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Macroscopic Schroedinger quantization of the early chaotic solar system

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Abstract. Classical mechanics is compatible with Schroedinger's equation without any approximation of the standard WKB semiclassical quantum type. In order to illustrate this property, we propose an exact Schroedinger theory for Nottale's macro-quantization of the solar system in which neither semiclassical approximations nor the probabilistic Born postulate are assumed. We point out the existence and properties of radial-velocity proto-planetary solitons that describe the swarm of chaotic orbits supposed to dominate at the later stage of the solar system formation, in accordance with Nottale's scale-relativity theory.

Key words: relativity – gravitation – chaos – quantization

1. Introduction

The Titius-Bode law is the most famous of the remarkable relationships among planetary and satellite parameters concerning the solar system. This simple geometric progression describes with good precision the distances of most, but not all, of the planets from the sun. Known for over two centuries, this law still lacks an explanation based upon physical laws. Although it has recently been shown that simply respecting both scale and rotational invariance can yield an endless collection of theoretical models predicting a Titius-Bode law, irrespective to their physical content (Dubrulle& Graner, 1994), early comparisons of the Titius-Bode law with the predictions of the Bohr theory for atoms immediately raised the question of the applicability of (some, at least) principles of Quantum Mechanics (*QM*) to orbital systems in astronomy (Corliss, 1986). About fifty years ago, or even more, the following ranking of the planets of the solar system according to Bohr's quantum number N and its related law $\bar{r} = a_B\,N^2$ became known with a weak percentage deviation for the observed mean distances $\bar{r}$ from sun if adopting the value $a_B = 0.0425\,A.U.$ for the Bohr radius a_B (Caswell, 1929; Penniston, 1930; Barnothy, 1946): Mercury ($N = 3$), Venus ($N = 4$), Earth ($N = 5$), Mars ($N = 6$), Planetoids ($N = 8$), Jupiter ($N = 11$), Saturn ($N = 15$), Uranus ($N = 21$), Neptune ($N = 27$). Bagby (1979) added the planet Pluto at $N = 31$. Then from Kepler's third law and the distance relation, the planet velocities were also expected to be quantized in $1/N$, and indeed it was found that they were in inverse proportion to simple integral numbers, while the periods of the planets were proportional to the cubes of the same integers (Malisoff, 1929). Moreover it was suggested that the innermost hypothetical planet *Vulcanus* expected to orbit at the distance $3\,10^7\,km$ from the sun, between the sun and Mercury, should rank $N = 2$ (Barnothy, 1946). Bagby (1979), in his extensive comparison of the Titius-Bode law with the Bohr atomic orbitals, even suggested that the Earth's moon might have been this intramercurial planet, since the capture possibility for the Moon was envisaged (Gerstenkorn, 1970).

Once the so-called 'inner solar system' (telluric planets) was quite reasonably given the increasing series of 'quantum Bohr numbers' $3 \leq N \leq 6$, which raises the problem of the existence of Vulcanus at $N = 2$, two questions remain unanswered: i) does anything correspond to $N = 1$ in the solar system? ii) why is the 'outer solar system' (i.e. J, S, U, N & P) described by Bohr numbers that are no more in sequence? The present paper provides a definite negative answer to the first question. Concerning the second question, one immediately notices that the mean quantum number interval between the planets of the outer solar system is 5. Therefore if a 'renormalized' quantum number $N' = N/5$ and a 'renormalized' Bohr radius $a'_B = 25a_B$ are defined, it seems that there is a two-stage quantization process according to the Bohr law written as $\bar{r}_N = a_B N^2 = a'_B N'^2 = \bar{r}_{N'}$ for $N \geq 11$. Hence Jupiter would now rank $N' = N/5 = 11/5 \sim 2$, and then Saturn ($N' = 3$), Uranus ($N' = 4$), Neptune ($N' = 5$) and Pluto at $N' = 6$ (note that, again, the $N = 1$ planet does not exist). Nottale (1993, 1996a,b, 1997) has proposed such a two-stage quantization in order to suggest a convincing explanation of the mass distribution in the solar system by a cascade mechanism for the planetesimals and their final accretion into planets.

But Nottale did more. He proposed an explanation for the amazingly good Bohr account of the main solar system parameters, which is indeed the great mystery of the problem (in fact, the situation concerning the 'macro-quantization' of the solar system right now is not far from what existed at the onset of the quantum description of the atomic world, where people had Balmer et al.'s very precise formulas about the spectral lines of, say, hydrogen, but lacked any convincing explanation for such good fits...). By applying his 'Scale Relativity' principle to the chaotic – and hence fractal – matter flow which is believed to have dominated the later stage of the solar system formation (Laskar, 1989) and by borrowing some technical tools from Nelson's Stochastic Quantum Mechanics (Nelson, 1966; Kyprianidis, 1992), Nottale was able to derive a macroscopic Schroedinger equation for the description of the Chaotic Proto-Solar-System, hereafter refered as the *CPSS* (Nottale 1993, 1996a,b, 1997). Therefore a convincing link was proposed for the first time between the CPSS and the Schroedinger equation, although the main scaling parameters of this latter (actually the Bohr radius a_B) had of course to be adapted to its macroscopic dimensions (hence the terminology *macro-quantization* used in the present paper).

Actually, a highly non-conventional approach to planetary dynamics that yields an equation very much like Schroedinger's one for the stationary states of n particles, with the same basic interpretation, was proposed as early as 1944 by Liebowitz. The corresponding theory drastically changes certain standard quantum mechanical ideas by constructing a "probability of presence" from first mechanical principles (Liebowitz, 1944). It is therefore amazing that one seems to get a much larger domain of relevance of the Schroedinger equation with respect to its reference atomic world as soon as one envisages new foundation principles for the corresponding quantum theory that remains otherwise (i.e. in all its full further technical developments) unchanged. Could it be a complete round-turn with respect to the early quantum theory where people tried to fit the classical laws of celestial mechanics at atomic scales? Could the Schroedinger equation actually yield some underlying physical principles to (part of) Classical Mechanics (CM) itself, in particular when chaos dominates and classical determinism disappears? We note with great interest the discovery, in the quite recent literature devoted to these questions, of deterministic chaos in the frame of the causal interpretation of QM (Parmenter & Valentine, 1995; Konkel & Makowski, 1998). Delivering definite answers to these fundamental questions is far too ambitious at the present state of the art, but the present paper intends to participate in this stimulating debate by pointing out indeed the existence of a macroscopic context for Schroedinger's equation.

2. Quantum probability and complex wavefunctions

Nottale's major statement is that the CPSS trajectories become nondifferentiable (and hence fractal) after their very short inverse Lyapounov exponent of about 5 Myr, thus forcing one to jump to a probabilistic description *à-la-Born/Schroedinger* (Nottale, 1993). This link between nondifferentiable particle trajectories (explicitely involving, or not, fractal space-time) and the Schroedinger equation, first pioneered by Feynman (Feynman & Hibbs, 1965), and shortly after followed by (amongst others) Nelson (1966), Abbot & Wise (1981), Ord (1983), Nottale (1989), Sornette (1990), is indeed a crucial progress in the conceptual frame of the Schroedinger equation, for it opens the possibility of applying the basic quantum rules, and amongst them the orbit and/or energy quantization formulas, to dynamical systems, the dimensions, energy and action of which lie far from those of the microscopical world. Schroedinger's equation is no longer restricted to atoms. It may concern much larger dynamical systems, such as astronomical systems in the present case.

Note that, recently, this link between the Schroedinger equation and fractal trajectories has been strengthened, although the two were not equated (Hermann, 1997).

2.1. Continuity equation

The question we wish to address in the present paper is the following: once we have the CPSS Schroedinger equation, what does it mean? How does the key-concept of quantum probability enter macroscopic quantization? Nottale's Scale-Relativity theory proposes that matter concentrates where the infinite set of geodesics of the fractal space-time get denser. Quantitatively, he relates this statement to the so-called Madelung-Bohm-DeBroglie equation of continuity (Madelung, 1926; Bohm, 1952a,b, 1953; DeBroglie, 1956; Nelson, 1966; Nottale, 1993, 1996a,b, 1997) which is but the imaginary part of the Schroedinger equation:

$$\frac{\partial \rho}{\partial t} + \frac{1}{\mu} \nabla . (\rho \nabla S) = 0 \quad , \tag{1}$$

once its wavefunction solution is assumed complex-valued:

$$\Psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)}\, e^{\frac{i}{\hbar} S(\mathbf{x}, t)} \quad . \tag{2}$$

The quantity $\rho = \Psi\Psi^*$ is Nottale's geodesics density, while μ is the (reduced, in the case of a central-field problem) particle mass. Making use of the continuity equation Eq. (1) in order to confirm the identification of ρ with a probability density, as Nottale does, is indeed very attractive and has been proposed by numerous authors (see Holland, 1993 for an extensive review). The pioneering works originate from Bohm et al.'s classical (causal) statistical interpretation of QM that has the dual sense of the probability distribution of the initial conditions characterizing an *ensemble* of particle trajectories in the configuration space, as well as of a real 'quantum-potential' field present in a *single* experiment (Bohm, 1952a,b, 1953; Bohm & Vigier, 1954; DeBroglie, 1956, Holland, 1993). However this statistical interpretation was criticized by Keller (1953) and Pauli (1953) amongst others, who showed that Eq. (1) is indeed a necessary condition for ρ to be a quantum probability density of the classical statistical type, but not a sufficient one.

