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On wave equations and cut-off frequencies of plane atmospheres

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Abstract. This paper deals with the one-dimensional vertical propagation of linear adiabatic waves in plane atmospheres. In the literature there are various representations of the standard form of the wave equation from which different forms of the so called cut-off frequency are inferred. It is not uncommon that statements concerning the propagation behavior of waves are made which are based on the height dependence of a cut-off frequency. In this paper, first we critically discuss concepts resting on the use of cut-off frequencies. We add a further wave equation to three wave equations previously presented in the literature, yielding an additional cut-off frequency. Comparison among the various cut-off frequencies of the VAL-atmosphere reveals significant differences, which illustrate the difficulties of interpreting a height dependent cut-off frequency. We also discuss the cut-off frequency of the parabolic temperature profile and the behavior of the polytropic atmosphere. The invariants of the four wave equations presented contain first and second derivatives of the adiabatic sound speed. These derivatives cause oscillations and peaks in the space dependent part of the invariants, which unnecessarily complicate the discussion. We therefore present a new form of the wave equation, the invariant of which is extremely simple and does not contain derivatives of the thermodynamic variables. It is valid for any LTE equation of state. It allows us to make effective use of strict oscillation theorems. We calculate the height-dependent part of the invariant of this equation for the VAL-atmosphere including ionization and dissociation. For this real atmosphere, there is no obvious correspondence between the behavior of the invariant and the temperature structure or the sound speed profile. The invariant of the wave equation is nearly constant around the temperature minimum. In the chromosphere, the invariant is almost linear. The case of the wave equation with a linear invariant is studied analytically.

Key words: Sun: atmosphere – Sun: oscillations – stars: atmospheres

1. Introduction

For one-dimensional vertical propagation of linear adiabatic waves in a plane isothermal atmosphere there is a characteristic frequency, the acoustic cut-off frequency $\omega = \gamma g/2c$, which separates acoustic waves and evanescent waves. Waves can propagate only if their frequency is greater than the cut-off frequency. Besides, this frequency is the resonance frequency of the isothermal atmosphere. A cut-off frequency exists also in the field of lattice oscillations in solids and electromagnetic wave guides. In the case of three-dimensional wave propagation the cut-off frequency depends on the horizontal wave number.

The time-independent adiabatic wave equation of a non-isothermal atmosphere can be solved only numerically, in particular when ionization and dissociation are included. This can be done readily, even for three-dimensional wave propagation. However, to understand particular effects or numerical results one often studies simple atmospheric models or separate layers and uses only the equation of state of the classical ideal gas.

In the following we restrict ourselves to purely vertical propagation. The problems tackled there are present also in the case of three-dimensional wave propagation. There, however, the problems are immensely complicated, so that the difficulties cannot be mastered as in the one-dimensional case.

The stationary wave equation is a second order ordinary differential equation. The form which does not contain the first derivative of the dependent variable is called the standard form. The invariant of this equation, its coefficient, contains a formal, height-dependent cut-off frequency. This frequency is often used to obtain statements on the propagation behavior of waves. As the representation of the wave equation and thus the invariant of the standard form depends on both the dependent and the independent variable there exist different cut-off frequencies. In a recent paper, Mosser (1995) has discussed this subject. In general, the frequency ω_1 given below (cf. Eq. 11) is used. This expression holds in the case of the classical ideal gas. It is obtained when the wave equation of the vertical displacement is used.

The existence of a cut-off frequency of a non-isothermal atmosphere is not solely due to the pressure stratification produced by the external gravity. A temperature stratification can generate a cut-off frequency as well. A gas without an external field which has constant pressure can have a cut-off frequency when a corresponding temperature gradient is present. Apparently, this fact was not generally noted, since Balmforth & Gough (1990) emphazise and discuss this issue.

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By transformations not only of the dependent variables but also of the independent ones new representations of the wave equation are obtained and one can define a number of cut-off frequencies or invariants. The invariants can become rather complicated, in particular they can have derivatives higher than the second order.

In the present paper we first compare different wave equations. We add a further wave equation to three wave equations already presented in the literature yielding a further cut-off frequency. We study these cut-off frequencies for the VAL temperature stratification, the parabolic temperature profile and a polytropic atmosphere. For the VAL-atmosphere we find significant differences between the various cut-off frequencies. In the case of the parabolic temperature profile a cut-off frequency exists which is not due to the gravity. We discuss the calculation of this frequency without solving the wave equation explicitly. The model of the polytropic atmosphere with outwards increasing temperature shows which wave equations are suitable for statements on the existence of a cut-off frequency. The two examples also show that the necessary condition of oscillatory behavior of the solution, namely the positivity of the invariant is not sufficient. In the case of real atmospheres such as the VAL-model, the effective application of mathematically strict criteria is not possible because of the complexity of the invariants. This is due to the occurence of first and second derivatives of the thermodynamic equilibrium quantities in the invariants.

One of the aims of the present work is to introduce a wave equation the invariant of which does not contain derivatives of the usual thermodynamic equilibrium quantities of an atmosphere. This is possible by choosing suitable independent and dependent variables. There are numerous possibilities; we chose one with a very simple invariant. We calculate and discuss this invariant for the VAL-model including ionization and dissociation. It appears that the presented form of our wave equation is most suitable in the case of the VAL-model.

Sect. 2 presents the general linear equations of vertically propagating adiabatic waves. Sect. 3 deals with four forms of the wave equation and the corresponding cut-off frequencies in the case of the classical ideal gas. In Sect. 4 we briefly present oscillation theorems which are referred to in the subsequent sections. In Sect. 5 we discuss the behavior of the various cut-off frequenies of the VAL-atmosphere, the parabolic temperature profile and the polytropic atmosphere. The new representations of the wave equation are presented in Sect. 6.

