s}\, \frac{\mathcal{N}_s^2}{2^{n_e-1}\,\Gamma(n_e+1)} - D_{2,s}\right] \\ &\left(\frac{n_e-1/2}{6}\, \frac{K_{1,1}}{K_{1,s}} + \frac{K_{6,1}}{K_{6,s}}\right) \left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2} K_{6,s}. \end{aligned} \tag{22}$$

Hence, at $r = 1$, the divergence of the tidal displacement is of order ε^2, while the radial component of the tidal displacement is of order ε^0 and contains no term of order ε^2.

Expressions (159)-(161) of Paper I remain valid. By means of Expressions (156) and (157) of Paper I for Ψ and $d\Psi/dr$, one derives to order ε^2 that

$$\Psi_{R_1} = -\,(\xi_0)_{R_1} + \frac{\varepsilon^2}{\ell(\ell+1)} \left[\left(\frac{d\xi_0}{dr}\right)_{R_1} + 2\,(\xi_0)_{R_1}\right], \tag{23}$$

$$\left(\frac{d\Psi}{dr}\right)_{R_1} = 2\,(\xi_0)_{R_1} - \left(\frac{d\xi_0}{dr}\right)_{R_1} +\varepsilon^2 \left[(\xi_0)_{R_1} + C_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2} K_{6,s}\right]. \tag{24}$$

Substitution of these expressions into boundary Condition (33) of Paper I leads to the following equation for the constant

$C_{2,s}$:

$$C_{2,s} \left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}\, K_{6,s} = -\frac{1}{\ell}\left[\left(\frac{d\xi_0}{dr}\right)_{R_1} + (\ell+2)\,(\xi_0)_{R_1}\right]. \tag{25}$$

Hence, the constant $C_{2,s}$ is determined by the condition ensuring the continuity of the gravitational potential and its gradient at $r = 1$. The constant $D_{2,s}$ does not appear in the boundary condition to order ε^2, since it is involved in terms of order ε^4 at $r = 1$, as it is seen from Equalities (20)–(22).

5. Matching of the second-order boundary-layer solutions valid near $r = 1$

In view of the matching, the Bessel functions are again determined by Hankel's first asymptotic approximations for sufficiently large values of their argument. On the analogy of Solution (8) for $v_2^{(c)}(\tau)$, Solution (14) for $v_2^{(s)}(\tau_s)$ can be approximated, for $\tau_s \geq \tau_N$, as

$$\begin{aligned} v_2^{(s)}(\tau_s) = {} & A'_{2,s}\left(\frac{2}{\pi}\right)^{1/2} \sin(\tau_s - \varphi) \\ & - I_Y^{(s)}(\tau_N)\sin(\tau_s-\varphi) - I_J^{(s)}(\tau_N)\cos(\tau_s-\varphi) \\ & - g_2^{(s)}(\tau_N)\cos(\tau_s-\tau_N) + g_2^{(s)}(\tau_s) \end{aligned} \tag{26}$$

with $\varphi = (n_e + 1/2)\,\pi/2$.

Similarly, Solution (18) for $w_2^*(\tau_s)$ can be approximated as

$$\begin{aligned} w_2^*(\tau_s) = {} & C_{2,s}\,\tau_s^{n_e+3/2} + D_{2,s}\tau_s^{n_e-1/2} \\ & + \frac{2\,\mathcal{N}_s^2}{\tau_s}\left[-A'_{2,s}\left(\frac{2}{\pi}\right)^{1/2}\cos(\tau_s-\varphi)\right. \\ & + I_Y^{(s)}(\tau_N)\,\cos(\tau_s-\varphi) - I_J^{(s)}(\tau_N)\,\sin(\tau_s-\varphi) \\ & \left. - g_2^{(s)}(\tau_N)\,\sin(\tau_s-\tau_N)\right] \\ & + 2\,\mathcal{N}_s^2\,\tau_s^{n_e-1/2}\int_0^{\tau_s} \tau_s'^{-(n_e+1/2)}\, g_2^{(s)}(\tau_s')\,d\tau_s'. \end{aligned} \tag{27}$$

Matching Condition (145) of Paper I is satisfied to order $\gamma_1^{(s)}(\varepsilon) = \varepsilon^2$ by the relations

$$\left.\begin{aligned} A_2^* = {} & \varepsilon^{n_e+3/2}\left\{\left[A'_{2,s}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(s)}(\tau_N)\right]\right. \\ & \sin(\tau_{R_1}-\varphi) - I_J^{(s)}(\tau_N)\,\cos(\tau_{R_1}-\varphi) \\ & \left. - g_2^{(s)}(\tau_N)\,\cos(\tau_{R_1}-\tau_N)\right\}, \\ B_2^* = {} & -\varepsilon^{n_e+3/2}\left\{\left[A'_{2,s}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(s)}(\tau_N)\right]\right. \\ & \cos(\tau_{R_1}-\varphi) + I_J^{(s)}(\tau_N)\,\sin(\tau_{R_1}-\varphi) \\ & \left. + g_2^{(s)}(\tau_N)\,\sin(\tau_{R_1}-\tau_N)\right\}. \end{aligned}\right\} \tag{28}$$

These relations replace Relations (146) of Paper I.

