Astron. Astrophys. 337, 966–974 (1998)

ASTRONOMY AND ASTROPHYSICS

# Influence of viscosity laws on the transition to the self-gravitating part of accretion disks

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Received 7 January 1998 / Accepted 12 June 1998

**Abstract.** In this paper we obtain a new solution for accretion disks influenced by self-gravity using the viscosity law proposed by Heyvaerts et al. (1996). We show how to solve analytically the problem taking into account self-gravity. Using this new viscosity law, we obtain a slightly different solution (compared with the standard one) for the transition between the non self-gravitating and the self-gravitating part of the disk but no solution in the self-gravitating part where additional viscosity mechanisms may play a role.

**Key words:** accretion disks – galaxies : active – hydrodynamics – turbulence

## 1. Introduction

Active galactic nuclei (AGNs) are commonly thought to be powered by accretion disks. A direct observation of an AGN accretion disk has recently been obtained in radio using the VLBA and VLA (Gallimore et al., 1997) and by the HST which shows a rotating gaseous disk on the scale of $\sim 20$ pc (Ford et al., 1994). But, the physical processes underlying the AGN phenomenon are poorly known and in particular we do not have a satisfactory description of the physical conditions prevailing in the accretion disks of such objects.

Using the standard model (Shakura and Sunyanev, 1973), widely used in X-ray binaries and cataclysmic variables, it can easily be seen that, with the typical parameters for AGNs, the outer part of the disk is likely to be self-gravitating. A disk can be vertically self-gravitating or fully self-gravitating. In the first case, the vertical equilibrium is dominated by the mass of the disk but the rotation remains quasi-keplerian: the disk is subject to the local Jeans instability but is not globally unstable. In the second case, the mass of the disk dominates in both directions and the rotation is not keplerian. In this paper, we will be interested in gaseous accretion disks with a transition from a non self-gravitating part to a vertically self-gravitating part which is believed to exist in AGNs.

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In the first part of this paper (Sect. 2), we discuss the different viscosity laws used in accretion disks. We then propose in Sect. 3 an analytical solution describing the transition between the non self-gravitating part of a disk and the part dominated by self-gravity using the formulation of Heyvaerts et al. (1996) for viscosity. We finally discuss this new solution (Sect. 4).

## 2. The viscosity law

Although angular momentum transport is central to the theory of accretion disks, the exact origin of it is still unknown. A mechanism more efficient than molecular viscosity is needed to redistribute angular momentum and to allow accretion in the observed time scales. Note also that no laminar flows can produce the observed amount of energy dissipation. In this section we will discuss turbulence and gravitational torques as possible mechanisms for angular momentum transport.

Accretion disks are thought to be turbulent because their Reynolds numbers based on molecular viscosity is large. Such a turbulence provides an effective viscosity. Within the turbulent viscosity assumption, one difficulty is to point out a plausible source of turbulence. A few local instabilities have been studied. Accretion disks subject to a keplerian rotation law are linearly stable to the hydrodynamic shear. But, it has been suggested that they could be non linearly unstable (Zahn, 1990; Dubrulle and Knobloch, 1992). This non linear instability is still a matter of debate (Hawley et al., 1995; Kato and Yoshikawa, 1997) and should not be turned down yet. Another source of instability could be the thermal vertical stratification. But convection is initiated only in sufficiently cold disks like proto-planetary ones. A third possible source of local instability is magnetic field. The magneto-rotational instability has been initially discovered by Velikov (1959) and Chandrasekhar (1961) and has been extensively studied in the context of accretion disks (Balbus and Hawley, 1991; Hawley and Balbus, 1991; Hawley et al., 1995; Stone et al., 1996). This instability is very efficient and the conditions for it to grow seem to be fulfilled in many disks.

Whatever the relevant instability, how it non linearly develops and leads to an effective turbulent viscosity is not completely understood. The usual approach to this problem is a statistical treatment of the Navier-Stokes equations. But, as in all non-linear stochastic systems, this leads to a closure problem. The basic assumption of most of the closure models is that all the moments can be expressed in terms of the Reynolds stresses which are second order moments (these models are called second order closures, see the review by Speziale (1991)): turbulence is thus assumed to be quasi-gaussian. Deviation from gaussianity due to anisotropic processes (rotation for example) has also been taken into account by Dubrulle (1992). The constants induced by these closure approximations can be calibrated using real experiments and, then, only the length scale of the turbulence which rests on the initial instability remains to be determined (Dubrulle, 1992).

Because of this difficulty of treating in a complete rigorous physical way the turbulence in accretion disks, empirical viscosity prescriptions have been broadly used. In particular, the most well-known is the $\alpha$-prescription (Shakura and Sunyaev, 1973). It assumes that the molecular viscosity can be replaced by an isotropic turbulent viscosity $\nu_\star$ given by

$$\nu_\star \simeq v_{\rm t}\, l \qquad (1)$$

where $v_{\rm t}$ is the turn-over velocity of the eddies and $l$ their typical scale. The authors argued that $v_{\rm t}$ must be less than the sound speed $C_{\rm S}$ and that $l$ is smaller than the thickness $2h$ of the disk. We thus get

$$\nu_\star = \alpha_{\rm ss}\, C_{\rm S}\, h \qquad (2)$$

where $\alpha_{\rm ss}$ is a free parameter in which all the unknowns have been put and which must be smaller than one. This prescription enabled a reasonable global description of stationary, thin and non self-gravitating disks (Frank et al., 1992).

Few improvements of the $\alpha$-prescription have been made. As the radial infall velocity is expected not to exceed the sound speed because of causality, Narayan (1992) proposed a modified prescription which has been used by Popham & Narayan (1992). The prescrition has also been modified to be valid in large gradient zones (Narayan et al., 1994) and in non-keplerian flows (Godon, 1995).

A different approach to this effective turbulent viscosity has recently been proposed by Heyvaerts et al. (1996). These authors propose that the turbulence level is such that it self-consistently adjusts the accretion rate of matter and thus they equal the energy dissipation rate in the disk to the energy transfer rate in the turbulent cascade. This argument does not necessitate a detailed knowledge of the turbulence development and can be applied whatever the local instability giving rise to turbulence. In this theory, the effective turbulent viscosity is thus given by

$$\nu_\star = \alpha\, \Omega\, h^2 \qquad (3)$$

where $\Omega$ is the rotation velocity which is supposed to be keplerian and $\alpha$ is a constant depending on fundamental constants and, in a quadratic way, on the injection scale of energy in the cascade. If this scale is equal to the half-thickness $h$ of the disk then $\alpha = 0.08$.

