Astron. Astrophys. 338, 111–121 (1998)

ASTRONOMY AND ASTROPHYSICS

Modelling of magnetic fields of CP stars

I. A diagnostic method for dipole and quadrupole fields from Stokes I and V observations

M. Landolfi[1], **S. Bagnulo**[2], **and M. Landi Degl'Innocenti**[3]

[1] Osservatorio Astrofisico di Arcetri, Largo E. Fermi 5, I-50125 Firenze, Italy (landolfi@arcetri.astro.it)
[2] Institut für Astronomie, Universität Wien, Türkenschanzstrasse 17, A-1180 Wien, Austria (bagnulo@sinead.ast.univie.ac.at)
[3] C.N.R., Gruppo Nazionale di Astronomia, Unità di Ricerca di Arcetri, Largo E. Fermi 5, I-50125 Firenze, Italy (mlandi@arcetri.astro.it)

Received 25 May 1998 / Accepted 21 July 1998

Abstract. This paper is the first of a series aimed at developing and applying a new modelling technique for magnetic fields of chemically peculiar (CP) stars. Newly introduced observational techniques provided us evidence that the "oblique rotator model" with a simple dipolar field is not sufficient to describe the magnetic morphology of CP stars, and that the problem of inverting data must be approached within a more sophisticated framework. In this paper we consider the configuration produced by the superposition of a dipole and a quadrupole, and we study in detail the behaviour of some observational quantities related to the magnetic field, i.e. the "mean longitudinal field", the "crossover" and the "mean square field", which are obtained from Stokes I and V observations. We illustrate the diagnostic content of the combined analysis of these three kinds of measurement, and we discuss some intrinsic limitations of the method. We present an algorithm for inverting the data, which has been tested against numerical simulations. We examine the possibility of recovering the stellar magnetic configuration from real measurements, by taking into account the problems related to the observational errors.

Key words: stars: magnetic fields – polarization – stars: chemically peculiar

1. Introduction

Our knowledge of the magnetic fields of CP stars originates from observations of different kinds, all based on phenomena related to the Zeeman effect. The most classical observations are those of mean longitudinal field and of mean field modulus (or mean surface field) which are based on the circular polarization and the splitting of spectral lines, respectively (Babcock 1947, 1960). Further observational quantities have been recently introduced by Mathys (1993): the crossover and the mean square field, which are derived from the second-order moments of the line profiles of the Stokes parameters V and I, respectively. Finally, Leroy (1995) performed broad band linear polarization (BBLP) observations on several CP stars.

The modelling of the magnetic structure of CP stars has been mainly based on the oblique rotator model (ORM) with a dipolar field (Babcock 1949). Although this model has generally appeared consistent with longitudinal field measurements, there are several indications that it could be inadequate in some cases. Discrepancies from the behaviour predicted by the model are found both in longitudinal field measurements (e.g., HD 137509: see Mathys 1994 and Mathys & Hubrig 1997) and in BBLP measurements (e.g., 49 Cam: see Leroy et al. 1994). In other cases, the *combined* interpretation of longitudinal field and BBLP measurements in the framework of the model is not possible (e.g., HD 71866: see Bagnulo et al. 1995). On the other hand, it has been shown that abundance inhomogeneities – which are known to exist in most CP stars – cannot explain by themselves such discrepancies (Leroy et al. 1995). On the whole, there are strong suggestions that the magnetic field of CP stars is not (or at least, not always) purely dipolar.

The problem of introducing a magnetic configuration more sophisticated than the classical centred dipole was considered, e.g., by Stift (1975) and by Landi Degl'Innocenti et al. (1981), who introduced the decentred-dipole model to describe the magnetic field of β CrB and 53 Cam, respectively. Leroy et al. (1996) proposed a modified dipolar model obtained by expanding outwards the lines of force in a belt around the magnetic equator.

A more general approach was followed by Bagnulo et al. (1996), who considered the expansion of the stellar magnetic field into multipoles of any order. The calculations presented in that paper make use of the powerful formalism of spherical tensors and Racah algebra, which allows one to find very compact expressions for the different observational quantities. However, such formalism is rather complicated, and the application of the results to actual measurements is not straightforward. On the other hand, consideration of multipoles of order higher than the second one (quadrupole) is probably an untimely sophistication because of the large errors which at present affect the measurements.

Send offprint requests to: M. Landolfi

In the present work, which is a first application of the results obtained by Bagnulo et al. (1996), we study in detail the expressions of the mean longitudinal field, crossover and mean square field both for the dipole and for the dipole plus quadrupole magnetic configuration. In Bagnulo et al. (1996) the mean field modulus and the BBLP were also considered; however, no analytical expression was derived for the former, and the expression obtained for the latter is valid only if the magnetic field is weak. It has been shown that the weak field assumption does not introduce substantial errors in the calculation of the BBLP if the magnetic configuration is purely dipolar (Bagnulo et al. 1995); but this property is probably related to the fact that, for the dipole configuration, the field modulus does not change much (only a factor 2) over the stellar surface, which is no longer true for the dipole plus quadrupole configuration. Therefore, a systematic analysis is needed to establish whether the analytical expression for the BBLP derived by Bagnulo et al. (1996) for the dipole plus quadrupole configuration can be applied to CP stars with typical fields of several kG; this matter is under investigation at present. On the other hand, measurements of longitudinal field, crossover, and square field are available for several CP stars where the BBLP has not yet been measured. For these reasons we restrict attention, for the time being, to the three quantities above, all of which can be expressed with simple analytical formulae, suitable to fit the observational data.