2.2. Divergence of ρ for stationary states

There is another serious difficulty related to the identification of ρ with a probability density, and it is the following. In the case of a stationary and separable quantum state, the quantity $\rho = \Psi\Psi^*$ where Ψ is defined by Eq. (2) and $\nabla S \not\equiv 0$ is *not* integrable. It diverges to infinity at the boundaries of the system (Reinisch, 1994; 1997). This divergence emphasizes the ambiguity of the probabilistic Born postulate of QM which has given rise to many of the conceptual difficulties associated with the subject (Rae, 1992). Indeed, while on the one hand this postulate yields as a general statement that $\Psi\Psi^*$ is the density of probability to find the particle at a given position (Born, 1926), on the other hand, the Born probability density is rather $[\Re\Psi]^2$ than $\Psi\Psi^*$ in the case of stationary and separable quantum systems that ultimately reduce to 1-dimensional Schroedinger eigenproblems, precisely for the sake of normalization of the probability density (Reinisch, 1994; 1997).

Tackling this problem actually amounts to recovering Einstein's definition of a *real* understanding of quantum physics, namely that the probabilistic description of the whole quantum process should naturally (i.e. *self-consistently*) emerge from the 'complete' dynamical description of the system itself (Pais, 1982). In our opinion, there is no doubt that, whereas this point of view opened sophisticated debates in quantum microphysics by the endless controversy between the Copenhagen interpretation of QM and its causal alternative (Einstein & Born, 1969), it is the clue to any convincing macro-Schroedinger model of chaos. Hence the credo of our present approach: *if it is macro-Schroedinger, then it should be possible to tell it in classical terms.*

2.3. Schroedinger equation's irregular eigenmode

The goal of the present paper is to tentatively resolve the conflict between the Schroedinger equation and CM on the scale of the Solar System. The argument is that the Schroedinger spectrum is also the spectrum of the nonlinear Madelung system that is obtained from the Schroedinger equation by choosing an ansatz solution of the type Eq. (2) and then separating the real and imaginary parts (Madelung, 1926). The corresponding system of two coupled Ordinary Differential Equations (ODE's) yields a single nonlinear second-order ODE, known as the Ermakov-Milne-Pinney (EMP) equation (Ermakov, 1880; Milne, 1930; Pinney, 1950), in the stationary and separable case (Reinisch, 1994; 1997). This EMP equation is related to the linear second-order Schroedinger eigenproblem by a nonlinear superposition formula making use of the fundamental set of solutions of Schroedinger's equation (Milne, 1930; Common & Musette, 1997). Indeed, to each normalized (regular) eigenfunction u_N of the discrete Schroedinger spectrum that is labeled by the (main) quantum number N, there exists a second linearly independent solution v_N to the same eigenvalue E_N, which is, however, non-normalizable, or 'irregular' (Richter & Wuensche, 1996). We show that this irregular solution v_N (which is divergent to infinity at the boundaries of the system) allows us to give $\rho \equiv \Psi\Psi^*$ a clear local classical statistical sense in terms of the time interval that is spent by the system between two spheres of radius r and $r + dr$, although the quantity $\Psi\Psi^*$ obviously diverges at the boundaries of the system since $\Psi\Psi^* = u_N^2 + v_N^2$. Note that such a simple classical statistical meaning of ρ has been suggested by White (1931; 1934) in the early thirties.

2.4. Schroedinger equation's Hamilton-Jacobi dynamics

Therefore the present theory makes an extensive use of this irregular Schroedinger solution v_N that is completely discarded in the 'standard' (i.e. Schroedinger-Born) interpretation of QM. However there is a strong tendency in the very recent literature to re-introduce these irregular Schroedinger solutions in the investigation of the properties of the discrete (Richter & Wuensche, 1996; Leonhardt & Raymer, 1996) as well as of the continuum (Leonhardt & Schneider, 1997) energy spectrum.

The technical interest of taking into account the fundamental set of solutions to the Schroedinger equation is that one can build the general wavefunction solution of the type Eq. (2) as $\Psi_N = u_N + iv_N$. This yields the phase shift $S_N/\hbar = tan^{-1}(v_N/u_N)$ that is related to the actual momentum field $\mathbf{p}_N = \nabla S_N$ by use of the classical Hamilton-Jacobi equation. Therefore, to each value ζ in the configuration space that parametrizes the 'trajectory' of the system in the $\{u_N(\zeta)\,,\, v_N(\zeta)\}$ plane, there exists the polar angle $S_N(\zeta)/\hbar$ that unambiguously defines the actual momentum field $\mathbf{p}_N$. And the classical (i.e. causal) dynamics that is derived from this momentum field allows us to give $\Psi\Psi^*$ its local classical statistical meaning which, thus, extrapolates Born's postulate to the case where $\rho = \Psi\Psi^*$ is not integrable.

2.5. Schroedinger equation's new parameter

The constant Wronskian A of the fundamental set of solutions $\{u_N(\zeta)\,,\, v_N(\zeta)\}$ appears to be the *fundamental new parameter* of the present theory. It is *independent of the energy eigenvalue* E_N and its choice, as a mere constant of integration, is free. Its actual physical dimension is a matter flux. It has been shown (Reinisch, 1994; 1997) that standard microphysics amounts to taking $A \sim 0\,;\, A \neq 0$. On the other hand, the present paper shows that the value $A \sim 1$ (in appropriate reduced units) corresponds to the macro-Schroedinger CM.

One might wonder whether the present theory is not a mere remake of the several Madelung-Bohm-DeBroglie hydrodynamical pictures of QM (see Holland, 1993, for an extensive account of these theories). The answer is clearly negative, for there is a crucial difference between *all* these stationary hydrodynamical theories and the present one. In the case of the Kepler problem, for instance, Madelung et al. assume $A = 0$ (Holland, 1993). Then $v_N \equiv 0$ and there is no irregular mode in the corresponding Madelung description. Therefore this latter trivially degenerates into the standard Schroedinger eigenproblem related to the single remaining regular eigenfunction u_N. As a consequence, the classical dynamical information that is provided by the gradient of the phase $S_N/\hbar = tan^{-1}(v_N/u_N)$ is *lost* and, in order to *replace* it, one is then forced to *artificially introduce Born's postulate* like a sort of *deus ex machina* (Reinisch, 1994).

On the other hand, when the Wronskian A is non-zero, it can be shown that the Born postulate can indeed be recovered by the purely classical interpretation of the steady-state matter-flow dynamics that is related to the momentum field $\mathbf{p}_N = \nabla S_N$

(Reinisch, 1997). Therefore the Born postulate is actually contained in the non-degenerated ($A \neq 0$) Madelung system.

Let us stress as a final remark that the Madelung-DeBroglie-Bohm "causal" description which is used in the present paper is rigorously equivalent to the complex Hamilton-Jacobi equation that is provided by Nottale's Scale Relativity theory, although their conceptual foundations are quite different. Indeed, while the former makes use of real quantities (amplitude $\sqrt{\rho}$ and action S) in the complex description (Eq. (2)) of the wave function Ψ, the latter simply assumes $\rho \equiv 1$ and S complex. The interest of such a formal complex action is its gradient. It yields a complex momentum field which, in turn, defines the complex covariant derivative operator associated with the scale covariance postulate (Nottale, 1997; Pissondes, 1997).

3. Madelung's central-field problem

For the sake of simplicity, we shall continue to make use of standard QM notations (hence the reduced quantum of action $\hbar = h/2\pi$). The translation of the results in terms of macroscopic quantization will simply be made in Sect. 7 by changing $\hbar$ into $\mu a_B w_0$ (cf. Eq. (62)). Here μ is the (reduced) mass of the particle and a_B is the Bohr radius of the system. The quantity w_0 has the dimension of a velocity. It is the major concept that is introduced by Nottale in his Scale-Relativistic treatment of the CPSS (Nottale 1993, 1996a,b, 1997). Recent observations of the Tifft effect suggest that it has an universal value quite close to $145\, km/s$ (Guthrie & Napier, 1996; Nottale, 1996a).

