

Research Note

On the calculation of the linear stability parameter of periodic orbits

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Abstract. In this paper we propose an improved method for calculating Hénon's stability parameter, which is based on the differential of the Poincaré map using the first variational equation. We show that this method is very accurate and give some examples where it gives correct results, while the previous method could not cope.

Key words: methods: analytical – methods: numerical – celestial mechanics, stellar dynamics – galaxies: kinematics and dynamics

1. Introduction

Over 30 years ago, in a now seminal paper, Hénon (1965) introduced the stability parameter α , which distinguished whether a given periodic orbit is stable or not. This distinction is essential since the properties of the two types of orbits differ considerably. Indeed stable periodic orbits trap regular orbits around them, while unstable ones trigger chaos. The use of Hénon's stability parameter is, however, not limited to that. By allowing us to find precisely the value of the energy for which a given family changes from stable to unstable, or vice-versa, it allows us to find the point from which new families bifurcate, since for a periodic orbit a transition from stability to instability results in a bifurcation of a new stable family, while a transition from instability to stability introduces a new unstable family.

To calculate stability, Hénon (1965) approached the differential of the Poincaré map by finite differences. This technique has been so far widely followed in galactic dynamics (e.g. Contopoulos & Grösbøl 1989 and references therein). Nevertheless it suffers from a number of disadvantages. We have found it quite adequate in regular regions, but found it could not cope with difficult chaotic regions. For this reason we present here an alternative approach, based on the differential of the Poincaré map using the first variational equation. We will hereafter refer to it as variational equation method. The variational equation technique is used in other domains that need very accurate results, like the study of the three-body problem, the Störmer problem or a few others problems in celestial mechanics (e.g. Deprit

& Price 1965; Markellos 1974; Markellos & Zagouras 1977), as well as to obtain the Lyapunov exponents (Benettin et al., 1976, 1980; Contopoulos et al., 1978; Udry & Pfenniger 1988). In this paper the variational equation will allow us to obtain the differential of the Poincaré map exactly. In Sect. 2 we give a very brief mathematical justification of this method, which is well known from other applications, and then we apply it to the particular case of a two-dimensional system which is stationary in a rotating frame of reference, a case often encountered in galactic dynamics problems. Sect. 3 gives an example demonstrating the accuracy of the method we propose and Sect. 4 another example for which this new method gives accurate results, while finite differences can not cope. We conclude in Sect. 5.

2. Method

2.1. Notation

Let us consider an autonomous dynamical system, expressed by the ordinary differential equation (hereafter ODE). $\dot{x} = F(x)$ with $x \in \mathbb{R}^n$ and where $\dot{x} \equiv \frac{dx}{dt}$. By a solution of this ODE we will mean a map $\phi : U \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\dot{\phi}(t, x) = F(\phi(t, x))$.

Using the precepts of Poincaré (1892) we can reduce the study of a continuous time system (ODE) to the study of an associated discrete time system (map) called the Poincaré map, $P : V \subset \Sigma \rightarrow \Sigma; x \mapsto \bar{x} = \phi(\tau(x), x)$, where Σ is a hyper-surface perpendicular to the flow F , which we will hereafter refer to as section, V is an open set in Σ , and $\tau(x)$ is the time necessary for the point x to return for the first time to the section with the same sense of transversal of Σ . With this technique the problem of calculating the stability of a periodic solution of a ODE is reduced to the problem of calculating the stability of a fixed point of a map. For this we must check if solutions starting close to a fixed point at a given time remain close to it for all later times (Lyapunov stability). We thus compute the Taylor series expansion and we study the linear term of such an expansion.