2. Notations and basic equations

Let z be the outwards directed spatial coordinate, t the time, g the constant gravity, $p(z)$ and $\rho(z)$ undisturbed pressure and density, $c(z)$ the adiabatic sound speed, $\xi(z,t)$ the vertical Lagrangian displacement, $\Delta p(z,t)$ the Lagrangian pressure perturbation. The column mass m is given by $dm = -\rho\, dz$, the frequency is denoted by ω. The equilibrium pressure p and the mass m are related by $p = mg$. We study stationary adiabatic waves with time dependence $exp(i\omega t)$. Written in terms of the variables Δp and ξ, the linearized one-dimensional hydrodynamic equations read:

$$\omega^2 \rho\, \xi = \frac{d}{dz} \Delta p \,, \tag{1}$$

$$\Delta p = -\,\rho\, c^2\, \frac{d}{dz}\, \xi \,. \tag{2}$$

From these equations we obtain a stationary wave equation for Δp:

$$\frac{d}{dz}\,[\,\frac{1}{\rho}\frac{d}{dz}\Delta p\,] + \frac{\omega^2}{\rho c^2}\,\Delta p = 0\,, \tag{3}$$

and a stationary wave equation for the displacement ξ:

$$\frac{d}{dz}\,[\,\rho c^2 \frac{d}{dz}\,\xi\,] + \omega^2 \rho\, \xi = 0\,. \tag{4}$$

Both equations hold for arbitray equations of state. We call a wave equation simple when its coefficients depend on the equilibrium quantities in a simple way. Derivatives or higher derivatives of the equilibrium quantities ρ or c should be avoided. The form of the wave equation depends on the variable. Besides Δp and ξ we may use the Lagrangian density perturbation $\Delta\rho$, Eulerian perturbations δp and $\delta\rho$ or variables as $\Delta p/\sqrt{p}$ and $\sqrt{\rho}\,\xi$. However, we should use only variables which are continuous at a contact discontinuity where the equilibrium pressure p is continuous, ρ and c are discontinuous quantities. Discontinuities of the variables or their derivatives are caused by corresponding discontinuities of the coefficients. For Δp and ξ which are continuous at contact discontinuities, only first derivatives of the equilibrium quantities ρ and c enter the corresponding wave equations. The wave equation of the (at a contact discontinuity) discontinuous Eulerian pressure perturbation δp has coefficients which contain the second derivatives of the equilibrium quantities ρ and c. To obtain a simple wave equation we therefore must use Δp and ξ or such variables as $\Delta p/\sqrt{p}$ or $\Delta p/p$.

3. Different representations of the wave equation

In the following section we use only the equation of state of the classical ideal gas. There, the adiabatic sound speed is given by $c^2(z) = \gamma p/\rho = \gamma R\, T(z)/\mu$ where γ is the constant adiabatic exponent and μ the constant mean molecular weight. Most analytical studies dealing with linear adiabatic waves in a plane atmosphere are restricted to this case or assume that the atmosphere consists of separate layers with constant γ and μ (cf. Chiuderi & Giovanardi 1979, Balmforth & Gough 1990, Worrall 1991, Schmitz & Fleck 1992, Hindman & Zweibel 1994, Price 1996).

For the classical ideal gas, the wave equation of the displacement ξ is

$$c^2(z)\,\frac{d^2}{dz^2}\,\xi - \gamma g\,\frac{d}{dz}\,\xi + \omega^2 \xi = 0\,, \tag{5}$$

and the wave equation of the Lagrangian pressure perturbation Δp is

$$c^2(z)\frac{d^2}{dz^2}\Delta p + [\,\gamma\, g + \frac{d\,c^2}{dz}\,]\,\frac{d}{dz}\Delta p + \omega^2 \Delta p = 0\,. \tag{6}$$

Only the wave equation of the pressure perturbation contains the first derivative of the sound speed. This fact is due to a special property of the classical ideal gas. There, we have $\rho\, c^2 = \gamma\, p$, so that $\frac{d}{dz}\,\rho\, c^2 = -\,\gamma\, g\, \rho$. For more general equations of state the wave equation of the displacement reads

$$c^2\,\frac{d^2}{dz^2}\,\xi + \frac{1}{\rho}\,\frac{d}{dz}\,(\rho\, c^2)\,\frac{d}{dz}\,\xi + \omega^2\xi = 0\,. \tag{7}$$

Eliminating the first derivatives of the wave equations we obtain standard forms of these differential equations. In the following we shall use 5 "cut-off frequencies": $\omega_i(z),\ i = 0, 1, 2, 3, 4$. Let

$$\omega_0^2 = \frac{\gamma^2\, g^2}{4\, c^2(z)}\,. \tag{8}$$

We begin with the equation of the displacement $\xi(z)$. By the transformation

$$y_1(z) = p^{1/2}\,\xi(z) \tag{9}$$

this equation reduces to the standard form

$$\frac{d^2}{dz^2}\,y_1 + \frac{1}{c^2}\,[\omega^2 - \omega_1^2(z)]\,y_1 = 0 \tag{10}$$

with the familiar expression:

$$\omega_1^2 = \omega_0^2 + \frac{\gamma\, g}{2}\,\frac{1}{c^2}\,\frac{d\,c^2}{dz}\,. \tag{11}$$