3.1. Micro-versus macroscopic quantization

Chaotic planetary systems and atoms are *not* identical copies of the same thing on different scales, although the present paper aims to show that both may derive their respective physical properties from the *same* Schroedinger equation. In the case of the atom, the quantum of action, as an universal constant, does of course not depend on the mass, but the Bohr radius $a_B = \hbar^2/\mu\alpha$ (where α defines the the Keplerian central potential $V(r) = -\alpha/r$) obviously does. In the case of the CPSS, the situation is reverse: the 'macro quantum of action' is now the mass-dependent quantity $\mu a_B w_0 \propto \mu$ while the 'macro Bohr radius' yields $a_B = 0.042\, A.U.$ as emphasized in Sect. 1, regardless of the planet that is under consideration. This simply means that the quantization will be made in terms of a *velocity field* describing the chaotic matter flow in the CPSS instead of a *momentum field* (cf Eqs. (63–64) and (69–72)). And this is indeed expected since the parallel we draw between the atom and the solar system has an obvious limit: while the orbiting particles in the atom (namely the electrons) have all the same mass, such is of course not the case for the planets of the solar system.

There is another huge difference that concerns the build-up of the CPSS with respect to standard atom theory. In first approximation, the way the atoms are built is simply sequentially filling up all the available quantum 'boxes' (which are characterized by the three quantum numbers N, l and m) by couples of electrons, because of the spin (hence the periodic table). Once the lower-energy 'quantum-boxes' are filled out, this very situation immediately affects the energy balance for the next available outer-electron couples which are to fill out the remaining boxes (obviously of higher energy) because these outer electrons do feel less Coulomb attraction from the nucleus, due to the partial electrostatic screening opposed by the already existing inner electrons.

Such is clearly not the case for the solar system where the inner orbiting planets do not seriously affect the gravitational attraction force between any outer planet and the sun. Moreover, although the planet spin and the Pauli exclusion principle have explicitely been considered in early investigations with the amazingly correct prediction that Venus' revolution should be retrograde (Barnothy, 1946), the interpretation of such spin quantization phenomenon in the planetary solar system remains problematic. Therefore, one should not be tempted to look for a sequential filling up of a sort of planet periodic table in the CPSS. The macroscopic Schroedinger theory should be indeed, as clearly stated by Nottale in his Scale-Relativity theory, a *global* theory of chaos: what will be quantized does not really concern *individual* (particle-like) physical properties such as energy, action and spin, as in (micro) QM; rather, it will mostly concern extended field properties such as the *CPSS velocity field of the matter flow*.

We now want to show that the independent planet histories occurring while they are being formed as stationary patterns in the chaotic stage of the proto solar system can be related to the phenomenon of *static solitons*. Indeed, solitons (and in the present case, one-dimensional radial-velocity solitons) are nonlinear fields which, due to their integrability by use of the inverse-scattering-transform, preserve their shape and, more generally, their identity during all their dynamical history, whatever the number of collisions, interactions and/or weak perturbations affecting them (Bullough & Caudrey 1980; Dodd et al., 1982; Reinisch, 1992). This dynamical stability is actually due to the one-to-one relationship between the set of soliton parameters and the eigenvalues of a corresponding (linear) ODE. In the following sub-sections, we now want to exhibit that relationship which is the hall-mark of the soliton phenomenon.

3.2. The Ermakov-Milne-Pinney (EMP) nonlinear differential equation

Consider a stationary quantum state described by its complex-valued wavefunction:

$$\Psi(\mathbf{x}, t) = a(\mathbf{x}) e^{\frac{i}{\hbar} S(\mathbf{x})}\, e^{-\frac{i}{\hbar} E t} \quad , \tag{3}$$

in agreement with Eq. (2) (where the amplitude $a = \sqrt{\rho}$ and the phase S are now real-valued functions of the space variable $\mathbf{x}$ only). Note that this ansatz agrees with Scale Relativity where $\Psi \propto e^{iS'/\hbar}$ in which S' is complex-valued ($S' = (\hbar/i)\, Log\, a + S$). The stationary Schroedinger equation corresponding to the applied potential $V(\mathbf{x})$ and the energy eigenvalue E yields the following Madelung system of two coupled PDE's:

$$-\frac{\hbar^2}{2\mu}\Delta a+\left[V-E+\frac{1}{2\mu}\left(\nabla S\right)^2\right]a\;=\;0\quad, \tag{4}$$

$$\nabla\cdot\left[a^2\nabla S\right]\;=\;0\quad. \tag{5}$$

Now assume that $V(\mathbf{x})\equiv V(r)$ is a confining central-field potential where r is the value of the radius. Using the spherical polar coordinates and taking into account both the single-valuedness of the wave function at $\varphi=2m\pi$ and the spherical symmetry of the system (all meridian planes through the φ-axis are equally probable), we have:

$$a(\mathbf{x})\equiv a(r,\theta)\;;\quad S(\mathbf{x})\equiv S(r,\theta,\varphi)\equiv\tilde{S}(r,\theta)+m\hbar\varphi\,, \tag{6}$$

where m is any integer. Then Eq. (5) becomes:

$$\nabla\cdot\left[a^2\nabla S\right]\;=\;\frac{1}{r^2}\frac{\partial}{\partial r}\left[a^2r^2\frac{\partial}{\partial r}\tilde{S}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[a^2\sin\theta\frac{\partial}{\partial\theta}\tilde{S}\right]=0\quad. \tag{7}$$

This equation defines the two following constants of integration:

$$a^2r^2\frac{\partial}{\partial r}\tilde{S}\;=\;K_1(\theta)\quad, \tag{8}$$

$$a^2\sin\theta\frac{\partial}{\partial\theta}\tilde{S}\;=\;K_2(r)\quad, \tag{9}$$

while Eqs. (4) and (6) yield:

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}a\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}a\right]\right]+\left[V-E+\frac{1}{2\mu}\left[\left(\frac{\partial\tilde{S}}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\tilde{S}}{\partial\theta}\right)^2+\frac{m^2\hbar^2}{r^2\,\sin^2\theta}\right]\right]a=0\quad. \tag{10}$$

The treatment of the angular part of Eq. (10) is fairly standard (White, 1931, 1934; Messiah, 1962; Rae, 1992). Let $P_l^{|m|}[\cos\theta]$ (with $|m|\le l$) be the associated Legendre polynomial defined by:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dP_l^{|m|}}{d\theta}\right]+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P_l^{|m|}=0\,, \tag{11}$$

and normalized according to the convention:

$$2\pi\int_0^\pi d\theta\left[P_l^{|m|}[\cos\theta]\right]^2\sin\theta=1\quad. \tag{12}$$

Now assume the following particular separation of the variables r and θ:

$$a(r,\theta)=R(r)P_l^{|m|}[\cos\theta]\quad. \tag{13}$$

Then Eqs. (8 – 11) yield:

$$\frac{1}{r^2}\left[\frac{d}{dr}\left[r^2\frac{d}{dr}R(r)\right]-l(l+1)R(r)\right]+\frac{2\mu}{\hbar^2}(E-V)R(r)-\frac{1}{\hbar^2r^2R^3(r)\left[P_l^{|m|}[\cos\theta]\right]^4}\left[\frac{K_1^2(\theta)}{r^2}+\frac{K_2^2(r)}{\sin^2\theta}\right]=0\,. \tag{14}$$

Now because of the nonlinear term $\propto\left[P_l^{|m|}[\cos\theta]\right]^{-4}$ in Eq. (14) the wave equation does not *a priori* separate. The unique transformation which restitutes this separation is defined by:

$$K_1^2(\theta)=K_1^2\left[P_l^{|m|}[\cos\theta]\right]^4\quad, \tag{15}$$

where K_1 is any (say, positive) constant that has the dimension of a momentum flux (momentum times surface), and

$$K_2(r)\equiv0\quad. \tag{16}$$

Then defining:

$$\Xi(r)=rR(r)\quad, \tag{17}$$

and

$$W(r)=V(r)+\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\quad, \tag{18}$$

Eqs. (14–16) become:

$$\frac{d^2}{dr^2}\Xi(r)+\frac{2\mu}{\hbar^2}\left[E-W(r)\right]\Xi(r)-\left(\frac{K_1}{\hbar}\right)^2\frac{1}{\Xi^3(r)}=0\,. \tag{19}$$