This new formulation is interesting because it is established using very few reasonable assumptions on turbulence while the $\alpha$-prescription is only empirical. Moreover, it is equivalent to the $\alpha$-prescription in the limit of non self-gravitating accretion disks. Indeed, if the disk is non self-gravitating we have

$$h = \frac{\sqrt{2} C_{\rm S}}{\Omega} \qquad (4)$$

and thus Eqs. (2) and (3) are equivalent in this limit:

$$\nu = \alpha \Omega h^2 \quad \Leftrightarrow \quad \nu = \alpha_{\rm ss}\, C_{\rm S} h \quad \text{no self-gravity} \;\cdot \qquad (5)$$

But if self-gravity plays a role, the formulation (3) will differ from the $\alpha$-prescription. The two formulae are no more equivalent because $h$ differs from its non self-gravitating value given by Eq. (4):

$$\nu = \alpha \Omega h^2 \quad \not\Leftrightarrow \quad \nu = \alpha_{\rm ss}\, C_{\rm S} h \quad \text{self-gravity} \;\cdot \qquad (6)$$

Hence, the formulation (3) of the effective turbulent viscosity gives rise to a totally different dependence of $\nu_\star$ on $C_{\rm S}$ and on $h$ than the formulation (2) if self-gravity plays a role. In this case, note that Eq. (3) can still be regarded as equivalent to Eq. (2) but with a free parameter $\alpha_{\rm ss}$ depending on the radial position.

When self-gravity dominates (such disks will be called self-gravitating) the origin of the effective viscosity is probably different than turbulence. Duschl et al. (1998) argue for a hydrodynamical viscosity based on Reynolds turbulence. However, such accretion disks where self-gravity dominates seem unstable to non axi-symetric gravitational disturbances which take the form of spiral waves and exert a gravitational torque on the disk (Larson, 1984; Boss, 1984). It was proposed that the mean gravitational torque can be parametrized by an effective viscosity (Lin and Pringle 1987, 1990). Such a parametrization is supported by numerical simulations (Laughlin and Bodenheimer, 1994; Laughlin, 1996). This effective gravitational viscosity seems to be well represented by a constant Toomre parameter (Laughlin and Bodenheimer, 1994; Laughlin, 1996) as initially proposed by Paczyński (1978), Kozlowski et al. (1979) and Larson (1984). See also a criticism of the use of the $\alpha$-prescription in self-gravitating disks by Duschl et al. (1998).

In this work, we want to explore the differences induced by the formulation (3) for viscosity on the solution for the transition between the non self-gravitating limit and the self-gravitating regime. Previous works were done only numerically by Sakimoto & Coroniti (1981) and Shore & White (1982) and using the $\alpha$-prescription in the form of the stress tensor proportional to pressure. The motivation here is two-fold. We first propose an analytical method for taking into account self-gravity. Second, we feel that Eq. (3) for viscosity is well established and thus that it is worth exploring the differences with the standard prescription taking into account self-gravity. Note that we do not claim to model the part of the disk where self-gravity dominates as viscosity is probably of gravitational origin (see above) while Eq. (3) is based on turbulence development only.

## 3. An analytical solution for disks influenced by self-gravity

In this section, we derive analytically the solution for the flow in a thin and keplerian disk with self-gravity included. We use a cylindrical coordinates system $(R, z)$. The viscosity law is written as Eq. (3). The disk is assumed to be optically thick, radiating as a black body. In steady state, the effective temperature $T_{\rm eff}$ is thus given by

$$\sigma_{\rm B} T_{\rm eff}^4 = \frac{9}{8}\,\nu_\star\,\sigma\,\Omega^2 \tag{7}$$

where $\sigma_{\rm B}$ is the Stefan-Boltzmann constant and $\sigma$ is the surface density. The central temperature $T_{\rm c}$ is related to the sound speed $C_{\rm S}$ by the usual relation

$$C_{\rm S}^2 = \frac{k_{\rm B} T_{\rm c}}{m_{\rm p}} \tag{8}$$

where $k_{\rm B}$ is the Boltzmann constant and $m_{\rm p}$ is the proton mass. The effective and central temperature are related by

$$T_{\rm c} = \tau^{1/4}\, T_{\rm eff} \tag{9}$$

where $\tau$ is the optical depth. We will assume that $\tau$ is given by a Kramers' law.

The thickness of the disk is given by hydrostatic equilibrium between the vertical forces. We take into account the vertical pressure gradient and the vertical gravitational force with the two contributions of the central object and of the disk. This general equation cannot be solved analytically. If the vertical gravitational force is dominated by the contribution of the central object, the length scale $z_{\rm c}$ writes

$$z_{\rm c}^2 = \frac{2C_{\rm S}^2}{\Omega^2} \tag{10}$$

and, if it is dominated by the self-gravitational contribution of the disk, the length scale $z_{\rm s}$ writes (Goldreich and Lzyndon-Bell, 1965)

$$z_{\rm s}^2 = \frac{C_{\rm S}^2}{2\pi G \rho_0} \tag{11}$$

where $G$ is the gravitational constant and $\rho_0$ the central density of the disk. We take the half-thickness $h$ of the disk to be described by an interpolating formula between these two limit solutions (Sakimoto and Coroniti, 1981)

$$\frac{1}{h^2} = \frac{1}{z_{\rm c}^2} + \frac{1}{z_{\rm s}^2}\ . \tag{12}$$

Eq. (12) is rewritten as

$$h = \sqrt{\frac{2C_{\rm S}^2}{\Omega^2 + 4\pi\rho_0 G}}\ . \tag{13}$$

Eqs. (3), (7), (8), (9) and (13) are the basic equations to which we will add the equation of mass diffusion to close the system.