The aim of this paper is to examine the possibility of determining the stellar magnetic configuration (and in particular, of identifying the presence of a quadrupolar contribution) combining mean longitudinal field, crossover, and mean square field observations; in a forthcoming paper the results of this study will be applied to the CP stars for which such data are available.

The paper is organized as follows. In Sect. 2 we write down the explicit formulae for the three quantities in the presence of a dipole plus quadrupole magnetic field. In Sect. 3 we discuss their symmetry properties and point out the degenerate cases, i.e. those magnetic configurations that cannot be distinguished from each other. In Sect. 4 we describe an inversion algorithm and, via numerical simulations, we analyze the actual invertibility of the formulae by allowing for observational errors. A few additional remarks on the inversion procedure are presented in Sect. 5, and the main conclusions are summarized in Sect. 6.

2. Expressions of the mean longitudinal field, crossover, and mean square field for the dipole plus quadrupole magnetic configuration

We consider the ORM with a magnetic configuration produced either by a dipole, or by a (non-linear) quadrupole, or by the superposition of both of them. The model is assumed to be centered, i.e. the elementary multipoles that produce the magnetic configuration at the stellar surface are located at the star's center. As illustrated in Bagnulo et al. (1996), the dipole field can be characterized by the magnetic mass $\mathcal{M}_{\rm d}$ plus the direction $\boldsymbol{u}$ and modulus s (having dimensions of length) of a vector $\boldsymbol{s} = s\boldsymbol{u}$; the quadrupole field can be characterized by the magnetic mass $\mathcal{M}_{\rm q}$ plus the directions and moduli of two vectors $\boldsymbol{s}_1 = s_1\boldsymbol{u}_1$ and $\boldsymbol{s}_2 = s_2\boldsymbol{u}_2$. Denoting by R_* the star's radius, the magnetic field vector at a given point $\boldsymbol{r} \equiv (R_*, \theta, \chi)$ of the stellar surface is given, respectively, by

$$\boldsymbol{\mathcal{B}}_{\rm d}(\boldsymbol{r}) \;=\; -\frac{B_{\rm d}}{2}\left[\boldsymbol{u} - 3\,\frac{\boldsymbol{u}\cdot\boldsymbol{r}}{R_*^2}\,\boldsymbol{r}\right] \qquad (1)$$

$$\boldsymbol{\mathcal{B}}_{\rm q}(\boldsymbol{r}) \;=\; -\frac{B_{\rm q}}{2}\left[\frac{\boldsymbol{u_2}\cdot\boldsymbol{r}}{R_*}\,\boldsymbol{u_1} + \frac{\boldsymbol{u_1}\cdot\boldsymbol{r}}{R_*}\,\boldsymbol{u_2} + \left(\frac{\boldsymbol{u_1}\cdot\boldsymbol{u_2}}{R_*} - 5\,\frac{(\boldsymbol{u_1}\cdot\boldsymbol{r})(\boldsymbol{u_2}\cdot\boldsymbol{r})}{R_*^3}\right)\boldsymbol{r}\right], \qquad (2)$$

where

$$B_{\rm d} = 2\,k_{\rm G}\,\mathcal{M}_{\rm d}\;s\big/R_*^3\,, \qquad B_{\rm q} = 6\,k_{\rm G}\,\mathcal{M}_{\rm q}\;s_1\,s_2\big/R_*^4\,,$$

with $k_{\rm G}$ the Gilbert constant.[1] The quantity $B_{\rm d}$ is the dipole field modulus at the magnetic pole. The quantity $B_{\rm q}$ is the quadrupole field modulus at the magnetic pole *only* when the unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ coincide (linear quadrupole); otherwise, as apparent from Eq. (2), the concept itself of magnetic pole becomes meaningless. If both a dipole and a quadrupole are present, the magnetic field vector at point $\boldsymbol{r}$ is obviously given by the vector sum $\boldsymbol{\mathcal{B}}(\boldsymbol{r}) = \boldsymbol{\mathcal{B}}_{\rm d}(\boldsymbol{r}) + \boldsymbol{\mathcal{B}}_{\rm q}(\boldsymbol{r})$.

The angles characterizing the magnetic configuration are illustrated in Fig. 1, which represents a star with centre C and radius R_*. The right-handed, orthogonal frame (x, y, z) has the z-axis along the line of sight (pointing to the observer) and the x-axis in the direction of the North celestial pole. The point R is the *positive* rotation pole (the star rotates, by definition, in the counterclockwise direction as shown by the arrows); the direction of the star's rotation axis is specified, in the frame (x, y, z), by the inclination angle i and the azimuth angle Θ. The dipole field, with positive magnetic pole at point M, is characterized by the unit vector $\boldsymbol{u}$, the orientation of which is specified by the angles β and $(f - f_0)$. The phase angle f varies linearly with time ($f = 2\pi\,t/P$, with P the rotation period); when $f = f_0$ the angle between $\boldsymbol{u}$ and the z-axis is minimum. Similarly, the quadrupole field is characterized by the unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, the orientation of which is specified by the angles (β_1, γ_1) and (β_2, γ_2), respectively. If the magnetic field is purely quadrupolar, the azimuth angles of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are reckoned from an arbitrary half-plane (fixed on the star) containing the rotation axis, and f_0 can be set to zero. It follows that the zero-phase point is specified by $(f + \gamma_1) = 0$ in agreement with the definition given in Bagnulo et al. (1996), and corresponds to the minimum value for the angle between $\boldsymbol{u}_1$ and the z-axis.