Let x_0 be a fixed point of P and $x = x_0 + \Delta x_0$ a point in its neighbourhood. The Taylor expansion of $P(x)$ is

$$P(x) = P(x_0) + DP(x_0)\Delta x_0 + O(\Delta^2) \quad (1)$$

If we denote $P(x) = x_0 + \Delta x_1$, we have the linear relation $\Delta x_1 = DP(x_0)\Delta x_0$. The eigenvalues of DP will determine the stability. In two dimensions, if P is an area preserving map, the periodic orbit $\phi(t, x_0)$ is stable if $|\alpha| < 1$, where α is the stability parameter introduced by Hénon (1965) and defined as $\alpha = \frac{1}{2}(a_{11} + a_{22})$ where $DP = (a_{ij})$.

2.2. Calculation of DP

Hénon (1965) approximated the elements of the Jacobi matrix $DP(x)$ using finite differences, i.e.

$$DP(x)_{ij} = \frac{P_i(x_j + \Delta x_j) - P_i(x_j)}{\Delta x_j} + O(\Delta) \quad (2)$$

As will be shown below, the above approximation gives sufficient accuracy in regular regions, but not in regions dominated by chaotic dynamics.

We can calculate DP exactly, as:

$$DP(x) = \dot{\phi}(\tau(x), x)D\tau(x) + D\phi(\tau(x), x) \quad (3)$$

The matrix $D\phi$ can be obtained as a solution of the first variational equation with initial condition $D\phi(0, x) = Id$.

$$\frac{d}{dt}D\phi(t, x) = DF(\phi(t, x))D\phi(t, x) \quad (4)$$

To compute $D\tau(x)$ we use the fact that we are on the hyper-surface. Defining the hyper-surface $\Sigma \equiv g(x) = 0$ and differentiating we obtain after some algebra:

$$D\tau(x) = -\frac{1}{(\nabla g(x_1), F(x_1))} Dg(x_1)D\phi(\tau(x), x) \quad (5)$$

where $x_1 = \phi(\tau(x), x)$ and where the symbol (\cdot, \cdot) represents the dot product. The denominator is different from zero since the hyper-surface Σ is perpendicular to the flow F .

2.2.1. Particular case

We will now apply the above general technique to a two dimensional system which is stationary in a rotating frame of reference. Let us consider an autonomous dynamical system, expressed by the following ODE

$$\left. \begin{aligned} \ddot{x} &= -\Phi_x + 2\Omega_b \dot{y} + \Omega_b^2 x \\ \ddot{y} &= -\Phi_y - 2\Omega_b \dot{x} + \Omega_b^2 y \end{aligned} \right\} \quad (6)$$

where $\Phi(x, y)$ is the potential, Ω_b is the pattern speed of the coordinate system in which the dynamical system is stationary, and Φ_x and Φ_y denote the partial derivatives of the potential with respect to x and y respectively.

Writing the above second order ODE as a system of four first order ODE, and using the momenta $X = \dot{x} - \Omega_b$ and $Y = \dot{y} + \Omega_b$ we get

$$\left. \begin{aligned} \dot{x} &= X + \Omega_b y \\ \dot{X} &= -\Phi_x + \Omega_b Y \\ \dot{y} &= Y - \Omega_b x \\ \dot{Y} &= -\Phi_y - \Omega_b X \end{aligned} \right\} \quad (7)$$

This can also be expressed in vectorial form $\dot{z} = F(z)$ with $z = (x, X, y, Y)$ and since below we will need the individual components of F , we will denote them as F_j with $j = 1$ to 4. Hereafter, numbers as subindex indicate a component.

Since on the Poincaré section $y = 0$ and since the energy in a rotating frame, $E_j(x, X, y, Y)$, is an integral of motion, we can express Y as a function of x and X , i.e. $Y = Y(x, X)$. This restricts the Poincaré map to the two-dimensional space (x, X) . since, however, the system of first order ODEs has four equations, the variational equation and the method to calculate the differential of the Poincaré map hold in a four-dimensional space. We must therefore express the Poincaré map in four dimensions in order to calculate the differential and finally project in the two-dimensional space (x, X) as shown schematically below

$$\left. \begin{array}{ccc} (x, X, 0, Y) & \xrightarrow{\Psi} & (\bar{x}, \bar{X}, 0, \bar{Y}) \\ i \uparrow & & \downarrow \pi \\ (x, X) & \xrightarrow{P} & (\bar{x}, \bar{X}) \end{array} \right\} \Rightarrow P = \pi \circ \Psi \circ i \quad (8)$$

Where i includes (x, X) in \mathbb{R}^4 using the section equation and the integral of motion, Ψ is the Poincaré map in \mathbb{R}^4 and π projects the two first coordinates in \mathbb{R}^2 . We can therefore write the differential of the Poincaré map as $DP = D\pi \circ D\Psi \circ Di$.