We first calculate the half-thickness $h$ and the effective viscosity $\nu_\star$ as functions of the surface density. Eq. (13) is a quadratic equation for $h$ as $\rho_0$ depends on $\sigma$ and $h$. We will here adopt the approximate relation $\sigma \simeq h\rho_0$. Eq. (13) thus leads to

$$h^2\Omega^2 + 4\pi G\sigma h = 2C_{\rm S}^2\ \cdot \tag{14}$$

Choosing the positive root of this quadratic equation we obtain

$$h = \frac{-2\pi G\sigma + \sqrt{4\pi^2 G^2\sigma^2 + 2\Omega^2 C_{\rm S}^2}}{\Omega^2}\ . \tag{15}$$

Using Eq. (15) to express $\rho_0 \simeq \sigma/h$, Eq. (13) now gives

$$h^2 = \frac{2C_{\rm S}^2}{\Omega^2}\,\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1}\ . \tag{16}$$

The variable $\xi$ is the square of the Toomre parameter $Q$ and is defined by the ratio

$$\xi \equiv \frac{\Omega^2 C_{\rm S}^2}{2\pi^2 G^2 \sigma^2} = Q^2\ . \tag{17}$$

It can be easily interpreted: in fact, the disk is self-gravitating if

$$\frac{GM_\star}{R^3} z \ll 2\pi G\sigma \tag{18}$$

where $M_\star$ is the mass of the accreting object and which, after vertical integration and using Eq. (15), leads to $\xi \ll 1$. As expected, Eq. (16) reduces respectively to Eq. (10) and (11) in the limit cases $\xi \gg 1$ and $\xi \ll 1$.

With the previously defined quantities the optical depth writes

$$\tau = \sigma\,\kappa_{\rm R} = \frac{\sigma^2}{h}\,\kappa_0\,T_{\rm c}^{-7/2} \tag{19}$$

where $\kappa_{\rm R} = \kappa_0\,\rho\,T_{\rm c}^{-7/2}$ is the absorption coefficient given by the Kramers' law. We take $\kappa_0 = 6.6\,.10^{21}\ {\rm m^5 K^{7/2} kg^{-2}}$.

Let us take dimensionless quantities (to simplify algebraic manipulations) using a reference mass $M_0$ and a reference radius $R_0$. We define the reference viscosity $\nu_0$, the reference temperature $T_0$, the reference surface density $\sigma_0$, the reference sound speed $C_0$ and the reference optical depth $\tau_0$ by

$$\nu_0 = R_0^2\Omega_0\ ;\ T_0 = \left(\frac{9}{8}\frac{M_0\Omega_0^3}{\sigma_{\rm B}}\right)^{1/4};$$
$$\sigma_0 = \frac{M_0}{R_0^2}\ ;\ C_0^2 = \frac{k_{\rm B}T_0}{m_{\rm p}};\ \tau_0 = \kappa_0\frac{\sigma_0^2}{R_0\,T_0^{-7/2}}\ ; \tag{20}$$

where $\Omega_0$ is the keplerian rotation at $R_0$

$$\Omega_0 = \sqrt{\frac{GM_\star}{R_0^3}}\ . \tag{21}$$

Using Eqs. (7), (9) and (19) to express the central temparature $T_{\rm c}$, we substitute Eq. (8) into Eq. (16). Using Eq. (20) into Eqs. (3) and (16) we can obtain the couple of equations which gives the dimensionless viscosity $\nu$ and the dimensionless half-thickness $H$

$$\nu\ =\ \alpha\,x^{-3/2}H^2\ , \tag{22}$$

$$H^2 = \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{15}{16}} \tau_0^{\frac{1}{8}}\,\Sigma^{3/8}\,\nu^{1/8}\,x^{39/16}\left(\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1}\right)^{\frac{15}{16}} \tag{23}$$

in which $\Sigma$ is the dimensionless surface density, $x$ is the dimensionless radius. We also have assumed a keplerian rotation law. This assumption will be checked *a posteriori*. Resolution of Eqs. (22) and (23) will give $\nu$ and $H$ as functions of $x$ and $\Sigma$.

Using Eq. (23) in Eq. (22) we obtain an implicit formulation of $\nu$ as a function of $\Sigma$ as $\xi$ depends on $\nu$

$$\nu = \alpha^{\frac{8}{7}} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{15}{14}} \tau_0^{\frac{1}{7}} \Sigma^{3/7} x^{15/14} \left(\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1}\right)^{\frac{15}{14}} . \tag{24}$$

To solve Eq. (24) we will first express $\nu$ as a function of $\xi$ and write an equation for $\xi$. Indeed we have

$$\xi \equiv \frac{\Omega^2 C_S^2}{2\pi^2 G\sigma^2} = \xi_0\, \tau_0^{2/15} \frac{\nu^{2/15}}{\Sigma^{8/15}\, x^{17/5}\, H^{2/15}} \tag{25}$$

with $\xi_0$ defined by

$$\xi_0 \equiv \frac{\Omega_0 C_0^2 R_0^4}{2\pi^2 G^2 M_0^2} \tag{26}$$

and with Eq. (23) we can obtain

$$\xi = \frac{\xi_0\, \tau_0^{1/8}\, \nu^{1/8}}{\Sigma^{13/8}\, x^{57/16}} \left(\frac{R_0^2\Omega_0^2}{2C_0^2}\right)^{\frac{1}{16}} \left(\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1}\right)^{-\frac{1}{16}} \tag{27}$$

which gives $\nu$ as a function of $\xi$, $\Sigma$ and $x$

$$\nu = \left(\frac{\xi}{\xi_0}\right)^8 \frac{\Sigma^{13}\, x^{57/2}}{\tau_0} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{1}{2}} \left(\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1}\right)^{\frac{1}{2}} . \tag{28}$$

Eqs. (24) and (28) together give the equation for $\xi$

$$\frac{\sqrt{1+\xi}-1}{\sqrt{1+\xi}+1} = \frac{1}{\alpha^2 \xi_0^{14} \tau_0^2} \frac{R_0^2\Omega_0^2}{2C_0^2}\; x^{48}\, \Sigma^{22}\; \xi^{14} . \tag{29}$$

Apart from the trivial solution $\xi = 0$, this equation always has a unique solution because the function in the left hand side is concave and the one in the right hand side is convex. In the case of a self-gravitating disk ($\xi \ll 1$), the solution $\xi_1$ of Eq. (29) writes

$$\xi_1 = \frac{\alpha^{2/13} \xi_0^{14/13} \tau_0^{2/13}}{4^{14/13}} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{1}{13}} \frac{1}{x^{48/13}\, \Sigma^{22/13}} . \tag{30}$$