Adopting the appropriate dimensionless radial variable:

$$\zeta=r\sqrt{\frac{2\mu(-E)}{\hbar^2}}\quad, \tag{20}$$

(since $E<0$ and $E=0$ for $r=\infty$), Eq. (19) becomes:

$$\ddot{\Xi}+\left[-1+\frac{\tilde{W}}{E}\right]\Xi-\frac{A^2}{\Xi^3}=0\quad, \tag{21}$$

where $\ddot{\Xi}=d^2\Xi/d\zeta^2$, $\tilde{W}(\zeta)=W\left[r(\zeta)\right]$ and

$$A=\frac{K_1}{\sqrt{2\mu(-E)}}\quad. \tag{22}$$

4. Classical interpretation of Schroedinger equation

4.1. Milne's nonlinear superposition formula

Eq. (21) is a nonlinear Ermakov-Milne-Pinney (EMP) ODE (Ermakov, 1880; Milne, 1930; Pinney, 1950) depending on the parameter A^2 which, for a stationary system, is the dimensionless constant of integration related to K_1 (cf. Eqs. (8), (15) and (22)). It can be recast into the well-known real-valued Schroedinger radial eigenproblem (Landau, 1966):

$$\ddot{u}_E+\left[-1+\frac{\tilde{W}}{E}\right]u_E=0\quad, \tag{23}$$

by use of the following transformation (Milne, 1930; Alijah et al., 1986; Reinisch, 1994):

$$u_E(\zeta)\;=\;\Xi_E^A(\zeta)\cos\left[S_E^A(\zeta)/\hbar\right]\;=\;\frac{1}{2}\Xi_E^A(\zeta)\left[e^{iS_E^A(\zeta)/\hbar}+e^{-iS_E^A(\zeta)/\hbar}\right]\quad, \tag{24}$$

through the ‘nonlinear phase’

$$S_E^A(\zeta)/\hbar = A\int_B^\zeta \frac{d\zeta'}{\left[\Xi_E^A(\zeta')\right]^2} \quad , \tag{25}$$

that depends on the arbitrary initial phase related to the parameter B. The dependence of the solution $\Xi(\zeta)$ to the nonlinear ODE (21) on the independent parameters E and A has been highlighted by ad-hoc labels.

By defining the irregular (i.e. exploding) solution:

$$v_E^A(\zeta) = \Xi_E^A(\zeta)\sin\left[S_E^A(\zeta)/\hbar\right] \quad , \tag{26}$$

to the Schroedinger equation (23), Eqs. (24) and (26) yield the following nonlinear superposition formula:

$$\Xi_E^A(\zeta) = \sqrt{u_E^2(\zeta) + \left[v_E^A(\zeta)\right]^2} \quad . \tag{27}$$

Therefore the solution Ξ_E^A to the nonlinear ODE (21) is defined by the superposition of the *regular* Schroedinger solution u_E (which is exponentially convergent because of the choice of E as an eigenvalue) and the *irregular* Schroedinger solution v_E^A (which is exponentially divergent at the bounday of the system). These two functions u_E and v_E^A form a fundamental set of solutions to the Schroedinger equation (23) (Leonhardt & Raymer, 1996; Richter & Wunsche, 1996). Therefore their Wronskian $u_E\dot{v}_E^A - \dot{u}_E v_E^A$ remains constant with respect to the spatial variable ζ. Moreover, this Wronskian invariant is equal to the constant of integration (22) (Milne, 1930):

$$u_E(\zeta)\frac{d\,v_E^A(\zeta)}{d\zeta} - v_E^A(\zeta)\frac{d\,u_E(\zeta)}{d\zeta} \equiv A \quad . \tag{28}$$

This yields the alternative definition of the fundamental free parameter A of our theory as the invariant Wronskian of the stationary Schroedinger equation (23).

Eqs. (25–28) show that the phase effects in a stationary quantum system are unambiguously related to the existence of the irregular solution (26) of the Schroedinger equation (23). We recover a general property of 1-dimensional stationary QM (Reinisch, 1994, 1997): a complete physical (i.e. dynamical) description of the eigenstate (3) demands the account of quantum phase effects; and these phase effects demand to take into account not only the *regular* (normalized) eigenfunction u_E that defines the particular discrete eigenvalue E, but also the *irregular* (divergent) solution v_E^A related to u_E by the Wronskian invariant (28). Then the ‘nonlinear phase’ $S_E^A(\zeta)/\hbar$ defined by Eq. (25), which is simply the polar angle of the ‘trajectory’ of the system in the complex plane $\{u_E\,,\,v_E^A\}$, allows us to build the momentum field $\mathbf{p} = \nabla S_E^A$ of the classical Hamilton-Jacobi type as shown below.

4.2. Hamilton-Jacobi definition of the radial matter flow

The real-valued regular radial Schroedinger eigenfunction $u_E(\zeta)$ appears through Eq. (24) to be the steady-state superposition of two unbound problems that consist of incoming and outcoming partial-scattering radial waves along the radial degree of freedom ζ. Hence the divergence of $|\Psi|^2$. Moreover the corresponding two-branch DeBroglie momentum field $p_\pm^{E,A}(\zeta)$, which describes the radial velocity field in the two equally probable outcoming and incoming radial directions, is simply the gradient of the action $\pm S_E^A$ of each of these waves, namely

$$\begin{aligned} p_\pm^{E,A}(\zeta) &= \pm\frac{dS_E^A}{dr} = \pm\frac{\sqrt{2\mu(-E)}}{\hbar}\frac{dS_E^A}{d\zeta} \\ &= \pm\frac{K_1}{\left[\Xi_E^A(\zeta)\right]^2} \quad , \end{aligned} \tag{29}$$

(cf. Eqs. (20), (22) and (25)).

This is the fundamental result of our theory and it defines the nonlinear radial-velocity soliton field:

$$c_{(r)}[\zeta(r)] = \left|\frac{dr}{dt}\right| = \frac{|\boldsymbol{p}_{(r)}(r)|}{\mu} = \frac{K_1}{\mu\left[\Xi_E^A[\zeta(r)]\right]^2} \quad , \tag{30}$$

in terms of the (square of the) nonlinear eigenstate Ξ_E^A that is the solution to the EMP ODE (21), i.e. in terms of the (squares of the) regular eigenfunction u_E *and* the irregular solution v_E^A to the Schroedinger eigenproblem (23), as shown by Eq. (27).

As a matter of fact, Ξ_E^A never vanishes because of Eqs. (27–28) and it always diverges at the boundary of the system, due to the presence of the irregular mode v_E^A in Eq. (27). As a consequence, the momentum field (29) has indeed a soliton profile whose ‘wings’ do actually describe the tunnel effect. Observe finally that Eq. (29) is but the well-known classical Hamilton-Jacobi definition:

$$\boldsymbol{p} = \nabla S \quad , \tag{31}$$

of the radial momentum field in terms of the action S defined by Eqs. (3) and (6). Indeed, by use of the spherical polar coordinates, we have:

$$\begin{aligned} &\boldsymbol{p}_{(r)} = \frac{\partial S}{\partial r} = \pm\frac{K_1}{\left[\Xi_E^A\right]^2} = p_\pm^{E,A} \quad ; \\ &\boldsymbol{p}_{(\theta)} \equiv 0 \quad ; \quad \boldsymbol{p}_{(\varphi)} = \frac{m\hbar}{r\sin\theta} \quad , \end{aligned} \tag{32}$$

(cf. Eqs. (6), (8), (13), (15) which provides the $\pm$ undeterminacy, and Eq. (17)). Note that the two last expressions are not to be confused with $\partial S/\partial\theta$ and $\partial S/\partial\varphi$ which are the canonical momenta respectively conjugate to the spherical polar coordinates θ and φ. Only $\boldsymbol{p}_{(r)}$ equals $\partial S/\partial r$, which will be of great importance for the quantization of the system along the radial degree of freedom (cf. Sect. 5 below).