Now we transform the equation of the Lagrangian pressure perturbation Δp. By

$$y_2(z) = c\, p^{-1/2}\,\Delta p(z) \tag{12}$$

the wave equation of the pressure perturbation Δp reduces to the standard form

$$\frac{d^2}{dz^2}\,y_2 + \frac{1}{c^2}\,[\omega^2 - \omega_2^2(z)]\,y_2 = 0 \tag{13}$$

with

$$\omega_2^2 = \omega_0^2 - \frac{1}{4c^2}\left(\frac{d\,c^2}{dz}\right)^2 + \frac{1}{2}\,\frac{d^2\,c^2}{dz^2}\,. \tag{14}$$

The frequency ω_2 depends on the second derivative of the sound speed c. The fact that ω_1 depends only on the first derivative of c is due to the above mentioned property of the ideal gas. It is noteworthy that $\omega_1 = 0$ for $g = 0$, but not ω_2. A cut-off frequency which is due to a temperature gradient is apparently represented by ω_2.

We now shall transform the independent variable z. The time-like variable $\tau = \tau(z)$ defined by $dz = c(z)d\tau$ was already introduced by Lamb (1909) to study waves in polytropic atmospheres. To eliminate the first derivative of the wave equation we have to transform also the dependent variables. By

$$y_3(\tau) = \left(\frac{p}{c}\right)^{1/2}\xi(z) \tag{15}$$

the wave equation (4) reduces to the standard form:

$$\frac{d^2}{d\tau^2}\,y_3 + [\omega^2 - \omega_3^2(\tau)]\,y_3 = 0 \tag{16}$$

where ω_3^2 written as a function of z (!) reads:

$$\omega_3^2 = \omega_0^2 + \frac{\gamma\, g}{2}\,\frac{1}{c^2}\,\frac{d\,c^2}{dz} + \frac{3}{16}\,\frac{1}{c^2}\left(\frac{d\,c^2}{dz}\right)^2 - \frac{1}{4}\,\frac{d^2\,c^2}{dz^2}\,. \tag{17}$$

To these representations of the wave equation already known we add a further version. By the transformation $dz = c\, d\tau$ and by

$$y_4(\tau) = \left(\frac{c}{p}\right)^{1/2}\Delta p(z) \tag{18}$$

the wave equation (3) reduces to the standard form

$$\frac{d^2}{d\tau^2}\,y_4 + [\omega^2 - \omega_4^2(\tau)]\,y_4 = 0 \tag{19}$$

where ω_4^2 written as a function of z (!) reads:

$$\omega_4^2 = \omega_0^2 - \frac{1}{16}\,\frac{1}{c^2}\left(\frac{d\,c^2}{dz}\right)^2 + \frac{1}{4}\,\frac{d^2\,c^2}{dz^2} \tag{20}$$

We can also formulate wave equations of Eulerian perturbations and transforms of these perturbations yielding numerous additional cut-off frequencies.

The transformation $\tau = \tau(z)$ can map the infinite interval $-\infty < z < +\infty$ to a finite interval $-a < z < +b$. If $c(z)$ behaves as z^α for $z \to \infty$ then $z = +\infty$ transforms to $\tau = 0$ if $\alpha > 1$. We have $\tau \to \infty$ for $z \to \infty$ if $\alpha \leq 1$.

$\omega_1(z)$ given by Eq. (11) is the most familiar form of the cut-off frequency of a non-isothermal atmosphere (cf. Beer, 1975). Tolstoi (1963) writes that Eq. (10) ".... is the usually separated form of the wave equation." Worrall's (1991) discussion of the generation of the 5-min peak by wave reflection is based to some extent on the frequency $\omega_1(z)$. The frequency $\omega_2(z)$ given by Eq. (14) has been used by Gough (1986), Balmforth and Gough (1990), and Hindman & Zweibel (1994) to discuss the propagation behavior of waves. Moore & Spiegel (1964), take $\omega_0(z)$ as the cut-off frequency. Recently, Mosser (1995) has presented the wave equation (16) and the corresponding cut-off frequency $\omega_3(z)$ defined by Eq. (16). Mosser has studied the propagation of waves in the Jovian atmosphere and compared the functions $\omega_i(z),\ i = 0, 1, 2, 3$. (His notation is different from that used here. He also formulates the frequency in terms of pressure and density scale heights instead of using the sound speed. The concepts are the same, however.) For the Jovian atmosphere, Mosser finds that $\omega_3(z)$ is better than $\omega_1(z)$ and $\omega_2(z)$. As his results finally are obtained by numerical integration of the differential equation they are uneffected by the choice of the cut-off frequency.

All the presented wave equations have the standard form:

$$y_i'' + Q_i\, y_i = 0 \tag{21}$$

where

$$Q_i = Q_i(z) = \frac{[\omega^2 - \omega_i^2(z)]}{c^2(z)}, \quad ' = \frac{d}{dz} \quad \text{for } i = 1, 2 \ , \tag{22}$$

$$Q_i = Q_i(\tau) = [\omega^2 - \omega_i^2(\tau)], \quad ' = \frac{d}{d\tau} \quad \text{for } i = 3, 4 \ . \tag{23}$$

For constant sound speed, because of $dz = c\, d\tau$ the variables z and τ coincide. The quantity Q_i is called the invariant of the differential equation.