In the opposite non self-gravitating case $\xi \gg 1$, we get

$$\xi_2 = \alpha^{1/7} \xi_0 \tau_0^{1/7} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{1}{14}} \frac{1}{x^{24/7}\, \Sigma^{11/7}} . \tag{31}$$

We found that the real solution of Eq. (29) is well approximated by

$$\left(\frac{1}{\xi}\right)^{14} \simeq \left(\frac{1}{\xi_1}\right)^{14} + \left(\frac{1}{\xi_2}\right)^{14} . \tag{32}$$

The approximation has been checked solving by dissection the Eq. (29). Comparison with the approximation given by Eq. (32) shows that the error is always less than 10%. The solution for $\xi$ given by Eqs. (30), (31) and (32) yields a solution for $\nu$. A simple expression for $\nu$ as a function of $\xi$ is obtained combining Eqs. (24) and (28)

$$\nu = \xi^{15} \frac{\Sigma^{24}\, x^{105/2}}{\alpha\, \xi_0^{15} \tau_0^2} . \tag{33}$$

We thus have $\nu$ as a function of $\Sigma$ and $x$

$$\nu = \frac{\Sigma^{24}\, x^{105/2}}{\alpha\, \xi_0^{15} \tau_0^2} \frac{1}{\left(4^{14/13} u^{14} + u^{13}\right)^{15/14}} \tag{34}$$

with $u$ defined by

$$u \equiv \left(\frac{R_0^2\Omega_0^2}{2C_0^2}\right)^{1/13} \frac{x^{48/13}\, \Sigma^{22/13}}{\alpha^{2/13} \xi_0^{14/13} \tau_0^{2/13}} . \tag{35}$$

The explicit solution for $H$ as a function of $x$ and $\Sigma$ can be obtained substituting Eq. (32) and (34) into Eq. (23).

We now need to specify the surface density. This comes from the resolution of the mass diffusion equation. We will assume here that the central object is much smaller than the disk radius and thus the mass diffusion equation leads to the classical relation (Pringle, 1981)

$$\nu\Sigma \simeq \frac{|\dot{M}|}{3\pi M_0 \Omega_0} \tag{36}$$

where $\dot{M}$ is the accretion rate. The direct substitution of Eq. (36) into Eq. (34) gives only an equation for $\nu(x)$ but a parametric solution can be found. Putting Eqs. (34) and (36) together to eliminate the variable $\nu$ we have

$$\frac{|\dot{M}|}{3\pi\Omega_0 M_0} = \frac{\Sigma^{25} x^{105/2}}{\alpha\, \xi_0^{15} \tau_0^2} \frac{1}{\left(4^{14/13} u^{14} + u^{13}\right)^{15/14}} . \tag{37}$$

Using Eq. (35) to express $\Sigma$ as a function of $x$ and $u$ we get

$$\frac{|\dot{M}|}{3\pi\Omega_0 M_0} = \frac{\alpha^{\frac{14}{11}} \xi_0^{\frac{10}{11}} \tau_0^{\frac{3}{11}}}{x^{45/22}} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{25}{22}} \frac{u^{65/77}}{\left(1+4^{14/13} u\right)^{15/14}} \tag{38}$$

which allows us to write the dimensionless radius $x$ as a function of $u$

$$x = \alpha^{\frac{28}{45}} \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{5}{9}} \left(\frac{3\pi\Omega_0 M_0}{|\dot{M}|}\right)^{\frac{22}{45}} \frac{\xi_0^{4/9} \tau_0^{2/15} u^{26/63}}{(1+4^{14/13} u)^{11/21}} , \tag{39}$$

and, then using Eq. (35), to write the surface density $\Sigma$ as a function of $u$

$$\Sigma = \left(\frac{R_0^2\Omega_0^2}{2C_0^2}\right)^{\frac{7}{6}} \left(\frac{|\dot{M}|}{3\pi\Omega_0 M_0}\right)^{\frac{16}{15}} \frac{(1+4^{14/13} u)^{8/7}}{\alpha^{19/15} \xi_0^{1/3} \tau_0^{1/5}\, u^{13/42}} . \tag{40}$$

Eqs. (39) and (40) constitute a parametric solution for the surface density. Eqs. (33), (39) and (40) give $\xi$ as a function of $u$

$$\xi = \frac{1}{u^{13/14}\, (1 + 4^{14/13} u)^{1/14}} . \tag{41}$$

Using Eqs. (39) and (40) into Eq. (23) gives the half-thickness of the disk as a function of $u$

$$\frac{H}{x} = \left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{4}{9}} \left(\frac{|\dot{M}|}{3\pi\Omega_0 M_0}\right)^{\frac{4}{45}} \frac{\tau_0^{1/15} \xi_0^{1/8}}{\alpha^{1/45}}$$

$$u^{\frac{13}{252}} (1+4^{\frac{14}{13}} u)^{\frac{19}{672}} \left(\frac{\sqrt{1+\frac{1}{u^{\frac{13}{14}} (1+4^{\frac{14}{13}} u)^{\frac{1}{14}}}}-1}{\sqrt{1+\frac{1}{u^{\frac{13}{14}} (1+4^{\frac{14}{13}} u)^{\frac{1}{14}}}}+1}\right)^{\frac{15}{32}} , \tag{42}$$

and Eqs. (36) and (40) give the viscosity as a function of $u$

$$\nu=\left(\frac{2C_0^2}{R_0^2\Omega_0^2}\right)^{\frac{7}{6}}\left(\frac{3\pi\Omega_0 M_0}{|\dot{M}|}\right)^{\frac{1}{15}}\frac{\alpha^{19/15}\tau_0^{1/5}\xi_0^{1/3}\,u^{13/42}}{(1+4^{14/13}u)^{8/7}}\ . \qquad (43)$$

The parametric solution for the half-thickness is thus given by Eqs. (39) and (42) and the parametric solution for the viscosity by Eqs. (39) and (43). The parameters $u$ and $\xi$ are related by Eq. (41) which shows that the disk is self-gravitating, *i.e.* $\xi \ll 1$, when $u \gg 1$ and is non self-gravitating, *i.e.* $\xi \gg 1$, when $u \ll 1$.