Eq. (30) yields:

$$\left[\Xi_E^A[\zeta(r)]\right]^2\,dr = \frac{K_1}{\mu}\,dt \quad . \tag{33}$$

Therefore $\left[\Xi_E^A[\zeta(r)]\right]^2\,dr$ measures the actual time interval that is spent by the particle between the spheres of radii r and $r + dr$ during either its incoming or its outcoming radial motion described by the momentum field (29–32). In terms of $a^2 = |\Psi|^2$ (cf. Eq. (3)), we have (since $d\tau = dxdydz = r^2\sin\theta drd\theta d\varphi$):

$$a^2 d\tau = \frac{K_1(\theta)}{r^2\,|\boldsymbol{p}_{(r)}|}\,d\tau = K_1\frac{\left[P_l^{|m|}\right]^2}{|\boldsymbol{p}_{(r)}|}\sin\theta dr\,d\theta\,d\varphi \quad , \tag{34}$$

(cf. Eqs. (8), (15) and (32)). Integrating Eq. (34) over the spherical polar angles θ, φ and taking into account the normalization condition (12), together with Eqs. (3) and (32), yields:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta |\Psi|^2 d\tau = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta [a^2 d\tau] = \frac{K_1}{|\boldsymbol{p}_{(r)}|} dr = \left[\Xi_E^A(\zeta)\right]^2 dr \quad . \quad (35)$$

Hence, by use of Eq. (33), we finally obtain:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta |\Psi|^2 d\tau = \frac{K_1}{\mu} dt \quad , \quad (36)$$

This demonstrates the (local) classical statistical information born by the wavefunction square modulus $|\Psi|^2$ in terms of the actual time interval dt that is being spent by the particle between the spheres of radii r and $r + dr$. Moreover, Eq. (34) displays the expected angular dependence of the statistical time-of-stay dt if the averaging over the polar angles is not performed.

4.3. Schroedinger equation and classical mechanics

Let us first briefly recall the present state of the art concerning the link between QM and CM. This is basically the so-called semi-classical limit of QM, and it is described by the WKB (Wentzel, Kramers, Brillouin) approximation that was elaborated the same year as the discovery of the Schroedinger equation itself (Messiah, 1962; Landau & Lifchitz 1966; Rae, 1992).

In the WKB description of QM, one *assumes* that Ψ is a function of $\hbar$ and that its phase may be asymptotically expanded as a polynomial in $\hbar$. Inserting this polynomial into the Schroedinger equation and equating powers of $\hbar$ yields a ‘classical wavefunction’ that is compatible with the classical equation of motion obtained from Eqs. (4) and (31) by neglecting the second-derivative term in Eq. (4) ($\propto \hbar^2$). The WKB approximation works all the better as the action S of the system is large compared with the quantum of action h, or equivalently as the energy eigenvalue E is large compared with the average energy gap between the energy levels.

It is, however, well-known that the conditions of validity of the WKB approximation are neither necessary nor sufficient to obtain classical motion: the basic drawback with the WKB method is that it attempts to formulate the classical limit in terms of properties of the external potential and the DeBroglie wavelength which do not make reference to the quantum state Ψ of the system (Holland, 1993). It is indeed fairly obvious that not all physically relevant Schroedinger wavefunctions Ψ do vary slowly within the space of a DeBroglie wavelength, and hence that their corresponding second-derivative terms in Eq. (4) cannot always be regarded as small compared with the rest of this equation. There is formally in QM a lack of such a parameter in the WKB approximation that would allow one to adjust the wavefunction amplitude to the WKB conditions of a slowly varying profile within a DeBroglie wavelength. Said otherwise: in standard QM, there is a formal missing link with CM.

We believe that this missing link can be provided by the irregular Schroedinger mode v_E^A that yields the additional parameter A in accordance with Eq. (28). Indeed, in the classically allowed region, the art of using the Milne transformation (23–28) is to choose both the free Wronskian parameter A and the initial conditions of the EMP equation (21) for Ξ_E^A in such a way that:

i) the normalized Schroedinger eigenfunction u_E satisfies regular physical boundary conditions;
ii) the mode v_E^A that is irregular at the boundary of the system oscillates just out of phase with u_E so as to keep the nonlinear eigenstate Ξ_E^A smooth and slowly varying (Milne, 1930; Alijah et al., 1986; Reinisch, 1994).

On the other hand, in the classically forbidden region, Ξ_E^A is obviously dominated by the exploding irregular solution v_E^A (cf. Eq. (27)). This yields the tunnel effect by use of Eqs. (29–32): there is indeed a non-zero, although vanishingly small, momentum field $p_\pm^{E,A}$ in the classically forbidden region where $\Xi_E^A \sim v_E^A \to \infty$.

4.4. Choice of initial conditions

The technical procedure that performs the Milne transformation is the following. Assume that, for a particular value of the parameter $A = A_*$, a specific choice of the initial conditions $\Xi_E^{A_*}(\zeta_*) = \Xi_*$ and $\dot{\Xi}_E^{A_*}(\zeta_*) = \dot{\Xi}_*$ at a given $\zeta = \zeta_*$ is performed that keeps the nonlinear mode $\Xi_E^{A_*}(\zeta)$ smooth and slowly varying in the classically allowed region. Therefore: $\ddot{\Xi} \sim 0$ there and Eq. (21) yields:

$$\Xi_E^{A_*}(\zeta) \sim \Xi_{cl}(\zeta) = \frac{\sqrt{A_*}}{\left[-1 + \dfrac{\tilde{W}(\zeta)}{E}\right]^{1/4}} \quad . \quad (37)$$

We shall call the right-hand-side of Eq. (37) ‘the classical wavefunction’ for reasons that are explained below.

The problem is now to determine A_*, ζ_*, Ξ_* and $\dot{\Xi}_*$.

Eqs. (22) and (29–32) define the reduced radial momentum field throughout the classically allowed region:

$$\frac{p_\pm^{E,A_*}(\zeta)}{\sqrt{2\mu(-E)}} \sim \pm \frac{A_*}{\Xi_{cl}^2(\zeta)} = \left[-1 + \frac{\tilde{W}(\zeta)}{E}\right]^{1/2} \quad . \quad (38)$$

Obviously, this yields the classical equation of motion (hence the label ‘cl’ for the function defined by Eq. (37)):

$$\frac{\left[p_\pm^{E,A_*}(\zeta)\right]^2}{2\mu} + \tilde{W}(\zeta) \sim E \quad . \quad (39)$$

Now recall that the normalized Schroedinger eigenfunction u_E, associated with the corresponding discrete eigenvalue E, satisfies the Schroedinger equation (23) with prescribed (regular) boundary values. Therefore this mode is completely defined and, in particular, we have $u_E(\zeta_*) = \gamma$ and $\dot{u}_E(\zeta_*) = \beta$ where γ and β are two finite real values. We shall regard the solution of the ODE (21) as an initial-value problem, according to the following choice of parameters:

$$\Xi_* = \Xi_E^{A_*}(\zeta_*) = \Xi_{cl}(\zeta_*) \quad ; \quad \dot{\Xi}_* = \dot{\Xi}_E^{A_*}(\zeta_*) = 0 \quad , \tag{40}$$

(thus we demand a local extremum of the nonlinear amplitude $\Xi_E^{A_*}$ at the abscissa ζ_*). Eqs. (27–28) and (40) yield the following system of three equations for the definitions of the two unknown $v_E^A(\zeta_*)$ and $\dot{v}_E^A(\zeta_*)$ at $\zeta = \zeta_*$:

$$\begin{aligned} &\gamma\beta + v_E^A(\zeta_*)\dot{v}_E^A(\zeta_*) \;=\; 0\,; \\ &\gamma\dot{v}_E^A(\zeta_*) - \beta v_E^A(\zeta_*) \;=\; A\,;\; \gamma^2 + \left[v_E^A(\zeta)\right]^2 = \Xi_*^2 \quad . \end{aligned} \tag{41}$$

There is no solution, except for the following particular choice of the parameters:

$$\gamma = \Xi_* \quad ; \quad \beta = 0 \quad , \tag{42}$$

which yields:

$$\begin{aligned} &u_E(\zeta_*) \;=\; \Xi_*\,; \quad \dot{u}_E(\zeta_*) = 0 \\ &v_E^A(\zeta_*) \;=\; 0\,; \quad \dot{v}_E^A(\zeta_*) = \frac{A}{\Xi_*} \quad . \end{aligned} \tag{43}$$

Therefore the initial conditions for the definition of the nonlinear mode $\Xi_E^{A_*}(\zeta)$ are taken at a local extremum $\zeta = \zeta_*$ of both the normalized Schroedinger eigenfunction $u_E(\zeta)$ and this nonlinear mode, with a common amplitude given by Eqs. (37) and (40) (Reinisch, 1994; 1997).