The opinion that the numerical value of ω_i or the sign of Q_i defines oscillatory behavior in the case of representation (22) seems to be widespread. The behavior of the solution of the equation $y'' + Q(x)y = 0$ is often described as follows: The equation has an oscillatory type solution when $Q > 0$ and an exponential or hyperbolic typ when $Q < 0$. However, $Q > 0$ is only a necessary criterion for oscillatory behavior. It is not sufficient. We did not found any publication where Kneser‘s oscillation theorem (see, Sect. 4) is used. The situation is clearer if the standard form represented by Eqs. (21) and (23) is used. There, Sturm's comparison theorems can be applied effectively.

As the kind of the solution is determined by the function Q_i, it is not sufficient to consider only the frequencies ω_1 and ω_2. Here, also the behavior of $c(z)$ is crucial. In the case of the Eqs. (21) and (23), the quantities ω_3 and ω_4 completely determine the behavior of the solutions. However, to predict the behavior of the solutions we need oscillation theorems. In the following, we present those theorems which we shall refer to.

4. Oscillation theorems

There is a series of oscillation theorems for linear second order differential equations. Most of these theorems concern infinite intervals, i.e. $[a, \infty)$. A solution is said to be oscillatory if there is an infinite number of zeros in the interval $[a, \infty)$, non-oscillatory, if there is only a finite number. Mathematically, the terms ”oscillatory” and ”non-oscillatory” always refer to an infinite interval. A non-oscillatory solution can have a finite number of oscillations in $[a, \infty)$. As regards the number of zeros in a finite interval $[a, b]$ there are only a few theorems. For real atmospheres this question is more important than the behavior for $z \to +\infty$. We need the following theorems:

1. For a differential equation $y'' + f(x)y = 0$ we have: if $f \leq 0$ on an interval (a, b) or on $(-\infty, +\infty)$, then $y(x)$ has at the most one zero in this interval. In particular, if $f(x) \to -\infty$ for $|x| \to \infty$, then $y(x)$ is non-oscillatory (for large $|x|$).

2. Sturm's comparison theorem (sufficient condition). If the solutions of $u'' + f(x)u = 0$ oscillate in $a \leq x \leq b$, and $g(x) \geq f(x)$ there, then also the solutions of $v'' + g(x)v = 0$ oscillate with at least the same number of zeros. Conversely, if $v(x)$ does not oscillate, then $u(x)$, too. For $f(x) = k^2$ we obtain a special case of this theorem.

3. Kneser's (necessary and sufficient) oscillation theorem: Consider the equation $y'' + f(x)y = 0$. Then, if $0 < f(x) \leq 1/(4x^2)$ for $0 < a \leq x < \infty$, the solution $y(x)$ is nonoscillatory. If $f(x) > (1 + \epsilon)/(4x^2)$ with $\epsilon > 0$, then the solution $y(x)$ is oscillatory.

These theorems show that the critical quantities are the functions $Q_i(z)$ and $Q_i(\tau)$, but not the frequencies $\omega_i(z)$ and $\omega_i(\tau)$. So the frequencies ω_1 and ω_2 alone cannot indicate the character of the solution. However, the frequencies ω_3 and ω_4 determine the functions Q_3 and Q_4 directly. The above theorems do not give information on the ratio and the behavior of the amplitudes, the reflection, and the transmission. The theorems are presented in many text-books of differential equations (e.g. Kamke 1983, Stepanow 1963, and also in the work of Gradshteyn & Rhyzik (1980).

Examples of non-oscillatory solutions with a finite number of oscillations are the wave functions of discrete energy niveaus in quantum mechanics. For instance, all the eigen-functions of the one-dimensional linear oscillator are called non-oscillatory as they behave exponentially at infinity.

5. Comparison of different cut-off frequencies

We study the cut-off frequencies of the temperature profile of the VAL-atmosphere and of two elementary non-isothermal models. In all cases we use the equation of state of a classical ideal gas with constant adiabatic exponent γ and constant molecular weight μ. Under these assumptions the cut-off frequencies depend only on the temperature $T(z)$ and thus only on the sound speed $c(z)$. We take $\gamma = 5/3$ and $\mu = 1.3$.

5.1. The VAL-atmosphere

We combine the VAL-atmosphere and the convection zone of Spruit (1977). For the resulting temperature stratification $T(z)$ we calculate the quantities $\omega_i^2(z)$ for $i = 0, 1, 2, 3, 4$. Fig. 1 shows these functions. The second derivative of the sound speed depends on the smallest details of the atmospheric structure and on its approximation. Therefore, the frequencies ω_2, ω_3 and ω_4 are burdened by uncertainties. The transition layer is not included as there the uncertainties of the second derivatives are unlimited. There are huge spikes in the transition layer. In the case of the wave equation (23) the frequencies ω_3 and ω_4 are functions of the variable τ. As the function $\tau = \tau(z)$ is monotonically increasing and smooth there is no need to consider the functions $\omega_3^2(\tau)$ and $\omega_4^2(\tau)$ in addition to the functions $\omega_3^2(z)$ and $\omega_4^2(z)$. To compare the frequencies we take the independent variable z. The transition to τ only stretches or contracts the abscissa.

The quantities $\omega_1^2(z)$ and $\omega_3^2(z)$ are similar, just as the quantities $\omega_2^2(z)$ and $\omega_4^2(z)$. In the convection zone where the temperature decreases linearly, and in the range of the chromospheric plateau where the temperature increases linearly all frequencies $\omega_i(z)$ are nearly equal as the second derivatives of c^2 vanish there.