After substituting the reference quantities using Eq. (20), we obtain the following numerical solution in S.I. units:

$$R=4.10^{14}\alpha^{\frac{28}{45}}\left(\frac{M_\star}{10^8M_\odot}\right)^{\frac{1}{3}}\left(\frac{10M_\odot/\mathrm{yr}}{|\dot{M}|}\right)^{\frac{22}{45}}\frac{u^{\frac{26}{63}}}{(1+4^{\frac{14}{13}}u)^{\frac{11}{21}}} \qquad (44)$$

$$\sigma = \frac{9.10^5}{\alpha^{19/15}}\left(\frac{|\dot{M}|}{10M_\odot/\mathrm{yr}}\right)^{\frac{16}{15}}\frac{(1+4^{14/13}u)^{8/7}}{u^{13/42}} \qquad (45)$$

$$\frac{h}{R} = \frac{6.10^{-3}}{\alpha^{\frac{1}{45}}}\left(\frac{|\dot{M}|}{10M_\odot/\mathrm{yr}}\right)^{\frac{4}{45}}\left(\frac{10^8M_\odot}{M_\star}\right)^{\frac{1}{3}}$$

$$u^{\frac{13}{252}}(1+4^{\frac{14}{13}}u)^{\frac{19}{672}}\left(\frac{\sqrt{1+\frac{1}{u^{\frac{13}{14}}(1+4^{\frac{14}{13}}u)^{\frac{1}{14}}}}-1}{\sqrt{1+\frac{1}{u^{\frac{13}{14}}(1+4^{\frac{14}{13}}u)^{\frac{1}{14}}}}+1}\right)^{\frac{15}{32}} \qquad (46)$$

$$\nu_\star = 8.10^{16}\alpha^{\frac{19}{15}}\left(\frac{10M_\odot/\mathrm{yr}}{|\dot{M}|}\right)^{\frac{1}{15}}\frac{u^{13/42}}{(1+4^{14/13}u)^{8/7}} \qquad (47)$$

## 4. Discussion

### 4.1. Global features of the new solution

The whole solution as given by Eqs. (44) to (47) is plotted for $|\dot{M}| = 10M_\odot/\mathrm{yr}$ and $M_\star = 10^8 M_\odot$ in Fig. 1a–f. As can be seen in these graphs, the radius $R$ is bounded by an upper value. In the limit of self-gravitating disks, $u \gg 1$ and $R$ decreases with $u$ as $u^{-1/9}$. In the non self-gravitating limit, it increases with $u$ as $u^{26/63}$. The maximum value $R_{\max}$ of $R$ is reached for $u \simeq 0.8$ and can be written as

$$\frac{R_{\max}}{R_\mathrm{s}} = 622\ \alpha^{28/45}\left(\frac{10M_\odot/\mathrm{yr}}{|\dot{M}|}\right)^{\frac{22}{45}}\left(\frac{10^8M_\odot}{M_\star}\right)^{\frac{2}{3}} \qquad (48)$$

where $R_\mathrm{s}$ is the Schwarzschild radius. It can be checked that $R_{\max}$ is larger than the external radius of the disk for objects like intermediate polars, X-ray binaries and T-Tauri stars (see Table 1). But for AGNs with central mass of order $M_\star = 10^8 M_\odot$ and with an accretion rate of order $|\dot{M}| = 10M_\odot/\mathrm{yr}$, the maximum radius $R_{\max}$ of $R$ is about 148 Schwarzschild radius ($\alpha = 0.1$). This is indeed smaller than the disk external radius of AGNs, believed to be of a few thousand Schwarzschild radii. Moreover, the solution found here is subject to the viscous instability near $R_{\max}$ as shown by the negative slope around $R_{\max}$ in the $(\sigma, T_\mathrm{c})$ diagram (Fig. 2). This maximum radius is also caracterized by a Toomre parameter of order 1 as will be discussed in Sect. 4.3.

It is interesting to mention that the opacity influences this maximum radius. If the disk is vertically isothermal (which is a non physical case), we calculate the maximum radius to be about the Schwarzschild radius.

The analytical resolution makes an approximation using Eq. (32) to solve Eq. (29). In Fig. 3a and b, we compare the exact numerical solution of the equations and our analytical solution. The agreement is satisfactory nearly everywhere. As expected, the error is stronger for $R \sim R_{\max}$ where the approximation given by Eq. (32) is the poorest. Quantitatively, the error on the value of $R_{\max}$ is about 8 %. The analytical solution found here is thus reliable and has the advantage of giving the dependence of the solution on the physical parameters.

In Fig. 4a and b, we compare the solution using the standard prescription (Eq. 2) and the new prescription used in this paper (Eq. 3). The main difference is that, with the standard prescription, there is only one branch of solution whereas the use of Eq. (3) for viscosity yields two branches of solution.

The topology of our new solution can be easily understood with physical arguments. As the disk becomes dominated by self-gravity, it becomes much thinner than in the non self-gravitating limit. Thus, the largest possible eddies in 3D turbulence which are scaled by the thickness of the disk becomes much smaller. Hence, the effective turbulent viscosity decreases strongly with radius (see Fig. 1e and f) and $R_{\max}$ is the radius at which turbulent viscosity cannot ensure a stationnary accretion rate of matter. As expected, when the accretion rate $\dot{M}$ decreases, $R_{\max}$ increases as shown by Eq. (48).

### 4.2. Self-consistency of the solution

Before discussing the new solution calculated here we must check whether the solution is self-consistent. In other words, we wonder if the assumptions used in this model remain verified by the solution.

We first assumed that the disk is thin and indeed the solution given here remains thin everywhere (see Fig. 1c and d).