The observational quantities we are interested in are the mean longitudinal field, the crossover, and the mean square field as defined by Mathys (1995a,b). Assuming a limb-darkening law of the form

$$1 - u + u\,\cos\theta \qquad (0 \le u \le 1),$$

[1] Given two magnetic masses m_1 and m_2 separated by the distance r, $k_{\rm G}$ is the constant appearing in the Coulomb law which gives the interaction force $F = k_{\rm G}\,m_1 m_2/r^2$.

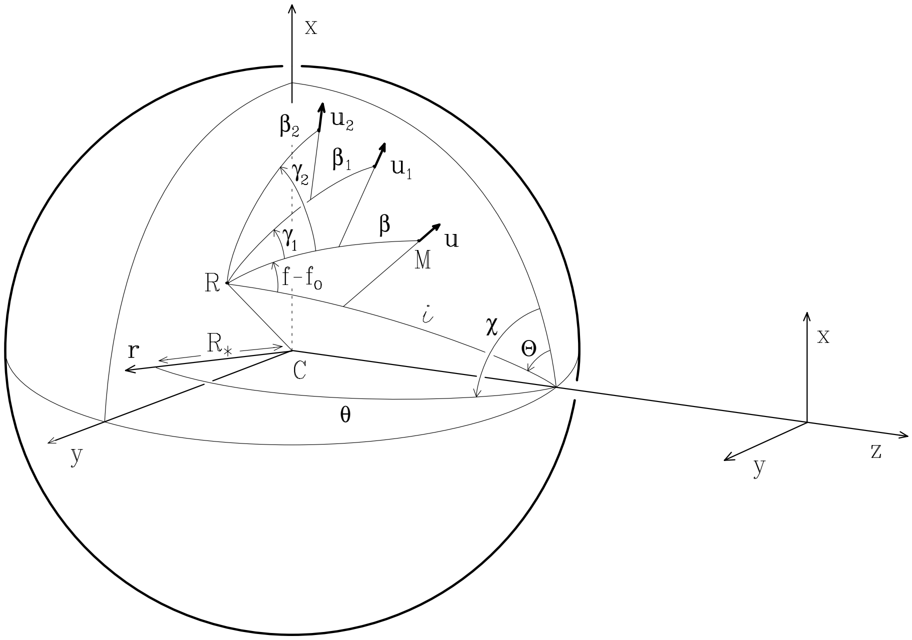

Fig. 1. Geometry of the oblique rotator for a dipole plus quadrupole magnetic field

they are given, respectively, by the expressions

$$
\begin{aligned}
F^{(1)} &= \langle \mathcal{B}_z \rangle \\
F^{(2)} &= v_{\rm e}\, \sin i \,\langle d\, \mathcal{B}_z \rangle \\
F^{(3)} &= \langle \mathcal{B}^2 + \mathcal{B}_z^2 \rangle \,,
\end{aligned}
\tag{3}
$$

where $v_{\rm e}$ is the (positive) equatorial velocity of the star, d the algebraic distance of a point on the stellar surface (where the magnetic field vector is $\boldsymbol{\mathcal{B}}$) from the plane containing the rotation axis and the line of sight (d is positive on the stellar side receding from the observer), and the symbol $\langle \cdots \rangle$ denotes the disk average, i.e., for an arbitrary function $\Phi(\theta, \chi)$,

$$
\langle \Phi \rangle = \frac{3}{\pi\,(3-u)} \times \int_0^{2\pi} {\rm d}\chi \int_0^{\pi/2} {\rm d}\theta \, \sin\theta \cos\theta \,(1-u+u\cos\theta)\; \Phi(\theta,\chi)\,.
$$

The three quantities $F^{(k)}$ defined in Eqs. (3) have been evaluated by Bagnulo et al. (1996) for an arbitrary multipolar configuration and an arbitrary value of the angle Θ. However, only the linear polarization is affected by Θ. The quantities $F^{(k)}$, which are derived from (unpolarized) intensity or circular polarization in spectral lines, turn out to be independent of Θ; thus we can consider, without losing in generality, their expressions for $\Theta = 0$. For the dipole plus quadrupole configuration we obtain