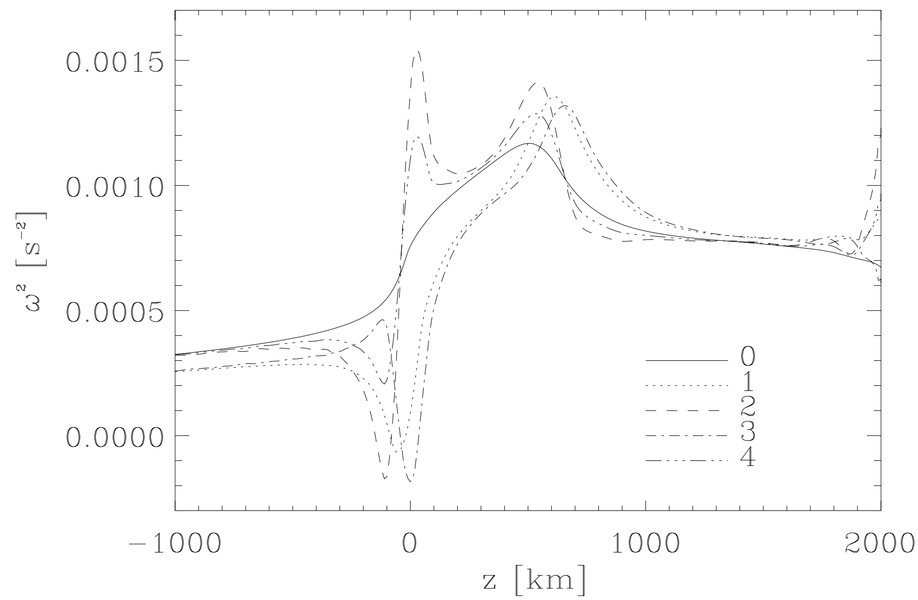

Fig. 1. The square of different cut-off frequencies $\omega_i, \quad i = 0,1,2,3,4$ calculated with the temperature stratification of the VAL-atmosphere and the equation of state of the classical ideal gas.

In the range $-200\,km < z < +200\,km$ round about the temperature minimum, however, the behavior of the frequencies is very different. Here, at $z = 0$ the similar-shaped quantities $\omega_2^2(z)$ and $\omega_4^2(z)$ are contrary to the similar-shaped quantities $\omega_1^2(z)$ and $\omega_3^2(z)$. The behavior of $\omega_1^2(z)$ corresponds to the behavior of the frequency $\omega_1(z)$ presented by Worrall (1991 Fig. 1). The quantity $\omega_0^2(z)$ is a smooth function. The differences of the quantities ω_i show the difficulty of a height-dependent cut-off frequency.

5.2. *The parabolic temperature profile*

Kahn (1962), Moore & Spiegel (1964), and Chiuderi & Giovanardi (1979) have approximated the temperature stratification of the whole solar atmosphere by a parabolic profile. In this case, for an ideal gas, the adiabatic sound speed is given by

$$c^2(z) = c_0^2\,(1 + \alpha^2 z^2)\ . \tag{24}$$

While the discussion presented by Moore & Spiegel was based on $\omega_0(z)$ as the cut-off frequency, Chiuderi & Giovanardi have reduced the wave equation (5) to the hypergeometric differential equation. From the asymptotic behavior of the solution they obtained a cut-off frequency ω_c:

$$\omega_c^2 = \frac{c_0^2\alpha^2}{4}\ . \tag{25}$$

This cut-off frequency is due to the parabolic temperature stratification at infinity, and occurs even if $g = 0$. Chiuderi & Giovanardi attribute the existence of the cut-off frequeny to the temperature gradient. Strictly speaking, however, the cut-off frequency is caused by the curvature of the function $c^2(z)$.

We now derive the cut-off frequency without solving the wave equation explicitly. As can be shown readily, we have $\omega_1(z) = \omega_2(z) = 0$ as $z \to \infty$. Thus, in the case of the parabolic profile the frequencies $\omega_1(z)$ and $\omega_2(z)$ have no meaning. This is clear as $Q_1(z),\ Q_2(z) \geq 0$ is no criterion for oscillatory behavior. However, the application of Kneser's theorem (3.) gives the cut-off frequency ω_c as

$$Q_1(z) = Q_2(z) = \left(\frac{\omega}{c_0\,\alpha\,z}\right)^2 \quad \text{for} \quad z \to \infty\ . \tag{26}$$

So the solutions $y_1(z)$ and $y_2(z)$ oscillate for $\omega > \omega_c$ as in this case with $Q_1(z)\ ,\ Q_2(z) > 1/(4\,z^2)$ Kneser's theorem is fulfilled.

We also obtain the constant value ω_c from both Eqs. (16) and (19). As can be shown readily, for $z \to \infty$ we obtain $\omega_3 = \omega_4 = \omega_c$, so that

$$Q_3 = Q_4 = \omega^2 - \omega_c^2\ . \tag{27}$$

In the case of the parabolic temperature profile, Eqs. (16) and (19) are appropriate to show the existence of the cut-off frequency, and to yield its value. Note that

$$\tau = \frac{1}{c_0\alpha}\ \mathrm{Arsinh}(\alpha z)\ ,\ c^2(\tau) = c_0^2\cosh^2(c_0\alpha\tau) \tag{28}$$

so that the interval $-\infty < z < +\infty$ corresponds to the interval $-\infty < \tau < +\infty$.