We also assumed a keplerian rotation. This hypothesis can be checked using the radial equilibrium equation which is written as

$$\left(\frac{\Omega}{\Omega_\mathrm{k}}\right)^2 - 1 = -\frac{1}{\sigma\, R\,\Omega_\mathrm{k}^2}\frac{\partial P_\mathrm{g}}{\partial R} - \frac{v_\mathrm{r}}{R\,\Omega_\mathrm{k}^2}\frac{\partial v_\mathrm{r}}{\partial R} \qquad (49)$$

where $\Omega_\mathrm{k}$ is a keplerian rotation law, $P_\mathrm{g}$ is the gas pressure given by $P_\mathrm{g} = \sigma C_\mathrm{S}^2/h$ and $v_\mathrm{r}$ is the radial infall velocity given by the conservation of mass

$$v_\mathrm{r} = -\frac{|\dot{M}|}{2\pi}\frac{1}{R\,\sigma}\ . \qquad (50)$$

The right hand side of Eq. (49) can be evaluated using the solution given by Eqs. (44) to (47). No significant deviation to keplerian occurs around $R_{\max}$: the deviation $|(\Omega/\Omega_\mathrm{k})^2 - 1|$ is about $10^{-11}$ for $\alpha = 0.1$ and $10^{-10}$ for $\alpha = 0.01$ at this point. The deviation increases for radii smaller than $R_{\max}$. On the non self-gravitating branch, it is about $10^{-10}$ at $R = R_\mathrm{s}$. On the self-gravitating branch, the deviation is stronger: about $10^{-2}$ at

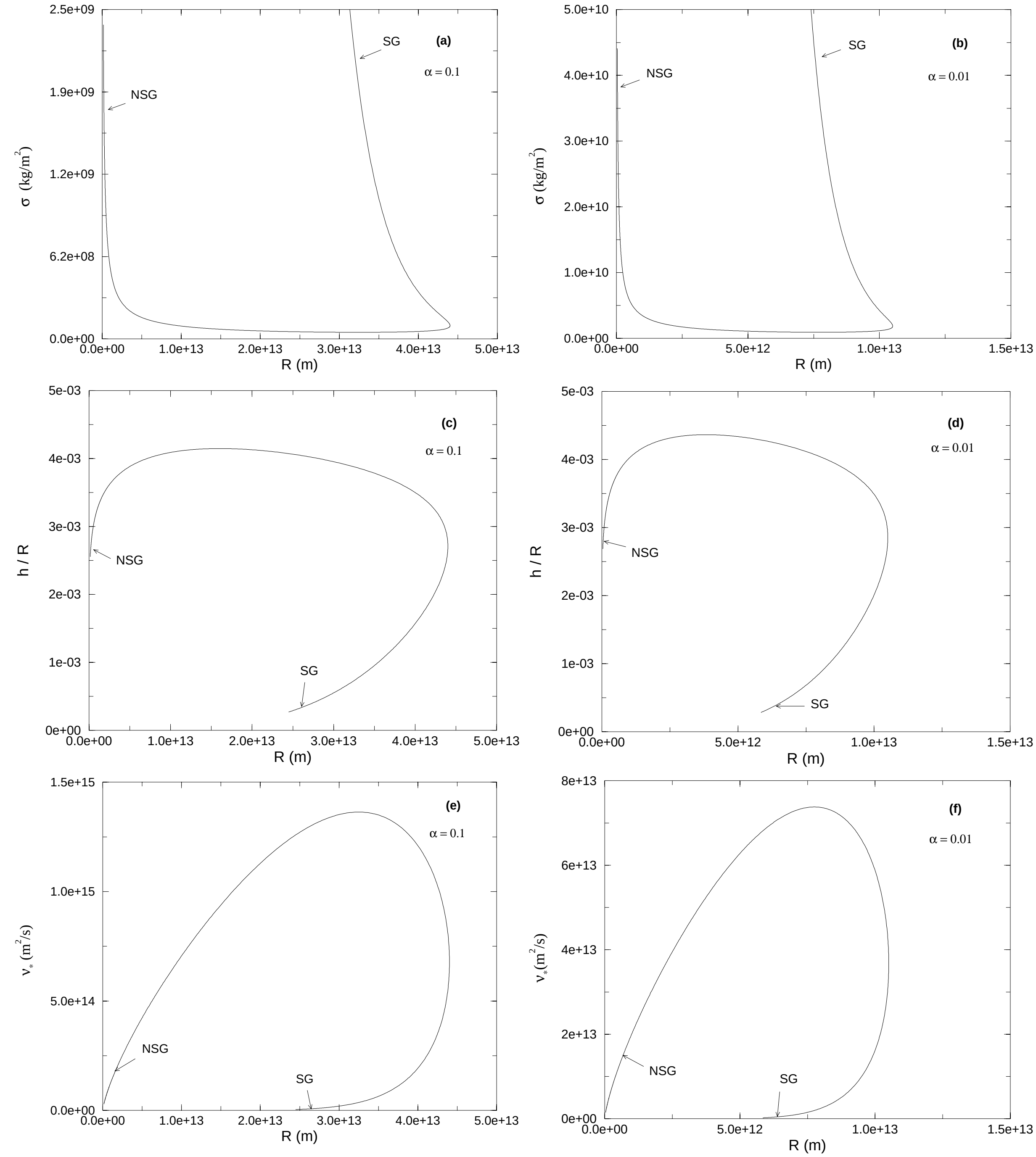


**Fig. 1a–f.** Surface density $\sigma$, $h/R$ and of the viscosity $\nu_\star$ as functions of the radius $R$ for two different values of $\alpha$ : on the left side **a**, **c** and **e** we have $\alpha = 0.1$ and on the right side **b**, **d** and **f** we have $\alpha = 0.01$. We have chosen here $|\dot{M}| = 10 M_\odot/\mathrm{yr}$ and $M_\star = 10^8 M_\odot$. The self-gravitating (SG) and non self-gravitating (NSG) branches are also indicated.

$R = 10 R_\mathrm{s}$ for $\alpha = 0.1$. Deviation from the keplerian rotation is thus significant on the self-gravitating branch for $R \simeq R_\mathrm{s}$ but deviation from keplerian cannot explain the fact that no solution can be found for $R > R_\mathrm{max}$.

Another assumption is that the angular momentum transport is due to an effective turbulent viscosity. We have thus neglected angular momentum transport by wind. This is justified by the fact that the magnetic Reynolds number $R_\mathrm{M}$ associated with the radial inward velocity is very small because the disk is thin (Heyvaerts et al., 1996)

$$R_\mathrm{M}(v_\mathrm{r}) = \frac{R\, v_\mathrm{r}}{\eta_\star} \sim \frac{1}{P}\frac{h}{R} \tag{51}$$

where $\eta_\star$ is the magnetic diffusivity which is related to the effective turbulent viscosity by the Prandtl number : $\eta_\star = P\nu_\star$. The Prandtl number is proved to be of order 1 in the case of