$$
\begin{aligned}
F^{(1)} &= B_{\rm d}\, c_1^{(1)} f_1^{(1)} + B_{\rm q}\, c_2^{(1)} f_2^{(1)} \\
F^{(2)} &= v_{\rm e} \left[B_{\rm d}\, c_1^{(2)} f_1^{(2)} + B_{\rm q}\, c_2^{(2)} f_2^{(2)} \right] \\
F^{(3)} &= B_{\rm d}^2 \left[c_1^{(3)} f_1^{(3)} + c_2^{(3)} f_2^{(3)} \right] \\
&\quad + B_{\rm d} B_{\rm q} \left[c_3^{(3)} f_3^{(3)} + c_4^{(3)} f_4^{(3)} \right] \\
&\quad + B_{\rm q}^2 \left[c_5^{(3)} f_5^{(3)} + c_6^{(3)} f_6^{(3)} + c_7^{(3)} f_7^{(3)} \right] ,
\end{aligned}
\tag{4}
$$

where the coefficients $c_j^{(k)}$ depend on the limb-darkening factor u, and the quantities $f_j^{(k)}$ depend on the magnetic geometry. Explicitly we have

$$
\begin{aligned}
c_1^{(1)} &= c_0\, \frac{15+u}{60} & c_1^{(3)} &= c_0\, \frac{255-79\,u}{360} \\
c_2^{(1)} &= c_0\, \frac{u}{24} & c_2^{(3)} &= c_0\, \frac{105+13\,u}{1260} \\
c_1^{(2)} &= c_0\, \frac{8-3\,u}{40} & c_3^{(3)} &= c_0\, \frac{-1392+377\,u}{1680} \\
c_2^{(2)} &= c_0\, \frac{35-3\,u}{420} & c_4^{(3)} &= c_0\, \frac{192-31\,u}{336} \\
& & c_5^{(3)} &= c_0\, \frac{9765-4517\,u}{40320} \\
& & c_6^{(3)} &= c_0\, \frac{819-227\,u}{2016} \\
& & c_7^{(3)} &= c_0\, \frac{3045-677\,u}{20160} \,,
\end{aligned}
\tag{5}
$$

where $c_0 = 3/(3-u)$. Denoting by (X,Y,Z), (X_1,Y_1,Z_1), (X_2,Y_2,Z_2) the components of the unit vectors $\boldsymbol{u}$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, respectively, in the reference frame (x, y, z) of Fig. 1, the quantities $f_j^{(k)}$ are given by[2]

$$
\begin{aligned}
f_1^{(1)} &= Z \\
f_2^{(1)} &= 2Z_1Z_2 - X_1X_2 - Y_1Y_2 \\
f_1^{(2)} &= -\sin i\ Y \\
f_2^{(2)} &= -\sin i\ (Y_1Z_2 + Y_2Z_1) \\
f_1^{(3)} &= 1 \\
f_2^{(3)} &= 3Z^2 - 1 \\
f_3^{(3)} &= -Z_1(XX_2 + YY_2) - Z_2(XX_1 + YY_1) \\
f_4^{(3)} &= Z\, f_2^{(1)} \\
f_5^{(3)} &= (1 - Z_1^2)(1 - Z_2^2) \\
f_6^{(3)} &= Z_1^2(1 - Z_2^2) + Z_2^2(1 - Z_1^2) \\
&\quad + 2Z_1Z_2(X_1X_2 + Y_1Y_2) \\
f_7^{(3)} &= \left[f_2^{(1)}\right]^2 ,
\end{aligned}
\tag{6}
$$

where, in terms of angles

$$
\begin{aligned}
X &= \sin i \cos\beta - \cos i \sin\beta \cos(f - f_0) \\
X_\ell &= \sin i \cos\beta_\ell - \cos i \sin\beta_\ell \cos(f - f_0 + \gamma_\ell) \\
Y &= -\sin\beta \sin(f - f_0) \\
Y_\ell &= -\sin\beta_\ell \sin(f - f_0 + \gamma_\ell) \\
Z &= \cos i \cos\beta + \sin i \sin\beta \cos(f - f_0) \\
Z_\ell &= \cos i \cos\beta_\ell + \sin i \sin\beta_\ell \cos(f - f_0 + \gamma_\ell) ,
\end{aligned}
\tag{7}
$$

with $\ell = 1, 2$. Note that, by definition, the different parameters are allowed to vary within the range

$$
\begin{aligned}
&B_{\rm d},\ B_{\rm q},\ v_{\rm e} \ge 0 \\
&0 \le i,\ \beta,\ \beta_1,\ \beta_2 \le \pi \\
&0 \le f_0,\ \gamma_1,\ \gamma_2 < 2\pi .
\end{aligned}
\tag{8}
$$

The quantities $F^{(k)}$ defined in Eqs. (4) are functions of the rotational phase and depend on the parameters appearing in Eqs. (8). Their dependence on the various angles is rather complicated, so it is hard to get a qualitative idea of how the three functions change when the angles are varied. Figs. 2 to 4 aim at pointing out, in a few sample cases, the differences between the functions generated by a dipole and by a quadrupole configuration; note that in Fig. 4 we plot $\sqrt{F^{(3)}}$ (the so-called mean quadratic field) rather than $F^{(3)}$, as usually done in the literature.