5.3. *The polytropic atmosphere with outwards increasing temperature*

In this case, the adiabatic sound speed is:

$$c^2(z) = \frac{\gamma g\,z}{|1{+}n|} \quad \text{with } n < -1 \text{ and } 0 < z < \infty\ . \tag{29}$$

By $dz = c\,d\tau$ the interval $0 < z < \infty$ is transformed to the interval $0 < \tau < \infty$. We obtain:

$$\omega_0^2 = \frac{\gamma g|1{+}n|}{4z}\ , \tag{30}$$

$$\omega_1^2 = \omega_0^2 [1 + \frac{2}{|1+n|}],\ \omega_2^2 = \omega_0^2 [1 - \frac{1}{|1+n|^2}], \tag{31}$$

$$\omega_3^2 = \omega_0^2 [1 + \frac{2}{|1+n|} + \frac{3}{4|1+n|^2}],\ \omega_4^2 = \omega_0^2 [1 - \frac{1}{4|1+n|^2}]. \tag{32}$$

It is known that the wave equation of a polytropic atmosphere can be reduced to Bessel's equation. The solutions are oscillatory for all frequencies, so that there is no cut-off frequency. This fact can be derived exactly from the oscillation theorems.

As $\omega_0 \to 0$ for $z \to \infty$ we have $\omega_i \to 0$ for $i = 1, 2, 3, 4$. Because of ω_3 and $\omega_4 \to 0$, we have Q_3 and $Q_4 \to \omega^2$. Then, Eq. (21) shows that the functions y_3 and y_4 always oscillate for $\tau \to +\infty$ and thus for $z \to +\infty$.Here, Eq. (16) and Eq. (19) are appropriate.

The quantities ω_1 and ω_2 have no meaning, as $Q_1(z)$ and $Q_2(z) \to 0$ for $z \to \infty$. However, in this case, application of Kneser's theorem immediately shows the oscillatory behavior. We have $Q_1(z) = Q_2(z) = |1+n|\,\omega^2 \,/\, \gamma g z$ for $z \to \infty$, so that $Q_1(z), Q_2(z) > 1/(4z^2)$ for $z \to \infty$. Similar conclusions hold also for atmospheres with positive index n. We have $\omega_4^2 < \omega_3^2$ and $\omega_2^2 < \omega_1^2$. This fact can be used to apply Sturm's comparison theorems.

6. Well-conditioned representations of the wave equation

The frequencies ω_1 and ω_2 alone do not determine the character of a wave. The wave equation should be written in a form to which certain oscillation theorems can be applied. The Eqs. (16) and (19) of Sect. 2 where a time-like coordinate τ is used are more appropriate than the equations with the geometrical height z as the independent variable. However, in the case of the solar atmosphere the coefficients Q_3 und Q_4 are too complicated and their numerical calculation is difficult (because of the second derivatives). Taking the mass as the independent variable and the Lagrangian pressure perturbation as the dependent variable we obtain an equation Knesers theorem is applicable to. Turning from the mass to a fictitious geometrical coordinate we obtain a wave equation Sturms oscillation theorems are applicable to.

6.1. The wave equation for the application of Kneser's theorem

Using the column mass m defined by $dm = -\rho\, dz$, the wave equation (3) of the pressure perturbation Δp reads:

$$\frac{d^2}{dm^2}\Delta p + \frac{\omega^2}{c^2\rho^2}\Delta p = 0\ . \tag{33}$$

Putting $\Delta p = m\, y$ and $\eta = 1/m$ we obtain the equation

$$\frac{d^2}{d\eta^2}\, y + \frac{\omega^2}{\eta^2}\Big(\frac{p}{\rho g c}\Big)^2 y = 0\ , \tag{34}$$

where we have used the relation $m = p/g$. For the classical ideal gas this equation reads

$$\frac{d^2}{d\eta^2}\, y + \frac{\omega^2}{\eta^2}(\frac{c^2}{\gamma^2 g^2})\, y = 0. \tag{35}$$

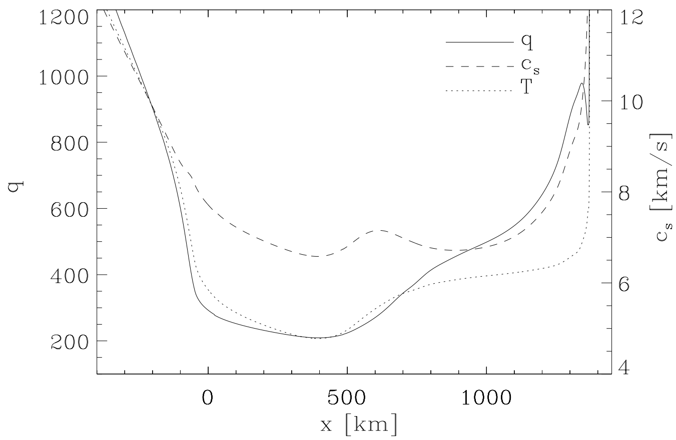

Fig. 2. The VAL-atmosphere: The temperature (dotted), the adiabatic sound speed (dashed), and the coefficient q as functions of the fictitious geometrical height x.

Eq. (34) has a form suitable for the application of Knesers oscillation theorem. From Eq. (35) and this theorem it follows that $\gamma g/2c$ is the cut-off frequency when c is constant.

The wave equation (33) has the coefficient $1/(c^2\rho^2)$ as a function of the column mass m, Eq. (34) the coefficient $p^2/(c^2\rho^2)$ as a function of the reciprocal mass $\eta = 1/m$. In the case of an isothermal atmosphere with a neutral ideal gas the wave equation reduces to an Eulerian differential equation. Such an equation is simple and compact, but in practice the mass is a coordinate difficult to handle. It is more familiar to use a wave equation which in the limiting case of an isothermal atmosphere reduces to a differential equation with constant coefficients. So instead of the mass we turn to a geometrical coordinate.