Elimination of the constants A_2^* and B_2^* between Relations (9) and (28) leads to a system of two linear, inhomogeneous algebraic equations in the constants $A'_{2,c}$ and $A'_{2,s}$ whose determinant is given by

$$\Delta = \varepsilon^{\ell+n_e+5/2}\,\sin\gamma \tag{29}$$

with

$$\gamma = \tau_{R_1} - \left(\ell + n_e + \frac{1}{2}\right)\frac{\pi}{2}. \tag{30}$$

When $\sin\gamma \neq 0$, the constants $A'_{2,c}$ and $A'_{2,s}$ are given by

$$\left.\begin{aligned} A'_{2,c} = {} & \frac{1}{\sin\gamma}\left(\frac{\pi}{2}\right)^{1/2} \\ & \left\{I_Y^{(c)}(\tau_M)\,\sin\gamma + I_J^{(c)}(\tau_M)\,\cos\gamma\right. \\ & + g_2^{(c)}(\tau_M)\cos(\tau_{R_1}-\tau_M-\varphi) \\ & - \varepsilon^{n_e-\ell+1/2}\left[I_J^{(s)}(\tau_N)\right. \\ & \left.\left. + g_2^{(s)}(\tau_N)\cos(\tau_N-\varphi)\right]\right\}, \\ A'_{2,s} = {} & \frac{1}{\sin\gamma}\left(\frac{\pi}{2}\right)^{1/2} \\ & \left\{I_Y^{(s)}(\tau_N)\,\sin\gamma + I_J^{(s)}(\tau_N)\,\cos\gamma\right. \\ & + g_2^{(s)}(\tau_N)\,\cos\left(\tau_{R_1}-\tau_N-\ell\,\frac{\pi}{2}\right) \\ & - \varepsilon^{-(n_e-\ell+1/2)}\left[I_J^{(c)}(\tau_M)\right. \\ & \left.\left. + g_2^{(c)}(\tau_M)\,\cos\left(\tau_M-\ell\,\frac{\pi}{2}\right)\right]\right\}. \end{aligned}\right\} \tag{31}$$

As a second matching condition, we impose, to some order $\gamma_2^{(s)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon\to 0\\ \tau_\eta \text{fixed}}} \frac{\varepsilon^2}{\gamma_2^{(s)}(\varepsilon)}\,\frac{1}{z^{(n_e-1/2)/2}} \left[\frac{G_2^{(o)}(r)}{K_6(r)} - f(r) - \varepsilon^{n_e+3/2}\,w_2^*(\tau_s)\right] = 0. \tag{32}$$

In terms of the intermediate variable τ_η, one has

$$\frac{1}{z^{(n_e-1/2)/2}}\,\frac{G_2^{(0)}(r)}{K_6(r)} = \frac{G_2^{(0)}(1)}{K_{6,s}}, \tag{33}$$

$$\frac{1}{z^{(n_e-1/2)/2}}\,f(r) = -\varepsilon^{2\eta}\,\frac{1}{4\,K_{1,s}^2}\,\frac{K_{4,s}}{K_{5,s}}\left(\frac{d\xi_0}{dr}\right)_{R_1}\tau_\eta^2, \tag{34}$$

$$\varepsilon^{n_e+3/2}\,C_{2,s}\,\frac{\tau_s^{n_e+3/2}}{z^{(n_e-1/2)/2}} = \varepsilon^{2\eta}\,C_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,\tau_\eta^2, \tag{35}$$

$$\varepsilon^{n_e+3/2}\, D_{2,s}\, \frac{\tau_s^{n_e-1/2}}{z^{(n_e-1/2)/2}} = \varepsilon^2\, D_{2,s}\, \left(2\, K_{1,s}^{1/2}\right)^{n_e-1/2}. \quad (36)$$

In the last term of Solution (27), the integral with respect to τ_s can be transformed into an integral with respect to z. In view of the matching of the boundary-layer solution for the radial component of the tidal displacement, it suffices to perform the integration to the lowest order of z. It then follows that

$$\int_0^{\tau_s} \tau_s'^{-(n_e+1/2)}\, g_2^{(s)}\left(\tau_s'\right) d\tau_s' = -\varepsilon^{-2}\, A\, z + O\left(z^2\right), \quad (37)$$

where A is a constant defined as

$$A = \frac{1}{K_{1,s}^{1/2}}\, \frac{K_{4,s}}{K_{5,s}} \left(2\, K_{1,s}^{1/2}\right)^{-(n_e+1/2)} \left(\frac{d\xi_0}{dr}\right)_{R_1}. \quad (38)$$