**Table 1.** Typical values of $R_{\max}$ for some objects.

| Typical values | $M_\star$ ($M_\odot$) | $\lvert\dot{M}\rvert$ ($M_\odot$/yr) | $R_{\rm ext}$ (m) | $R_{\max}$ (m) |
|---|---|---|---|---|
| Intermediate polars | 0.8 | $10^{-9}$ | $5.\,10^{7}$ | $3.\,10^{16}\,\alpha^{28/45}$ |
| X-ray binaries (low mass) | 1 | $10^{-8}$ | $10^{10}$ | $10^{16}\,\alpha^{28/45}$ |
| T-Tauri stars | 1 | $10^{-7}$ | $2.\,10^{12}$ | $3.\,10^{15}\,\alpha^{28/45}$ |

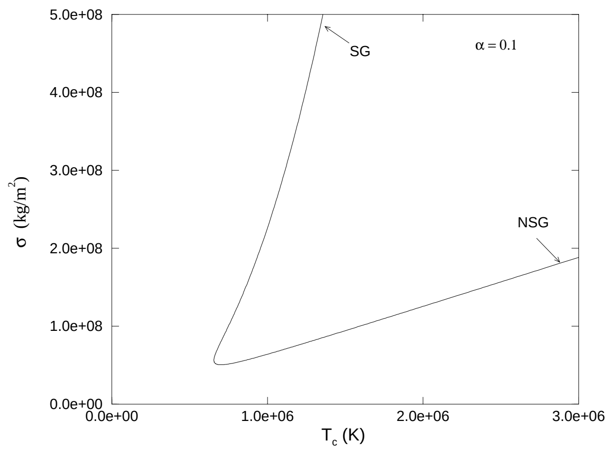


**Fig. 2.** Surface density as a function of the central temperature for $\alpha = 0.1$. The self-gravitating and non self-gravitating branches are identified by respectively SG and NSG.

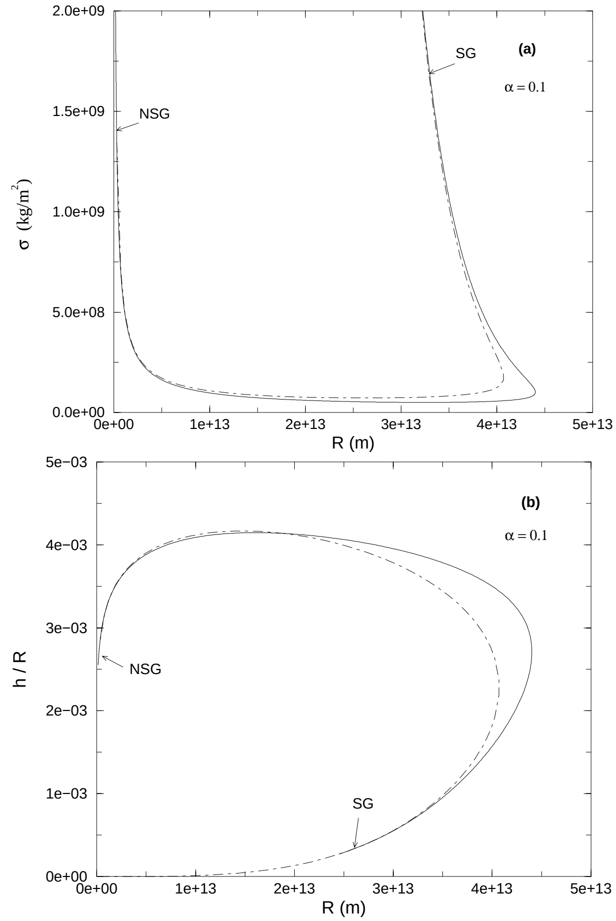


**Fig. 3a and b.** Comparison of the analytical solution (solid lines) and of the exact numerical solution (dot-dashed lines) for $\alpha = 0.1$.

a viscosity and magnetic diffusivity both due to 3D turbulence (Pouquet et al., 1976). As mentioned before, in our solution, the disk remains thin ($h/R \ll 1$) and thus neglecting angular momentum transport by wind is a self-consistent assumption.

The last assumption is that the radiation pressure $P_{\rm r}$ can be neglected compared to the gas pressure $P_{\rm g}$. The radiation pressure is calculated using

$$P_{\rm r} = \frac{1}{3}\, a\, T_{\rm c}^4 \qquad (52)$$

where $a$ is the radiation density constant. The ratio between these two pressures is plotted in Fig. 5 for $\alpha = 0.1$. For radii of physical relevance, we always have $P_{\rm g}/P_{\rm r} \gg 1$.

To sum up, the solution given by Eqs. (44) to (47) is self-consistent.

### *4.3. Transition to a disk dominated by self-gravity*

As stressed in Sect. 2, angular momentum transport in a disk dominated by self-gravity is probably carried out by gravitational torque and not by turbulence. The maximum radius $R_{\max}$ is located at $u \simeq 0.8$ where $Q = 1.05$ and thus where self-gravity begins to dominate. This means that we find a solution in the domain of the disk where the prescription (3) is valid ($R < R_{\max}$ and $Q > 1$) but we do not find a solution where it is not valid ($R > R_{\max}$) because self-gravity dominates. The $\alpha$-prescription does not have this behaviour of 'stopping' where it is not valid. This study thus strengthens the concept of the formulation (3) for the effective turbulent viscosity. Note that the second branch of solution we find and which is very thin with a high surface density is unphysical because it is gravitationally unstable as $Q \ll 1$.

As mentioned in Sect. 2, the effective viscosity of the part of the disk dominated by self-gravity is likely to be given by a constant Toomre parameter. Such a solution can be easily calculated replacing Eq. (3) by the condition $Q =$ cste in Eq. (17) and is given by:

$$\sigma = \frac{6.10^{35}}{R^{57/28}} \left(\frac{M_\star}{10^8 M_\odot}\right)^{19/28} \left(\frac{\dot{M}}{10 M_\odot/yr}\right)^{1/14} Q^{-8/7} \left(\frac{\sqrt{1+Q^2}+1}{\sqrt{1+Q^2}-1}\right)^{1/28} \qquad (53)$$

$$\frac{h}{R} = \frac{2.10^{-2}}{R^{1/28}} \left(\frac{10^8 M_\odot}{M_\star}\right)^{9/28} \left(\frac{\dot{M}}{10 M_\odot/yr}\right)^{1/14} Q^{-1/7} \left(\frac{\sqrt{1+Q^2}+1}{\sqrt{1+Q^2}-1}\right)^{13/28} \qquad (54)$$