6.2. The final representation of the wave equation

We start from Eq. (33) and define a new independent variable x by

$$d\,m = h(x)\, dx \tag{36}$$

with arbitrary $h(x)$. To suppress the occurence of the first derivative of the dependent variable Δp in the differential equation by this very general transformation we have to transform Δp, too. Putting $\Delta p = u\, v$ with v as the new dependent variable and u as a factor, the first derivative is omitted if $u = \sqrt{h}$. Thus we have to assume that

$$\Delta p = \sqrt{h(x)}\, v(x)\ . \tag{37}$$

The equation of $v(x)$ then reads:

$$\frac{d^2}{dx^2}v + \Big[\,\frac{1}{2h}\,\frac{d^2h}{dx^2} - \frac{3}{4h^2}\,\Big(\frac{dh}{dx}\Big)^2\,\Big]\, v + \omega^2\,\frac{h^2}{\rho^2 c^2}\, v = 0 \tag{38}$$

with $h = h(x)$, $\rho = \rho(x)$, and $c = c(x)$. The function $h(x)$ is arbitrary. Because of $dm = h(x)dx$ the transformation $x = x(m)$ depends on the form of $h(x)$. The strategy to obtain an adequate wave equation by choosing $h(x)$ for a given atmosphere, so that

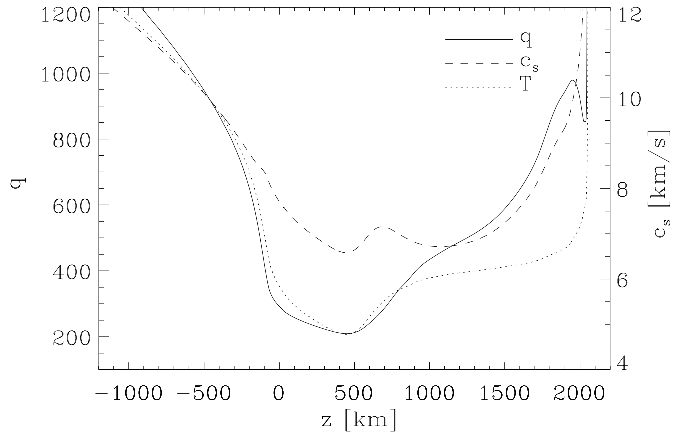

Fig. 3. The VAL-atmosphere: The temperature (dotted), the adiabatic sound speed (dashed), and the coefficient q as functions of the real geometrical height z.

both the expression in brackets and the coefficient $h^2/\rho^2\,c^2$ become in a sense optimal, shall not be discussed here. Instead, we consider h as a function of m, which, because of $p = mg$, is equivalent to $h(p)$. Other relations are not useful, as only m or p are continuous at contact discontinuities of the atmosphere. For $h = h(\rho)$ first and second derivatives of $\rho(x)$ would enter the differential equation. With $h = h(m)$ and because of $\frac{dh}{dx} = \frac{dh}{dm}\,\frac{dm}{dx} = \frac{dh}{dm}\,h$ we obtain the differential equation

$$\frac{d^2}{dx^2}v + [\,\frac{h}{2}\,\frac{d^2h}{dm^2} - \frac{1}{4}\left(\frac{dh}{dm}\right)^2\,]\,v + \frac{\omega^2 h^2}{\rho^2 c^2}v = 0\,. \tag{39}$$

It makes sense to define the function h so that the expression in brackets becomes constant. It can be shown readily (cf. Kamke, 1983) that the assumption

$$[\,\frac{h}{2}\,\frac{d^2h}{dm^2} - \frac{1}{4}\left(\frac{dh}{dm}\right)^2\,] = A = const. \tag{40}$$

is solved only by

$$h(m) = a_2\,m^2 + a_1\,m + a_0 \tag{41}$$

with the condition $A = a_2 a_0 - a_1^2/4$, and arbitrary coefficients a_2, a_1, and a_0. In this case, the transformation $x = x(m)$ can be performed analytically. With $h = 1$ we obtain $x = m$, and Eq. (39) redudes to Eq. (33). For $h = m^2$ we get $m = -1/x$, and then Eq. (35).

Putting $h = m/H$ whith an arbitary scale height H we obtain

$$x = H\ln\left(\frac{m}{m_0}\right) \tag{42}$$

This relation introduces a fictitious geometrical coordinate x. The dependent variable becomes $y = \Delta p/\sqrt{p}$. We obtain the equation

$$\frac{d^2}{dx^2}\,y + \frac{1}{H^2}\,[\left(\frac{p}{g\rho c}\right)^2\omega^2 - \frac{1}{4}\,]\,y = 0 \tag{43}$$

which holds for a general equation of state. So we have

$$\frac{d^2}{dx^2}\,y + \frac{1}{H^2}\,[\,\omega^2\,q(x) - \frac{1}{4}\,]\,y = 0 \tag{44}$$

with

$$q = \left(\frac{p}{\rho g c}\right)^2. \tag{45}$$

For a real atmosphere, the coefficient q given as a function of the real height z or of the mass m is now represented as a function of the fictitious geometrical height x. For the classical ideal gas we have $p/(\rho c) = c/\gamma$. Further, if c is constant Eq. (44) reduces to the usual wave equation of the isothermal atmosphere and x becomes the real geometrical coordinate z. Eq. (44) has no derivative of an undisturbed quantity. As $p = mg$ and $m = m_0\exp(x/H)$ we have

$$\Delta p = y(x)\,\exp(x/2H)\,. \tag{46}$$

Both Eqs. (34) and (43) contain the coefficient q. As the fictitious geometrical height x is only the logarithm of the column mass m, it is a simple, physical quantity. As for an atmosphere $p = mg$, the coordinate x can be calculated.