In terms of the intermediate variable τ_η, one then has

$$\varepsilon^{n_e+3/2}\, \frac{\tau_s^{n_e-1/2}}{z^{(n_e-1/2)/2}} \int_0^{\tau_s} \tau_s'^{-(n_e+1/2)}\, g_2^{(s)}\left(\tau_s'\right) d\tau_s'$$
$$= -\varepsilon^{2\eta}\, A \left(2\, K_{1,s}^{1/2}\right)^{n_e-5/2} \tau_\eta^2. \quad (39)$$

Matching condition (32) is satisfied to order $\gamma_2^{(s)}(\varepsilon) = \varepsilon^2$, if

$$G_2^{(o)}(1) = 0. \quad (40)$$

This condition allows one to fix the yet unknown constant $C_2^{(0)}$ appearing in Solution (66) of Paper I for the function $G_2^{(o)}(r)$. Subsequently, the constant $C_{2,c}$ appearing in Solution (89) of Paper I for the function $w_2^{(c)}(\tau)$ is fixed by means of Eq. (104) of that paper.

The term

$$\varepsilon^{n_e+7/2}\, D_{2,s}\, \tau_s^{n_e-1/2}$$

of the function $\varepsilon^{n_e+7/2}\, w_2^*\left(\tau_s\right)$ plays not any more role in the matching condition than it does in boundary Condition (25), and remains undetermined. It may be left out of consideration at the degree of approximation considered.

6. Concluding remarks

Asymptotic Expansions (23) and (24) for Ψ_{R_1} and $(d\Psi/dr)_{R_1}$ differ from the corresponding asymptotic expansions (164) and (165) in Paper I. Moreover, boundary Condition (25) and matching Conditions (40) stand for Conditions (154), (155), and (166) of Paper I.

We have verified the validity of asymptotic Expansion (23) for the polytropic models with indices $n_e = 2, 3, 4$ in the range of values of ε^2 from 0.01 to 0.001. The relative errors of the values of Ψ_{R_1} found in comparison to the exact values are presented in Fig. 2. For the three models, the relative errors are smaller than 1% from $\varepsilon^2 = 0.01$ on and decrease below 0.1% as ε^2 becomes smaller than 0.001.

A main motivation for the construction of the asymptotic representation of low-frequency, non-resonant dynamic tides has been the derivation of an asymptotic expansion for the Eulerian perturbation of the gravitational potential that is generated at the star's surface by the tidal distortion. This asymptotic expansion is useful for studies of dynamic effects of tides in close binaries as the apsidal motion.

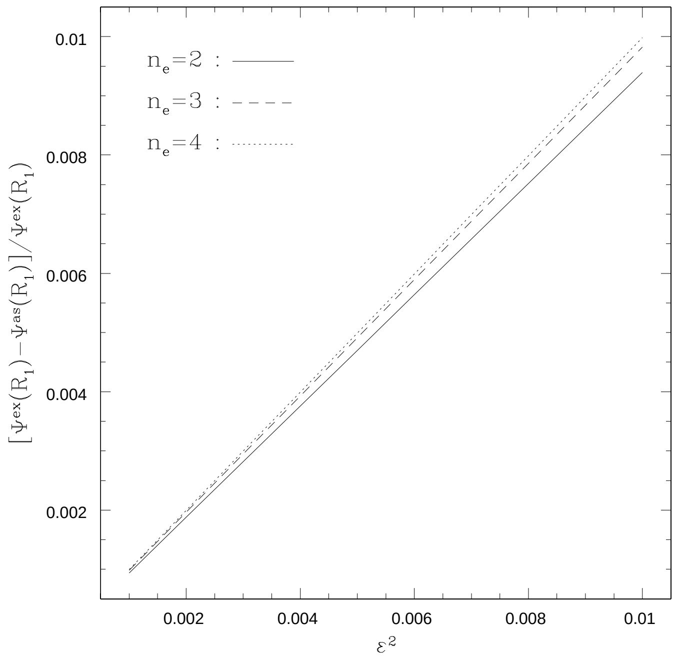

Fig. 2. Relative errors of the values Ψ_{R_1} in comparison to the exact values determined by integration of the full fourth-order system of equations governing forced oscillations.