Let us now proceed as in the general case, taking the derivative of $\Psi(z)$

$$D\Psi(z) = \dot{\phi}(\tau(z), z)D\tau(z) + D\phi(\tau(z), z) \quad (9)$$

Here $\dot{\phi}(\tau(z), z) = F(\phi(\tau(z), z))$, $D\phi(\tau(z), z)$ is the solution of the first variational equation with initial condition $D\phi(0, z) = Id$. To obtain $D\tau(z)$ we use the section equation $y = 0$, which in \mathbb{R}^4 is equivalent to $\Psi_3(z) = 0$. We differentiate it and we obtain that the third vector of the matrix equation (9) should be equal to zero, i.e. $D\Psi_3(z) = \mathbf{0}$, from which it follows that

$$D\tau(z) = -\frac{D\phi_3(\tau(z), z)}{F_3(\phi(\tau(z), z))} \quad (10)$$

Now we can express DP in matrix form

$$DP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \Psi_1 & \partial_X \Psi_1 & 0 & \partial_Y \Psi_1 \\ \partial_x \Psi_2 & \partial_X \Psi_2 & 0 & \partial_Y \Psi_2 \\ 0 & 0 & 0 & 0 \\ \partial_x \Psi_4 & \partial_X \Psi_4 & 0 & \partial_Y \Psi_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ Y_x & Y_X \end{pmatrix} \quad (11)$$

Where Y is, as discussed above, considered as a function of x and X . We see that in order to obtain DP we need to apply $D\Psi$ only to Di , that is, we need to apply $D\phi$ only to Di . Then, instead of solving the variational equation with initial condition $D\phi(0, z) = Id$, which gives us $D\phi(t, z)$, we solve it applied to the vectors in Di , which means solving $\dot{v} = DF(v)$ with initial condition $v^0 = (1, 0, 0, Y_x)^T$ and $w^0 = (0, 1, 0, Y_X)^T$. We will now show how it can be computed in practice.

We integrate simultaneously the orbit and the variational equation for the two vectors in Di .

$$\left. \begin{aligned} \dot{x} &= X + \Omega_b y \\ \dot{X} &= -\Phi_x + \Omega_b Y \\ \dot{y} &= Y - \Omega_b x \\ \dot{Y} &= -\Phi_y - \Omega_b X \end{aligned} \right\} \text{ with } \phi(0, z_0) = \begin{pmatrix} x_0 \\ X_0 \\ 0 \\ Y_0 \end{pmatrix} \quad (12)$$

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \Omega_b & 0 \\ -\Phi_{xx} & 0 & -\Phi_{xy} & \Omega_b \\ -\Omega_b & 0 & 0 & 1 \\ -\Phi_{yx} & -\Omega_b & -\Phi_{yy} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \quad (13)$$

with $v = v^0$ and with $v = w^0$

Finally with those vectors we write DP as

$$DP = \begin{pmatrix} \bar{v}_1^0 - \frac{F_1}{F_3} \bar{v}_3^0 & \bar{w}_1^0 - \frac{F_1}{F_3} \bar{w}_3^0 \\ \bar{v}_2^0 - \frac{F_2}{F_3} \bar{v}_3^0 & \bar{w}_2^0 - \frac{F_2}{F_3} \bar{w}_3^0 \end{pmatrix} \quad (14)$$

Remember that \bar{v}_j indicates the first return to the section - in the same sense of traversal - after the point v_j . This method involves the integration of a system of 12 rather than 4 equations, because we don't move only the point, but also two vectors. This of course takes more computational time, but it gives us an accurate value for DP , since the error in DP is of the same order as that of the section points.