6.3. The VAL-atmosphere

We take the temperature stratification $T(z)$ and the pressure $p(z)$ of the VAL-atmosphere, extented inwards by the convection zone of Spruit (1977), and calculate the density $\rho(z)$ and the adiabatic sound speed $c_s(z)$ including dissociation and ionization of hydrogen and ionization of Helium: H_2, H, H^+, He, He^+, He^{++}, e^-. We have used the code described by Wolf (1983). The scale height H was taken as $H = c_T^2/g$ where c_T is the isothermal sound speed at the temperature minimum. We have $H = 100\,km$. From these quantities we obtain $q(z)$ and finally $q(x)$.

Fig. 2 displays the result $q(x)$. Besides, Fig. 2 shows the temperature and the adiabatic sound speed as functions of x. We see that the function $q(x)$ does not behave like the temperature or the sound speed. Fig. 3 displays these quantities as functions of the geometrical coordinate z. For conventional wave equations using the coordinate z this figure is crucial. There the sound speed and its derivatives play a role. For wave propagation described by our wave equation, there is no chromospheric plateau. The behavior of the wave is determined alone by the coefficient $q(x)$.

We see that in the range $0 < x < 500km$ corresponding to the range $0 < z < 500km$ the function q varies only slightly. In the range $500km < x < 1200km$, the function $q(x)$ is roughly linear. Therefore, we present the solution for linear $q(x)$ in an appendix.

7. Conclusions

In this paper we have dealt with representations of the time-independent equation of purely vertically propagating linear adiabatic waves in a plane atmosphere with arbitrary equation of state. To study and discuss wave equations by analytical methods, by the application of oscillation theorems or WKB-like strategies, the wave equations are written in standard form, i.e. a form not containing the first derivative of the dependent variable with respect to the geometrical coordinate. Then, the invariant of this equation, i.e. the only remaining coefficient contains a formal, height-dependent cut-off frequency. When trying to draw conclusions from the form of the invariant or the cut-off frequency with respect to the propagation behavior of waves one must regard that the positivity of the invariant is not a sufficient criterion for oscillatory behavior of the solutions of the wave equation — a fact which often has been disregarded in the past. The fact that invariants and cut-off frequencies are not invariant with respect to transformations of both the independent and dependent variables is not significant if oscillation theorems are used correctly. However, in practise the use of such mathematical theorems is hampered by the complicated behavior of the invariant originating in first and second derivatives of the sound speed. The presentation of a wave equation, the invariant of which depends only on the equilibrium quantities ρ and c, but not on the first and second derivatives of these quantities, clears up this problem. The wave equation presented in this paper has no derivatives of the sound speed or other thermodynamic variables. It is very simple and seems to be appropriate in the case of the chromosphere of the sun. In the case of other atmospheres, other strategies might proof more successful. Such possibilites have been pointed out in this paper. In the case of the three-dimensional wave equation we have made encouraging progress. It seems difficult, however, to put the formalism into practise.

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Appendix A: solutions of the wave equation for linear $q(x)$

Let

$$q(x) = \alpha + \beta\, x\ . \tag{A1}$$

Then, by the substitution

$$\zeta = (H\omega^2\beta)^{-2/3}\ (\,\omega^2\alpha - \frac{1}{4} + \omega^2\beta\, x\,) \tag{A2}$$

Eq. (44) reduces to

$$\frac{d^2}{d\zeta^2}\, y + \zeta\, y = 0\ . \tag{A3}$$

The solutions of this equation are cylinder functions of fractional order:

$$y(\zeta) = \sqrt{\zeta}\; Z_{\frac{1}{3}}\Big(\frac{2}{3}\,\zeta^{\frac{3}{2}}\Big) \tag{A4}$$

where Z denotes the Bessel functions J or Y for real argument and the modified Besselfunctions K and I for imaginary argument. Finally, we may write:

$$y(x) = \sqrt{\omega^2 q(x) - \frac{1}{4}}\; \mathcal{Z}_{\frac{1}{3}}\Big(\frac{2}{3H\beta\omega^2}\sqrt{\omega^2 q(x) - \frac{1}{4}}^{\,3}\Big) \tag{A5}$$

for $\omega^2\, q(x) > 1/4$ where $\mathcal{Z}_{\frac{1}{3}}$ denotes the (oscillating) Besselfunctions $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$, and

$$y(x) = \sqrt{\frac{1}{4} - \omega^2 q(x)}\; \mathcal{C}_{\frac{1}{3}}\Big(\frac{2}{3H\beta\omega^2}\sqrt{\frac{1}{4} - \omega^2 q(x)}^{\,3}\Big) \tag{A6}$$

for $\omega^2\, q(x) < 1/4$ where $\mathcal{C}_{\frac{1}{3}}$ denotes the (non-oscillatory) modified Besselfunctions $K_{\frac{1}{3}}$ and $I_{\frac{1}{3}}$. Thus, the frequency

$$\omega(x) = \frac{1}{2\sqrt{q(x)}} \tag{A7}$$

separates two types of solutions: the exponential and the oscillatory ones. Therefore, in the case of linear $q(x)$ the frequency

$$\omega = \frac{g\rho c}{2p} \tag{A8}$$

may be called cut-off frequency. For the classical ideal gas, it reduces to ω_0.

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