The quadrupole curves appear to be considerably different from the dipole ones. In some cases, the $F^{(1)}$ and $F^{(2)}$ curves show up a second-order harmonic which is always absent for the dipole. In the general case of dipole plus quadrupole (not shown in the figures) the symmetry character (for $F^{(1)}$ and $F^{(3)}$) and the antisymmetry character (for $F^{(2)}$) about phases 0 and 0.5, typical of the pure dipole, are lost. The ratio between the magnitude of the quadrupole and dipole contributions depends considerably on the individual function. From Eqs. (4) and (5) we have approximately

$$
\frac{F_{\rm q}^{(1)}}{F_{\rm d}^{(1)}} \simeq 0.1\,\frac{B_{\rm q}}{B_{\rm d}}, \qquad \frac{F_{\rm q}^{(2)}}{F_{\rm d}^{(2)}} \simeq 0.5\,\frac{B_{\rm q}}{B_{\rm d}}, \qquad \sqrt{\frac{F_{\rm q}^{(3)}}{F_{\rm d}^{(3)}}} \simeq \frac{B_{\rm q}}{B_{\rm d}} .
$$

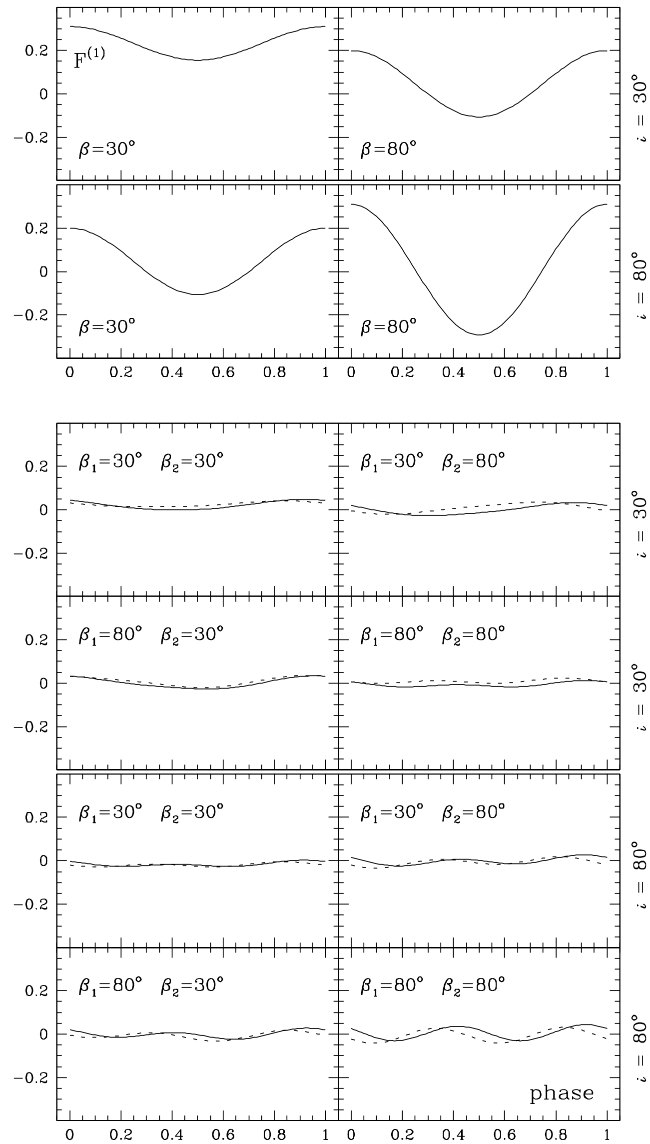

Fig. 2. The quantity $F^{(1)}$ (mean longitudinal field) vs. rotational phase, for a dipole field (upper panel; in units of $B_{\rm d}$) and for a quadrupole field (lower panel; in units of $B_{\rm q}$). The values of the fundamental angles are shown in each figure. The remaining angles are: upper panel, $f_0 = 0°$; lower panel, $\gamma_1 = 0°$, $\gamma_2 = 60°$ (solid line), $\gamma_1 = 0°$, $\gamma_2 = 120°$ (dashed line). The limb-darkening coefficient is $u = 0.5$

[2] Note the minus sign in the expressions of $f_1^{(2)}$ and $f_2^{(2)}$. In Bagnulo et al. (1996) the sign convention for the distance d appearing in Eqs. (3) was, incorrectly, opposite to the convention adopted by Mathys (1995a).