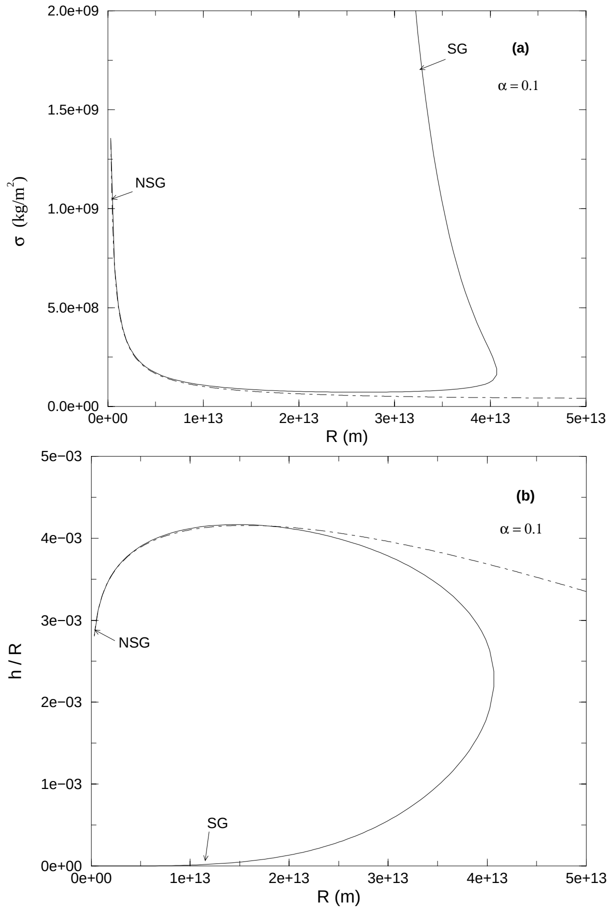


**Fig. 4a and b.** Comparison of the solution with the standard prescription (dot-dashed lines) and of the solution with the new prescription given by Eq. (3) (solid lines) for $\alpha = 0.1$.

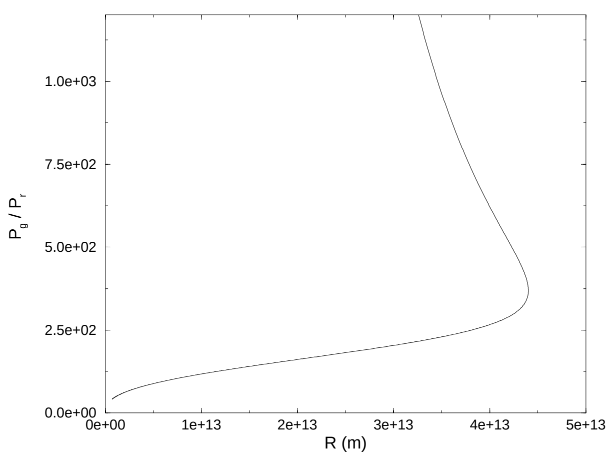


**Fig. 5.** Ratio of the gaz pressure $P_{\rm g}$ to the radiation pressure $P_{\rm r}$ as a function of $R$ for $\alpha = 0.1$.

This self-gravitating solution (for two different values of $Q$) and the solution obtained in Sect. 3 are reported in Fig. 6a and b. The global solution for the disk would thus be constituted of our non self-gravitating branch of solution until the radius where $Q \simeq 1$ is reached and of the branch of solution with $Q \simeq 1$ afterwards.

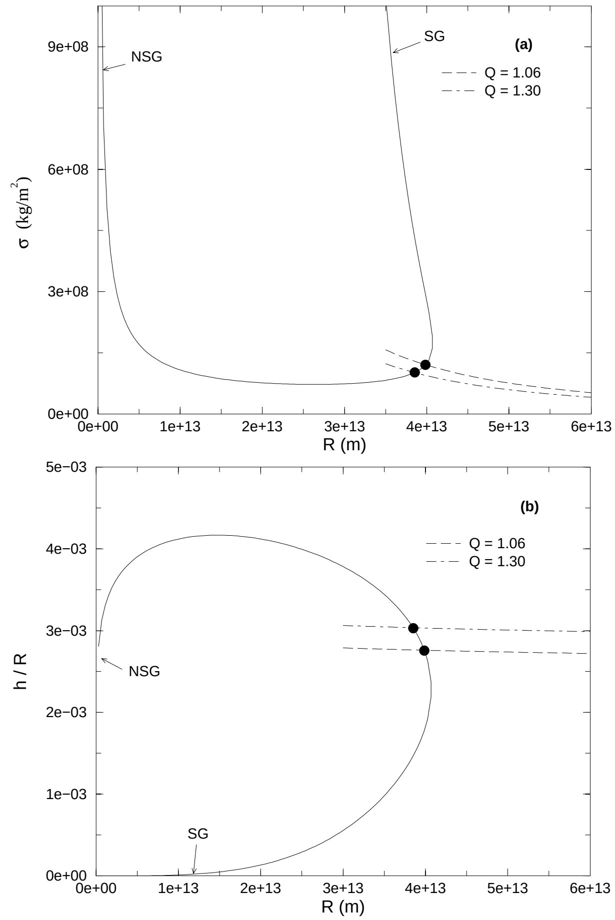


**Fig. 6a and b.** Surface density $\sigma$ and $h/R$ as functions of the radius $R$ for $\alpha = 0.1$. The two points show the place where $Q = 1.06$ and $Q = 1.30$. The corresponding solutions $Q =$ constant are also plotted.

## 5. Conclusion

We propose an analytical method to calculate the accretion disk structure including self-gravity and with a viscosity based on turbulence. We find that the formulation of the effective turbulent viscosity by Heyvaerts et al. (1996) has a natural satisfactory behaviour of giving a solution only where it is valid ($R < R_{\rm max}$ where $Q > 1$). The solution is thus bounded and at the maximum radius we have $Q \simeq 1$ which is the place where the effective viscosity is not due anymore to turbulence because gravitational torques exerted by spiral waves become dominant.

*Acknowledgements.* Part of this work was done when A.B. was financially supported by SFB 328. We thank S. Collin-Soufrin, J.-M. Hameury and J.-M. Huré for fruitful discussions and in particular for pointing out the importance of opacity. We thank an anonymous referee for constructive remarks which helped to improve the paper.

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