4.5. Classical quantization

The angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ of the system, defined in accordance with Eq. (31), yields:

$$L_z = x\,\boldsymbol{p}_{(y)} - y\,\boldsymbol{p}_{(x)} = x\,\frac{\partial S}{\partial y} - y\,\frac{\partial S}{\partial x} = \frac{\partial S}{\partial \varphi} = m\hbar \quad , \tag{44}$$

by use of Eqs. (6) and of the spherical polar coordinates. By considering for the solar system the largest possible z-component of the angular momentum $\boldsymbol{L}$, namely $m\hbar = l\hbar = (N-1)\hbar$ where N is the main quantum number of the Kepler system, and by changing $\hbar$ into $\mu a_B w_0$ (cf. Eq. (62)), Eq. (44) allows us to recover Nottale's remarkable conclusion that it is the quantization of the ratio L_z/μ, rather than that of L_z, which yields the distribution of angular momentum in the solar system (Nottale, 1993, 1996a,b, 1997).

On the other hand, the quantization along the radial degree of freedom yields, by use of Eq. (29):

$$\begin{aligned} \oint \boldsymbol{p}_{(r)}\,dr \;&=\; \int_0^\infty p_+^{E,A}(r)\,dr + \int_\infty^0 p_-^{E,A}(r)\,dr \\ &=\; A\hbar\left[\int_0^\infty \frac{1}{\left[\Xi_E^A\right]^2}d\zeta + \int_\infty^0 \frac{-1}{\left[\Xi_E^A\right]^2}d\zeta\right] \\ &=\; 2A\hbar\int_0^\infty \frac{1}{\left[\Xi_E^A\right]^2}d\zeta = (n_r+1)\,h \quad \forall A \quad , \end{aligned} \tag{45}$$

where the radial quantum number $n_r = 0,\, 1,\, 2,\, 3,\, ...$ is equal to the number of nodes of the Schroedinger eigenfunction u_E defined by Eq. (23) (Reinisch, 1994; 1997).

This radial quantization is of course reminiscent of the 'good old' Bohr-Sommerfeld quantization that reads $\oint \boldsymbol{p}_{(r)}\,dr = (n_r + 1/2)h$ (White, 1931, 1934; Messiah, 1962; Landau & Lifchitz 1966). But recall that this latter is obtained as the result of the semiclassical WKB approximation of QM (cf. Sect. 4.3) that assumes the action S of the system to be large compared with the quantum of action h (or the system energy E to be large compared with the energy gap between two adjacent levels), while such approximations are *not* being done here. Therefore the radial quantization Eq. (45) is *exact*, regardless of the values of S or E. Hence the remarkable conclusion that the 'vacuum state' corresponding to $n_r = 0$ bears one single quantum of action h instead of the well-known semiclassical value $h/2$. This is due to the proper account of the tunnel-effect divergence of the radial nonlinear mode Ξ_E^A at the boundary of the system (Reinisch, 1994; 1997).

5. The Kepler-Madelung system

Let us illustrate the above theory by considering the Kepler problem:

$$V(r) = -\frac{\alpha}{r} \quad . \tag{46}$$

5.1. The nonlinear radial EMP equation

In terms of the reduced variable ζ defined by Eqs. (20), we have:

$$\frac{\tilde{W}}{E} = C_E\frac{1}{\zeta} - l(l+1)\frac{1}{\zeta^2} \quad , \tag{47}$$

where

$$C_E = \frac{\alpha}{\hbar}\sqrt{\frac{2\mu}{(-E)}} \quad . \tag{48}$$

It is well-known that there exists a regular (normalized) eigenstate u_E for Eq. (23) provided that the energy E takes any of the following negative discrete eigenvalues (White, 1931, 1934; Rae, 1992):

$$E = E_N = -\frac{\mu\alpha^2}{2\hbar^2 N^2} \quad \longrightarrow \quad C_E = C_N = 2N \quad . \tag{49}$$

The radial quantum number n_r that defines the quantum condition (45) is equal to (Messiah, 1962):

$$n_r = N - l - 1 \quad , \tag{50}$$

where the azimuthal quantum number l related to the orbital angular momentum is defined by:

$$0 \,\le l \le N-1 \quad . \tag{51}$$

When the energy parameter E is tuned in accordance with Eq. (49), Eq. (21) becomes:

$$\begin{aligned} &\ddot{\Xi}_N^A + \left[-1 + \frac{2N}{\zeta} - \frac{l(l+1)}{\zeta^2}\right]\Xi_N^A - \frac{A^2}{\left[\Xi_N^A\right]^3} = 0\,; \\ &N \ge 1\,;\; l = 0,\, 1,\, ...,\, N-1 \quad . \end{aligned} \tag{52}$$

Equivalently, one can measure the radius r in units of the Bohr radius:

$$a_B = \frac{\hbar^2}{\mu\alpha} \quad . \tag{53}$$

Therefore, defining:

$$z = \frac{r}{a_B} = N\,\zeta \quad , \tag{54}$$

Eq. (52) becomes:

$$\frac{d^2}{dz^2}\,\Xi_N^A + \left[-\frac{1}{N^2} + \frac{2}{z} - \frac{l(l+1)}{z^2}\right]\Xi_N^A - \frac{A^2}{N^2\left[\Xi_N^A\right]^3} = 0 \quad , \tag{55}$$

while the discrete energy levels (49) now read:

$$E_N = -\frac{\alpha}{2a_B N^2} \quad . \tag{56}$$

5.2. The pseudo-circular case $l = N - 1$

For $l = N - 1$, there are no radial nodes ($n_r = 0$ according to Eq. (50)) and, hence, a *single* maximum for the regular eigenfunction u_E that is located at:

$$\zeta_* = N \quad \rightarrow \quad z_* = N^2 \quad \rightarrow \quad r_* = N^2\,a_B \quad , \tag{57}$$

(cf. Eq. (54)). Indeed the abscissa ζ_* that defines the initial conditions Eqs. (40–43) is then given by Eq. (57).

Moreover, the radial action (45) has the remarkable property to be independent of both the Wronskian A and the level N. Therefore it appears as a fundamental invariant of the pseudo-circular Kepler system since it is equal to one single quantum of action h:

$$\oint \boldsymbol{p}_{(r)}\,dr = h \quad \forall A\,\&\,N \quad . \tag{58}$$

The case where the azimuthal quantum number l has its largest possible value $N - 1$ is of importance here for it semiclassically corresponds to the quasi-circular orbits which one would like to associate with the planetary orbits of the solar system (Nottale, 1993, 1996a,b, 1997). However this correspondence only plays for large quantum numbers N. For such low values of N as those being considered here (for instance $N \leq 6$ for the 'inner solar system'), it is irrelevant to make use of it. This point is clearly illustrated by White (1934) who displays the four models used by different investigators in order to accommdate the newly discovered Schroedinger equation with classical orbit descriptions. Their 'effective azimuthal quantum number' k, which is basically the angular momentum of the system, ranges from $\sqrt{l(l+1)}$ to $l+1$, which means for $l = N-1$ an uncertainty of about 20% for, say, the $N = 4$ energy level. A crude (i.e. of the semiclassical type) application to QM of the CM formula $e^2 = 1 - (J_2/J_3)^2$ for the orbit eccentricity e, where $J_{2,3}$ are the appropriate action variables (Goldstein, 1980), yields $e^2 = 1 - (k/N)^2$. Accordingly the square eccentricity would range from $e^2 = 1 - [l(l+1)/N^2] = 1/N$ to $e^2 = 1 - [(l+1)/N]^2 = 0$ for $l = N - 1$. Nottale proposes a third expresssion, namely $e^2 = 1 - [l(l+1)/N(N-1)]$, that also yields zero for $l = N - 1$ (Nottale, 1993, 1996b, 1997). These few examples illustrate the ambiguity of the very concept of orbit eccentricity and, hence, of its value at low quantum numbers.

It would at least be more approriate to use the well-known concept of quantum expectation value and define the expectation value $\langle r\rangle$ of the radial distance and its corresponding mean square deviation $\Delta\,r = \sqrt{\langle r^2\rangle - \langle r\rangle^2}$, namely (White, 1934; Messiah, 1962):

$$\langle r\rangle = N\left(N + \frac{1}{2}\right)a_B \quad \text{and} \quad \frac{\Delta\,r}{\langle r\rangle} = \frac{1}{\sqrt{2N+1}} \quad . \tag{59}$$

Note that the relative mean square deviation $\Delta\,r/\langle r\rangle$ may reach 40% or more for values of N about 3 or less. In the present nonlinear QM theory, this mean square deviation is basically the width of the static radial-velocity soliton.