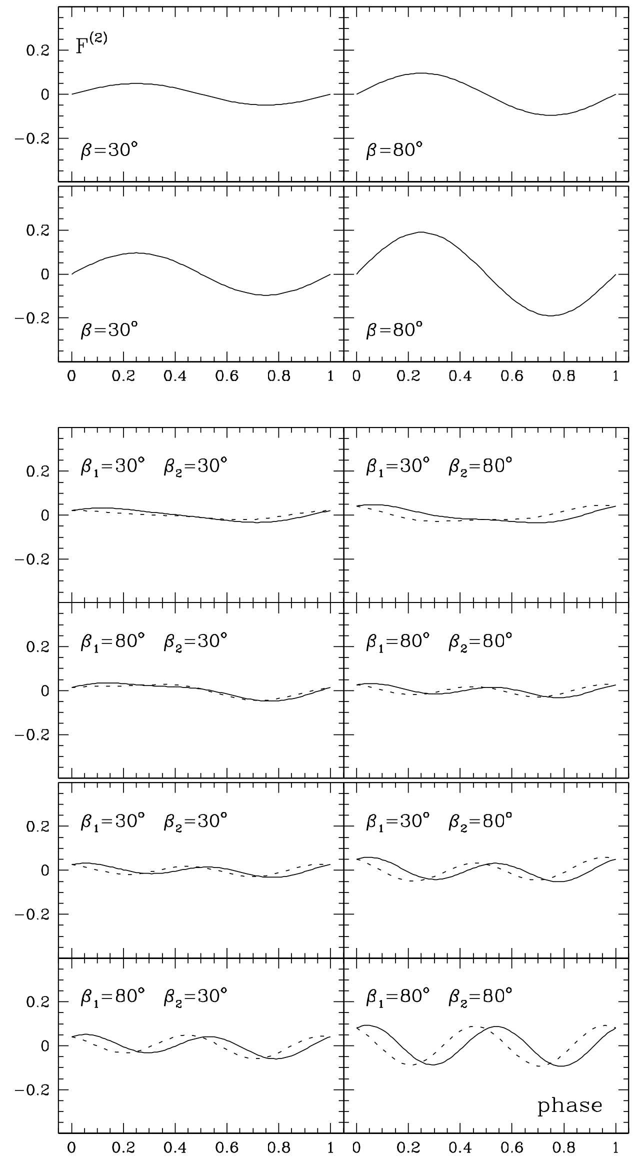

Fig. 3. Same as Fig. 2 for the quantity $F^{(2)}$ (crossover) in units of $v_{\mathrm{e}} B_{\mathrm{d}}$ (upper panel), and in units of $v_{\mathrm{e}} B_{\mathrm{q}}$ (lower panel)

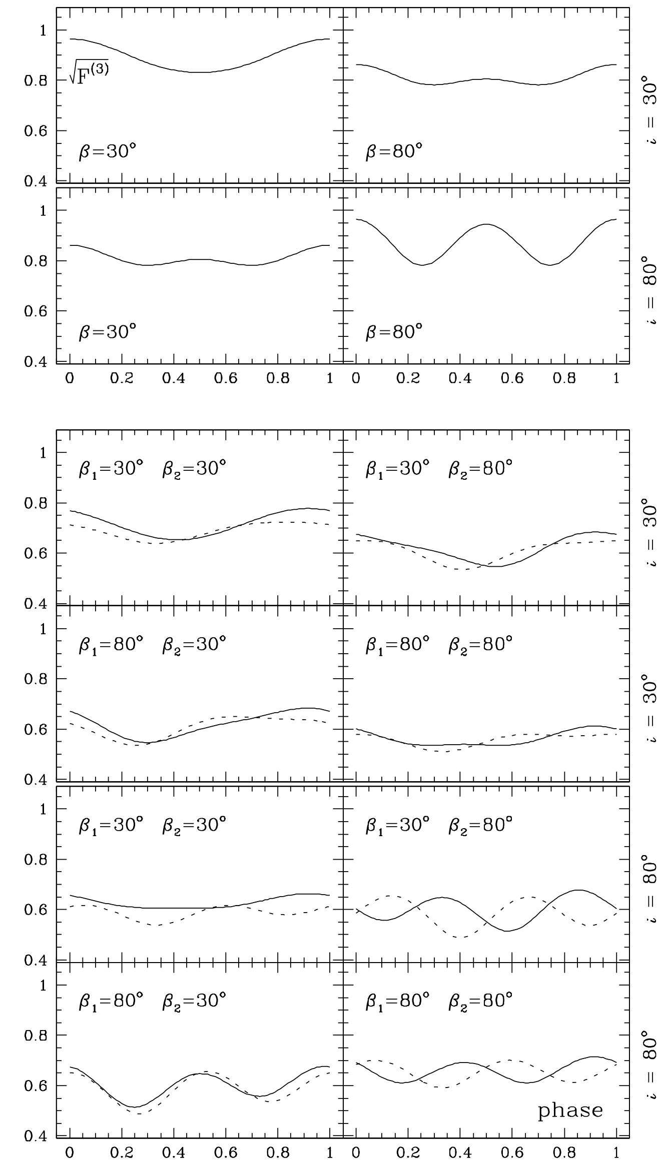

Fig. 4. Same as Fig. 2 for the quantity $\sqrt{F^{(3)}}$ (mean quadratic field)

The last two relations are almost independent of the value of the limb-darkening coefficient, while the first one is rather sensitive to it, ranging from 0.16 for $u = 1$ to zero for $u = 0$: in this extreme case, $F^{(1)}$ is unaffected by the presence of a quadrupole contribution.

A major point related to the use of the quantities $F^{(k)}$ to diagnose the stellar magnetic configuration concerns the invertibility of Eqs. (4). In other words, we should ascertain whether the various parameters on which Eqs. (4) depend can be unambiguously recovered from a set of measurements of the quantities $F^{(k)}$.

This question contains two distinct aspects. The first one, basically a "mathematical" one, concerns the possible degeneracies – or invariance properties – of Eqs. (4): there might exist different sets of parameters yielding the same values for the functions $F^{(k)}$ at all rotational phases. Obviously, the answer to this question depends on the specific function ($F^{(1)}$, $F^{(2)}$, or $F^{(3)}$) and magnetic configuration (dipole, quadrupole, or dipole plus quadrupole).