The asymptotic expansion for the Eulerian perturbation of the gravitational potential at the star's surface can be derived by means of Expression (32) of Paper I and Expression (23) of this paper. It follows that, to order ε^2,

$$\Phi'_{R_1} = \varepsilon_T\, c_{\ell,m,k} - (\xi_0)_{R_1}$$
$$+ \frac{\varepsilon^2}{\ell(\ell+1)} \left[\left(\frac{d\xi_0}{dr}\right)_{R_1} + 2\, (\xi_0)_{R_1}\right]. \quad (41)$$

The Eulerian perturbation of the gravitational potential of order ε^2 that is generated at the star's surface by a low-frequency, non-resonant dynamic tide turns out to be determined simply by the values of the function $\xi_0(r)$ and its gradient at the star's surface.

From the asymptotic representation of low-frequency, non-resonant dynamic tides to order ε^2, one passes on to the first asymptotic representation of low-frequency *free oscillation modes* g^+ in the star by setting the mass M_2 of the companion equal to zero. According to Definition (2) of Paper I, the small parameter ε_T then becomes equal to zero. Consequently, boundary Condition (40) of Paper I, which relates the function $\xi_0(r)$ and its first derivative at the star's surface, becomes homogeneous and can generally no more be satisfied by a solution $\xi_0(r)$ different from zero. The fact that the solution for $\xi_0(r)$ is identically zero greatly simplifies the asymptotic representation in the various regions of the star.

First, from Condition (25) it now results that

$$C_{2,s} = 0. \tag{42}$$

Secondly, matching Condition (40) remains valid and is equivalent with Condition (129) of Smeyers et al. (1995). From the matching condition, it follows that

$$C_2^{(o)} = 0, \tag{43}$$

so that the function $G_2^{(o)}(r)$ too is identically zero, and

$$C_{2,c} = 0. \tag{44}$$

Hence, the asymptotic solutions for the divergence and the radial component of the Lagrangian displacement are purely oscillatory. From $r = 0$ to a sufficiently large distance from $r = 1$, they take the form

$$\left.\begin{aligned} \alpha^{(c,u)}(r;\varepsilon) &= \varepsilon^{\ell+3}\, A'_{2,c}\, K_5(r)\, \tau^{1/2}\, J_{\ell+1/2}(\tau), \\ \xi^{(c,u)}(r;\varepsilon) &= \frac{c^2}{g}\, \alpha^{(c,u)}(r;\varepsilon), \end{aligned}\right\} \tag{45}$$

and, in the boundary layer near $r = 1$, the form

$$\left.\begin{aligned} &\alpha^{(s)}(r;\varepsilon) = \varepsilon^{n_e+7/2}\, A'_{2,s}\, K_5(r)\, \tau_s^{1/2}\, J_{n_e+1}\left(\tau_s\right), \\ &\xi^{(s)}(r;\varepsilon) = \frac{c^2}{g}\, \alpha^{(s)}(r;\varepsilon) \\ &\quad -\varepsilon^{n_e+7/2}\, A'_{2,s}\, 2\,\mathcal{N}_s^2\, K_6(r)\, \tau_s^{-1/2}\, J_{n_e}\left(\tau_s\right). \end{aligned}\right\} \tag{46}$$

From asymptotic Expansion (41), it follows that the Eulerian perturbation of the gravitational potential at the star's surface is equal to zero at the order of asymptotic approximation considered, as was observed by Smeyers et al. (1995).

The same authors also noted that asymptotic Solutions (45) and (46) even apply to the g^--modes of the equilibrium sphere with uniform mass density when one replaces the frequency σ^2 by $-\sigma^2$ and N^2 by $-N^2$ from the starting equations on. The asymptotic solutions can then be compared with the known *analytical* solutions (see Ledoux and Walraven 1958, Sect. 76). From their analysis, Smeyers et al. concluded that the asymptotic approximation is excellent.

One point may be stressed here. From the analytical solution for the radial component of the Lagrangian displacement, the following approximation of the lowest order in ε can be derived:

$$\xi(r) = \frac{\Gamma_1}{2}\, \frac{1 - r^2}{r}\, \alpha(r). \tag{47}$$

Hence, in the lowest-order approximation, the radial component of the Lagrangian displacement is equal to zero at $r = 1$, while the divergence of the Lagrangian displacement is different from zero at that point. This is reproduced by our asymptotic Solutions (20) and (21) at the same point. The lowest-order solution for a free g^--mode of the equilibrium sphere with uniform mass density is given by the terms of order ε^2. At that order, it is seen that indeed $\alpha_{R_1} \neq 0$ and $\xi_{R_1} = 0$.

References

Kevorkian, J., Cole, J.D. 1981, Perturbation Methods in Applied Mathematics, Springer-Verlag, New York

Kevorkian, J., Cole, J.D. 1996, Multiple Scale and Singular Perturbation Methods, Springer-Verlag, New York

Ledoux, P., Walraven, Th. 1958, Variable Stars, in: Handbuch der Physik 51, Springer, Berlin

Smeyers, P., 1997, A&A 318, 140

Smeyers, P., De Boeck, I., Van Hoolst, T., Decock, L., 1995, A&A 301, 105