6. Schroedinger equation on the scale of the solar system

Following Nottale who gives an impressive list of astronomical results supporting his assumption (Nottale, (1993, 1996a,b, 1997), we adopt the following characteristic velocity of the CPSS:

$$w_0 = \sqrt{\frac{GM}{a_B}} = \sqrt{\frac{\alpha}{\mu a_B}} = 145\,km/s \quad , \tag{60}$$

where $\alpha = G\mu M$ defines the gravitational potential given by Eq. (46), $M = 1.99\,10^{30}\,kg$ is the mass of the sun and $G = 6.67\,10^{-11}\,m^3/s^2\,kg$ is the gravitational constant. Eq. (60) yields the following mass-independent Bohr radius for the solar system:

$$a_B = \frac{GM}{w_0^2} = 6.31\,10^6\,km = 0.0422\,AU \quad . \tag{61}$$

Equivalently, one may consider in the above theory of the hydrogen atom a *formal mass-dependent CPSS "quantum of action"*, namely:

$$\text{“}\hbar\text{”} = \mu\,a_B\,w_0 \quad , \tag{62}$$

in accordance with Eqs. (53) and (60). As a consequence, for $l = N - 1$, the radial-action quantization given by Eq. (58) yields:

$$\oint \boldsymbol{p}_{(r)}\,dr = 2\pi\,\mu\,a_B\,w_0 \quad , \tag{63}$$

regardless of the energy level N. Therefore *the radial-velocity planetary solitons have all the same norm*:

$$\int_0^\infty \frac{c_{(r)}}{w_0}\,dr = \pi a_B \quad \forall\,A\,\&\,N \quad . \tag{64}$$

6.1. Radial-velocity solitions

When considering the Kepler system described by Eqs. (46–51) for $l = N - 1$, we have seen in Sect. 5.2 that the abscissa

ζ_* that defines the initial conditions given by Eqs. (40–43) is defined by Eq. (57). Since Eqs. (37), (47–49) and (54) yield, together with $l = N - 1$, the following classical wavefunction (which is singular, as expected, at the classical turning points $z_{1,2} = N^2 \pm N^{3/2}$):

$$\Xi_{cl}(z) = \frac{\sqrt{A_*}}{\left[-1 + \frac{2N^2}{z} - \frac{N^3(N-1)}{z^2}\right]^{1/4}} \quad , \tag{65}$$

the initial amplitude Ξ_* of both the regular Schroedinger eigenfunction $u_E(z_*)$ and the nonlinear mode $\Xi_E^{A_*}(z_*)$ at the initial abcissa $z_* = N\zeta_* = N^2$ is:

$$\Xi_* = \Xi_{cl}(z_*) = \sqrt{A_*}N^{1/4} \quad . \tag{66}$$

Now we note an interesting symmetry of both Eqs. (21) and (55), namely multiplying the solution by λ amounts to dividing the Wronskian A by λ^2. In particular, choosing $\lambda = \sqrt{A_*}$ reduces the ODE problem defined by Eqs. (40), (55) with $l = N - 1$, (57) for the choice of the initial abscissa ζ_* and (66) to the following one:

$$\frac{d^2}{dz^2}\Xi_N + \left[-\frac{1}{N^2} + \frac{2}{z} - \frac{N(N-1)}{z^2}\right]\Xi_N - \frac{1}{N^2\left[\Xi_N\right]^3} = 0 \quad , \tag{67}$$

$$\Xi_N\Big]_{z=N^2} = N^{1/4} \quad ; \quad \dot{\Xi}_N\Big]_{z=N^2} = 0 \quad . \tag{68}$$

Therefore, as for the (an)harmonic oscillator (Reinisch, 1994; 1997), the classical context for the Schroedinger equation in the case of the Kepler system is equivalent to choosing $A = 1$.

The radial-velocity soliton field that is defined by Eqs. (22), (30), (56) and (60) and that corresponds to the nonlinear mode defined by Eqs. (67–68) reads:

$$\frac{c_{(r)}(z)}{v_N} = \frac{1}{\left[\Xi_N(z)\right]^2} \quad , \tag{69}$$

where v_N is the Kepler-Bohr velocity:

$$v_N = \frac{w_0}{N} \quad , \tag{70}$$

simply resulting from the combination of Kepler's third law for the revolution period $\mathcal{T}_N$ at the energy level (56) and Bohr's quantization formula $r_{Bohr} = N^2\, a_B$ given by Eq. (57) (Malisoff, 1929; Nottale, 1997). Indeed:

$$v_N = \frac{2\pi r_{Bohr}}{\mathcal{T}_N} \quad \text{where} \quad \mathcal{T}_N = \frac{2\pi a_B}{w_0}N^3 \quad . \tag{71}$$

The radial velocity $c_{(r)}(z)$ defined by Eq. (69) must be complemented by the orbital velocity field $\boldsymbol{p}_{(\varphi)}/\mu$ that is defined by Eq. (32). By use of Eqs. (57) and (62), this latter field becomes at $z \sim z_* = N^2$:

$$c_{(\varphi)}(z)\big]_{z\sim z_*} \sim \frac{\boldsymbol{p}_{(\varphi)}(z_*)}{\mu} = \frac{N-1}{N}v_N \quad , \tag{72}$$

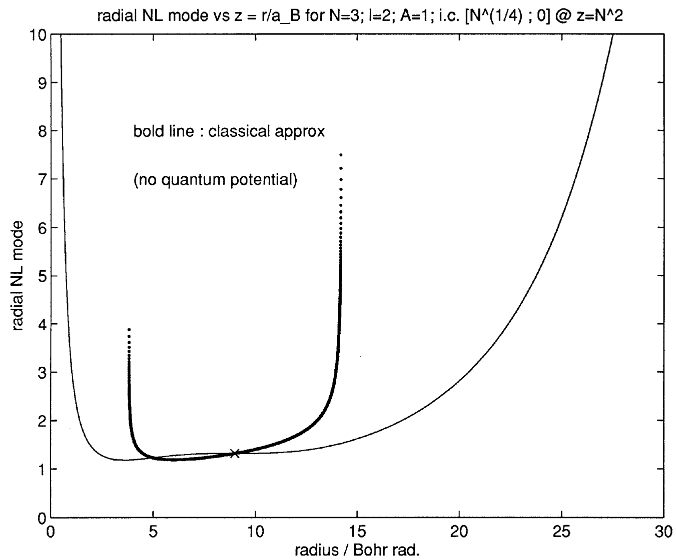

Fig. 1. The classical-like nonlinear radial eigenstate Ξ_3 defined by the EMP differential equation (67) together with its initial conditions (68) versus $z = r/a_B$ for the $N = 3, l = 2$ Kepler state (with $A_* = 1$). The classical mode $\Xi_{cl}(z)$ defined by Eq. (65) is displayed in bold line. The cross defines the initial conditions (68) at the location $z_* = N^2 = 9$.

for $m = l = N - 1$ and $\theta = \pi/2$. Eqs. (69) and (72) completely define the quantized velocity field in terms of its radial and orbital components. The case $N = 1$ is singular since it corresponds to a purely radial motion. We conclude that it does not yield any orbiting planetary motion in the CPSS.

6.2. Quantized patterns

Fig. 1 shows the radial nonlinear mode $\Xi_3(z)$ that is solution to Eq. (67) and corresponds to $N = 3$ (Mercury), $l = N - 1 = 2$ and to the initial conditions defined by Eqs. (68) at $z_* = N^2 = 9$ (cross). The classical counterpart $\Xi_{cl}(z)$ defined by Eq. (65) (with $A_* = 1$) is displayed in bold line. As expected, it diverges at the classical turning points $z_{1,2} = 9 \pm 3^{1.5}$. Note that the corresponding eccentricity $\sim N^{3/2}/N^2 = 1/\sqrt{N}$ agrees to an order of magnitude with Eq. (59). Clearly, the nonlinear mode $\Xi_3(z)$ is both quite flat and quite close to the classical state $\Xi_{cl}(z)$ within the classically allowed region.

Fig. 2 shows, in units of the Kepler-Bohr velocity v_N, the corresponding radial-velocity static soliton field that is defined by Eq. (69), together with its classical counterpart defined by Eqs. (38–39) and (65) and displayed in bold line. Figs. 3 and 4 display the same plots for $N = 11$ (Jupiter). There is clearly a 'quantum dressing' about the classical radial velocity field that is localized between the classical turning points $z_{1,2}$. It mostly accounts for the tunnel effect which extends fairly beyond the classical turning points and which is related to the divergence of the radial nonlinear mode about the boundaries $z = 0$ and $z = \infty$ of the system.