3. Application to an axisymmetric potential

As a first example let us take the axisymmetric logarithmic potential

$$\Phi_L = 0.5v_0^2 \ln(R_c^2 + R^2) \quad (15)$$

where v_o and R_c are constants taken, in our example, to be equal to 1. and 0.1 respectively (Binney & Tremaine 1987). The orbits of the main, circular family present no difficulty, so both finite differences and the variational equation method give roughly the same results. We can, however, compare the accuracy of the two methods by calculating the determinant of DP , which we will hereafter refer to as D . Its elements for finite differences are given by Eq. (2), while for the variational equation method by Eq. (14).

A perfectly accurate calculation would of course give $D = 1$, and for less accurate calculations the determinant will deviate more from unity than for more accurate ones. We calculated finite differences for $\Delta x = 10^{-3}$, 10^{-4} , 10^{-5} and 10^{-6} and we found that 10^{-3} gives the most accurate results. They are compared with those of the variational equation method in Fig. 1, which shows for both the value of $\log |D - 1|$. The difference in accuracy between the two methods is striking! The finite difference method gives an accuracy between 10^{-5} and 10^{-1} , while the accuracy of the variational equation method is bound only by the accuracy of the orbit calculation. It is this limiting accuracy that results also in the "quantisation" of the resulting values.

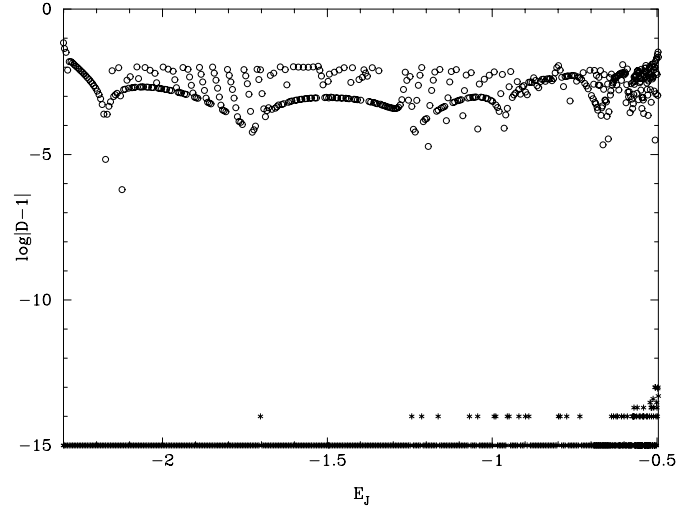


Fig. 1. Value of the $\log |D - 1|$ as a function of the Jacobi energy E_J for the axisymmetric logarithmic potential. The values obtained with finite differences are given by an open circle and those obtained with the variational equation method by a star.

Thus the variational equation method gives, in most cases, an accuracy of at least 10^{10} better than that obtained with finite differences.

The improvement in accuracy depends on the case considered and is more important in more chaotic regions.

4. Potential of a barred galaxy

In the previous section we discussed an example where the variational equation method gives quantitatively more accurate results. In other cases, however, the differences between the two methods can be even qualitative.

For our second example we will use the model 1 of Athanassoula (1992, hereafter A92). For most principal families the results of the two methods are in rough agreement, although the accuracy of the variational equation method is always better by at least as much as what we saw in the previous example. However, for families whose characteristic curve approaches the curve of zero velocity asymptotically, the differences can be much more important and there can be disagreement even as to whether a given orbit is stable or unstable. As an example let us take a Lagrangian family, the second one from the right in the lower panel of Fig. 2 of A92. We calculated its stability using finite differences as well as using the variational equation method and compare the results in Fig. 2.