The second aspect is more "physical", and concerns the *actual* invertibility of Eqs. (4): when the observational errors are taken into account, is it still possible to recover (within the limitations due to degeneracy) the set of parameters? Here the answer depends on the size of the errors and, most likely, on the region in the parameters space.

These two aspects will be considered in the following sections, in order to clarify the usefulness of the quantities $F^{(k)}$ to diagnose stellar magnetic fields, and in particular to establish whether the existence of a quadrupolar contribution can be singled out.

Table 1. Invariance properties of the functions $F^{(k)}$ for a dipole field (see text)

			$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	
i	β	f_0	$*$	$*$	$*$	
i	$\pi-\beta$	f_0		$*$		c
i	$\pi-\beta$	$\pi+f_0$			$*$	b
$\pi-i$	β	f_0		$*$		
$\pi-i$	β	$\pi+f_0$			$*$	
$\pi-i$	$\pi-\beta$	f_0	$*$	$*$	$*$	a
β	i	f_0	$*$	$*$	$*$	d
β	$\pi-i$	f_0		$*$		
β	$\pi-i$	$\pi+f_0$			$*$	
$\pi-\beta$	i	f_0		$*$		
$\pi-\beta$	i	$\pi+f_0$			$*$	
$\pi-\beta$	$\pi-i$	f_0	$*$	$*$	$*$	

Table 2. Same as Table 1 for a quadrupole field

					$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	
i	β_1	β_2	γ_1	γ_2	$*$	$*$	$*$	
i	$\pi-\beta_1$	β_2	$\pi+\gamma_1$	γ_2			$*$	b_1
i	β_1	$\pi-\beta_2$	γ_1	$\pi+\gamma_2$			$*$	b_2
i	$\pi-\beta_1$	$\pi-\beta_2$	$\pi+\gamma_1$	$\pi+\gamma_2$	$*$	$*$	$*$	e
$\pi-i$	$\pi-\beta_1$	$\pi-\beta_2$	γ_1	γ_2	$*$	$*$	$*$	a
$\pi-i$	$\pi-\beta_1$	β_2	γ_1	$\pi+\gamma_2$			$*$	
$\pi-i$	β_1	$\pi-\beta_2$	$\pi+\gamma_1$	γ_2			$*$	
$\pi-i$	β_1	β_2	$\pi+\gamma_1$	$\pi+\gamma_2$	$*$	$*$	$*$	
i	β_2	β_1	γ_2	γ_1	$*$	$*$	$*$	f
i	$\pi-\beta_2$	β_1	$\pi+\gamma_2$	γ_1			$*$	
i	β_2	$\pi-\beta_1$	γ_2	$\pi+\gamma_1$			$*$	
i	$\pi-\beta_2$	$\pi-\beta_1$	$\pi+\gamma_2$	$\pi+\gamma_1$	$*$	$*$	$*$	
$\pi-i$	$\pi-\beta_2$	$\pi-\beta_1$	γ_2	γ_1	$*$	$*$	$*$	
$\pi-i$	$\pi-\beta_2$	β_1	γ_2	$\pi+\gamma_1$			$*$	
$\pi-i$	β_2	$\pi-\beta_1$	$\pi+\gamma_2$	γ_1			$*$	
$\pi-i$	β_2	β_1	$\pi+\gamma_2$	$\pi+\gamma_1$	$*$	$*$	$*$	

3. Invariance properties of the functions $F^{(k)}$

In the cases of pure dipole, pure quadrupole and dipole plus quadrupole, respectively, the functions $F^{(k)}$ depend on the angles

$$i,\ \beta,\ f_0$$

$$i,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2$$

$$i,\ \beta,\ \beta_1,\ \beta_2,\ f_0,\ \gamma_1,\ \gamma_2\ .$$

It can be easily shown that, in each case, there exist sets of values of the angles which lead to the same value for one or several $F^{(k)}$ at all rotational phases. Inspection of Eqs. (4), (6), and (7) yields the results in Tables 1, 2, and 3.

Let us consider Table 1, which refers to the pure dipole field. The first row is the "reference case": for assigned values of i, β, f_0 (plus assigned values of B_d and v_e), each of the functions $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ takes, at a given phase f, a definite value denoted by the asterisk. Each of the asterisks in the following rows means that the corresponding function has the same value as in the reference case if (leaving unchanged B_d, v_e and f) the values of i, β, f_0 are changed into the values appearing in that row. For example, the third row means that, in general

$$F^{(1)}(f;\, i,\, \pi-\beta,\, \pi+f_0,\, B_\mathrm{d}) \neq F^{(1)}(f;\, i,\, \beta,\, f_0,\, B_\mathrm{d})$$

$$F^{(2)}(f;\, i,\, \pi-\beta,\, \pi+f_0,\, B_\mathrm{d},\, v_\mathrm{e}) \neq F^{(2)}(f;\, i,\, \beta,\, f_0,\, B_\mathrm{d},\, v_\mathrm{e})$$

$$F^{(3)}(f;\, i,\, \pi-\beta,\, \pi+f_0,\, B_\mathrm{d}) = F^{(3)}(f;\, i,\, \beta,\, f_0,\, B_\mathrm{d})\,.$$

Tables 2 and 3 have a similar meaning.