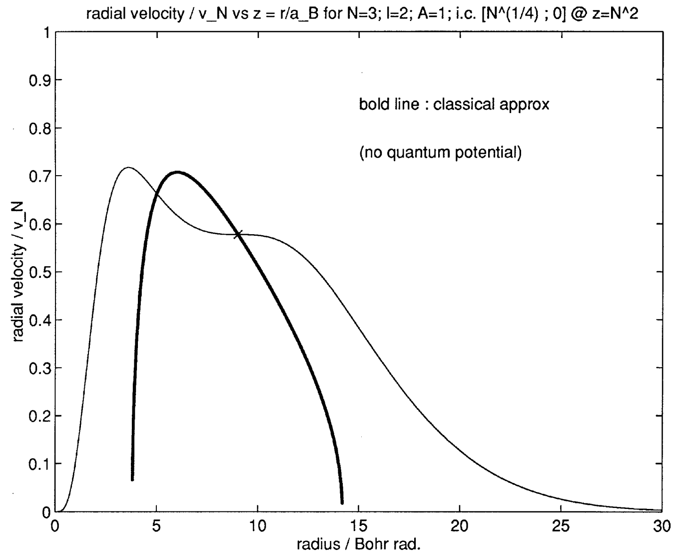

Fig. 2. The classical-like static soliton mode $c_{(r)}$ (divided by the Kepler-Bohr velocity v_N) that describes the steady-state radial-velocity matter flow versus $z = r/a_B$, as defined from Ξ_3 (displayed by Fig. 1) by use of Eq. (69). The cross defines the radial velocity that corresponds to the initial conditions (68) and to the orbital velocity (72) at the location $z_* = N^2 = 9$.

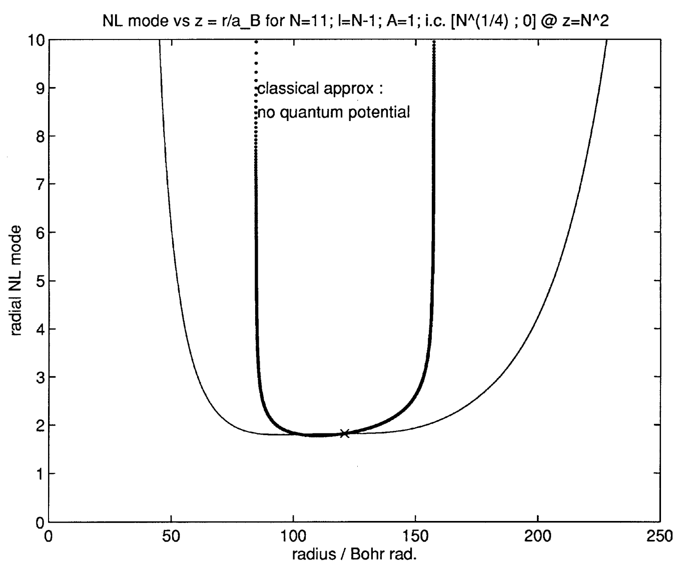

Fig. 3. The same as Fig. 1, but for $N = 11$.

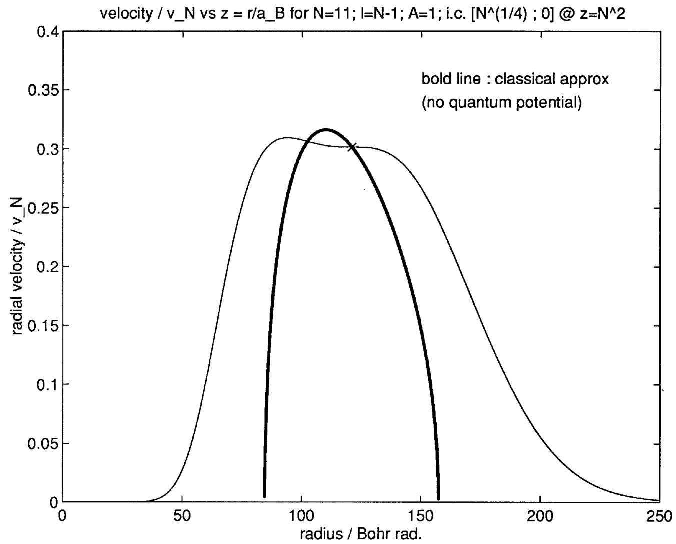

Fig. 4. The same as Fig. 2, but for $N = 11$.

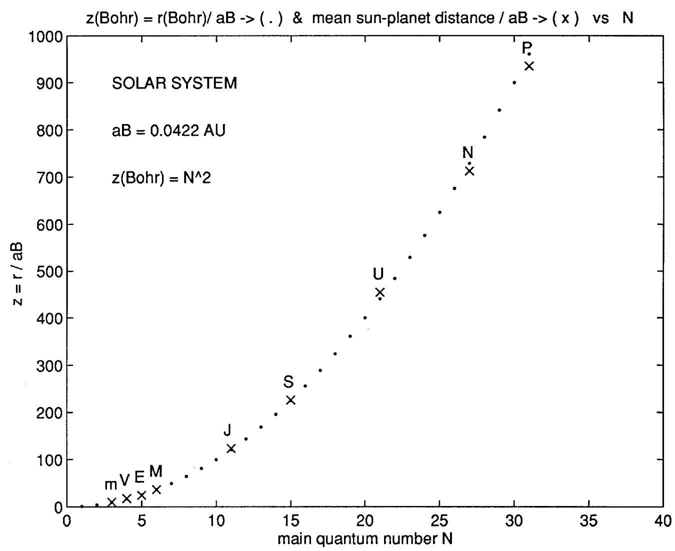

Fig. 5. The mean sun-planet distances (semimajor axis) versus the Bohr formula $z_{Bohr} = N^2$ for the nine planets of the solar system that are given the following ranks: $N = 3,\ 4,\ 5,\ 6,\ 11,\ 15,\ 21,\ 27,\ 31$.

6.3. Conjecture about planetary orbit quantization

Because the present macro-quantum theory is, like the stationary Schroedinger equation itself, conservative and reversible, we believe that it is not able to account for the final stage of planet accretion in the CPSS. Indeed, the process by which the static radial-velocity solitons would concentre into singularity-planets seems to be highly irreversible. Therefore, in the frame of the present theory (i.e. without making use of the standard Born postulate related to u_E^2), the Bohr formula (57) for the planet accretion could tentatively be explained by assuming that the radial CPSS matter flow, which is by the definition (40) fairly uniform about $z_* = N^2$ (see Figs. 2 and 4), ultimately concentres through irreversible processes about the center-part of the classically allowed region, namely $(z_1+z_2)/2 = N^2$. Fig. 5 displays the locations of the nine planets of the solar system divided by a_B (crosses = semimajor axis: McGraw-Hill, 1983), compared with their respective corresponding prediction, namely $z = N^2$ (points). The average of the absolute error that this formula yields with respect to the nine planet semimajor-axis locations is 3 and its standard deviation is 11, while the sequence of relative errors between $z = N^2$ and the nine planet locations is: $2\,\%$, $7\,\%$, $5\,\%$, $0.3\,\%$, $2\,\%$, $0.5\,\%$, $3\,\%$, $2\,\%$ and $3\,\%$.

Let us recall as a final result that *the case* $N = 1$ *is singular* (cf. Eq. (72)). Therefore the orbital quantization of the CPSS begins at $N = 2$, possibly corresponding to the hypothetical intramercurial planet 'Vulcanus'. *No orbiting pattern is expected at $N = 1$.*

7. Conclusion

As a summary account of this paper that investigates a macroscopic context for Schroedinger equation at the scale of the Solar System, let us emphasize the following new results:

1) There is a 'dynamical degeneracy' of any stationary quantum state Ψ. Indeed, to each such state defined by the energy eigenvalue E, one can associate an infinity of dynamical states of the system that are defined through Eqs. (22) and (29–32) by a particular value of the Wronskian A of the Schroedinger equation.
2) These dynamical states, the existence of which is guaranteed by i) the Planck-DeBroglie postulate stating that the phase of the wavefunction (3) is Hamilton's characteristic (or principal) function S of the system, and ii) the canonical Hamilton-Jacobi equation (31), must be understood in the classical (causal) sense.
3) The quantity $|\Psi|^2\, dr$ (averaged over the polar angles) is the local classical statistical probability distribution (36) of the particle position in terms of the corresponding velocity flow derived from the radial momentum field (29–32).
4) There exists for any energy level E of the system a particular state defined by Eqs. (40–43). It describes a classical-like dynamical behaviour of the system within the classically allowed region in accordance with Eq. (39). We suggest to call this quantum state the 'classical-like quantum state', and the corresponding steady-state radial-velocity matter flow the 'classical-like static soliton mode'.

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