Using the variational equation method we were able to show that this family is stable for as long as we could follow it. On the other hand finite differences with $\Delta x = 10^{-3}$ find that the orbits are unstable if $E_J > -126283$, while for $\Delta x = 10^{-4}$, $\Delta x = 10^{-5}$ and $\Delta x = 10^{-6}$ the values of the Jacobi constant for which the family changes from stable to unstable are -126150, -126000 and -126500 respectively. Thus finite differences give results that are qualitatively different from those of the variational equation method and that depend heavily on the

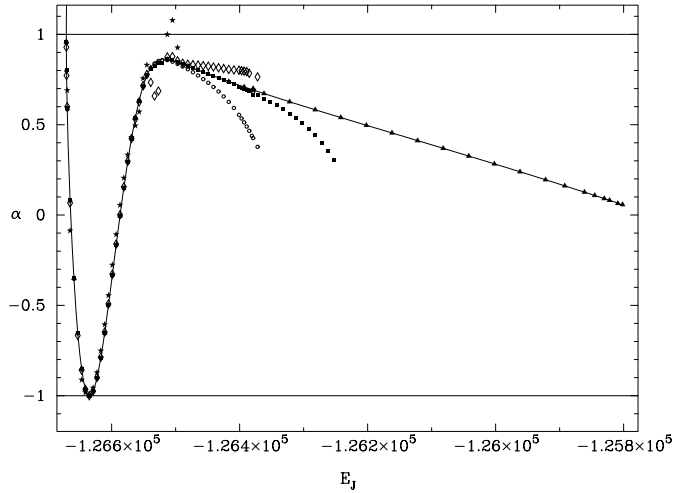


Fig. 2. Stability parameter α as a function of the Jacobi energy E_J for a Lagrangian family calculated in the model 1 of A92. The results obtained with finite differences and a Δx of 10^{-3} , 10^{-4} , 10^{-5} and 10^{-6} are shown respectively by open circles, filled squares, open diamonds and stars. Results obtained with the variational equation method are shown with a solid line and filled triangles. In order to make the figure clearer we have plotted symbols only at one out of every five calculated points. For finite differences we have only plotted points for which $|D - 1| < .5$. By applying a stricter criterion we can eliminate successively more points, but in this way we get no information on the stability of the higher energy orbits.

adopted value of Δx . With the help of Poincaré sections we confirmed that indeed this family is stable and therefore that it is the variational equation method that gives the correct result. We next repeated the calculation again using finite differences but this time keeping only the points for which the value of the determinant D is sufficiently close to unity and noted that in this way the erroneous values disappeared. Although we are thus able to remove erroneous values, we are not able to find the correct value for the stability parameter, unless we use the variational equation method.

5. Conclusions

In essence the difference between the two techniques is that finite differences involve the calculation of $\lim_{\Delta x \rightarrow 0}$ for every element in the Jacobian matrix of the Poincaré map. Numerically it is not possible to calculate the $\lim_{\Delta x \rightarrow 0}$, so we have to calculate this expression for different Δx in order to find the optimum one that gives the best approach to the limit. It is, however, well known that calculating the derivative from finite differences can be unstable, and that it may not be possible to obtain a good approximation of $\lim_{\Delta x \rightarrow 0}$. Thus this method may need several trials (i.e. more computing time), an *a posteriori* check of the results, and, in certain cases, it may not be possible to find an appropriate value of Δx .

The new method here calculates the Jacobian matrix accurately and doesn't depend on any Δx . It may involve some extra computing time because it involves the integration of more differential equations, but this is, more often than not, compensated

by the fact that it needs only one trial. Its assets are that it involves no tunable parameter, needs no *a posteriori* checking of the results and its accuracy is only limited by the method used to find the periodic orbits. In regular regions the two methods give similar results, although the variational equation method is always more accurate. In chaotic or "difficult" regions, however, e.g. regions involving small stable islands in a chaotic sea, using the variational equation method may prove essential for obtaining the correct value of the stability parameter. Furthermore, a study of such regions may be very time consuming and the fact that with the variational equation method one can find precisely the bifurcation points of new families can be a big help. We thus recommend it for all calculations such as the ones presented here.

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