It can be seen that the function $F^{(1)}$ is the least degenerate. It also appears that the combined use of all three functions cannot remove, neither reduce, the degeneracy of $F^{(1)}$. This does not mean, however, that the information contained in $F^{(2)}$ and $F^{(3)}$ is useless. For example, it is well known that, for the dipole configuration, the only parameter that can be recovered from $F^{(1)}$ measurements is the angle f_0; on the contrary, if $F^{(1)}$, $F^{(2)}$ and $F^{(3)}$ measurements are used together, the remaining parameters i, β, B_d, v_e can also be recovered (see Sect. 4).

The transformations listed in the three tables can be reduced to a few independent transformations having a simple meaning, as illustrated below:

— Set 1 a (row (a) of Table 1) describes a star whose magnetic field is, at any point, the same as the reference star, but whose rotation direction is opposite and whose azimuth angle is $\Theta' = \Theta + \pi$ (or $\Theta' = \pi$, as we have set $\Theta = 0$ in Eqs. (4)-(7)): for such star the three functions $F^{(k)}$ are unchanged. Set 2 a is partly analogous: it describes a star where the rotation direction is opposite and $\Theta' = \pi$, but where the magnetic field is different from the reference star: the unit vectors $\boldsymbol{u}_1'$ and $\boldsymbol{u}_1$ coincide, while $\boldsymbol{u}_2'$ is the mirror image of $\boldsymbol{u}_2$ about the plane containing $\boldsymbol{u}_1$ and the stellar rotation axis, i.e. $(\gamma_2' - \gamma_1') = -(\gamma_2 - \gamma_1)$. Set 3 a is similar: $\boldsymbol{u}_1'$ and $\boldsymbol{u}_2'$ are the mirror images of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, respectively, about the plane containing $\boldsymbol{u}$ and the rotation axis, i.e. $\gamma_1' = -\gamma_1$, $\gamma_2' = -\gamma_2$.

— Set 1 b describes a star where, at any point, the modulus of the magnetic field is the same as the reference star but its direction is reversed: $F^{(3)}$ is unchanged – as obvious from Eqs. (3) – while $F^{(1)}$ and $F^{(2)}$ are different (in fact they have the opposite sign). Sets 2 b_1 and 2 b_2 are similar (as apparent from Eq. (2), the direction of the quadrupole field $\boldsymbol{\mathcal{B}}_\mathrm{q}(\boldsymbol{r})$ is reversed if either of the unit vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ is reversed): again $F^{(3)}$ is unchanged

Table 3. Same as Table 1 for a dipole plus quadrupole field

							$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	
i	β	β_1	β_2	f_0	γ_1	γ_2	$*$	$*$	$*$	
i	$\pi-\beta$	β_1	β_2	f_0	γ_1	γ_2		$*$		c
i	$\pi-\beta$	β_1	$\pi-\beta_2$	$\pi+f_0$	$\pi+\gamma_1$	γ_2			$*$	b_1
i	$\pi-\beta$	$\pi-\beta_1$	β_2	$\pi+f_0$	γ_1	$\pi+\gamma_2$			$*$	b_2
i	β	$\pi-\beta_1$	$\pi-\beta_2$	f_0	$\pi+\gamma_1$	$\pi+\gamma_2$	$*$	$*$	$*$	e
i	$\pi-\beta$	$\pi-\beta_1$	$\pi-\beta_2$	f_0	$\pi+\gamma_1$	$\pi+\gamma_2$		$*$		
$\pi-i$	β	$\pi-\beta_1$	$\pi-\beta_2$	f_0	γ_1	γ_2		$*$		
$\pi-i$	$\pi-\beta$	$\pi-\beta_1$	$\pi-\beta_2$	f_0	γ_1	γ_2	$*$	$*$	$*$	a
$\pi-i$	β	$\pi-\beta_1$	β_2	$\pi+f_0$	$\pi+\gamma_1$	γ_2			$*$	
$\pi-i$	β	β_1	$\pi-\beta_2$	$\pi+f_0$	γ_1	$\pi+\gamma_2$			$*$	
$\pi-i$	β	β_1	β_2	f_0	$\pi+\gamma_1$	$\pi+\gamma_2$		$*$		
$\pi-i$	$\pi-\beta$	β_1	β_2	f_0	$\pi+\gamma_1$	$\pi+\gamma_2$	$*$	$*$	$*$	
i	β	β_2	β_1	f_0	γ_2	γ_1	$*$	$*$	$*$	f
i	$\pi-\beta$	β_2	β_1	f_0	γ_2	γ_1		$*$		
i	$\pi-\beta$	β_2	$\pi-\beta_1$	$\pi+f_0$	$\pi+\gamma_2$	γ_1			$*$	
i	$\pi-\beta$	$\pi-\beta_2$	β_1	$\pi+f_0$	γ_2	$\pi+\gamma_1$			$*$	
i	β	$\pi-\beta_2$	$\pi-\beta_1$	f_0	$\pi+\gamma_2$	$\pi+\gamma_1$	$*$	$*$	$*$	
i	$\pi-\beta$	$\pi-\beta_2$	$\pi-\beta_1$	f_0	$\pi+\gamma_2$	$\pi+\gamma_1$		$*$		
$\pi-i$	β	$\pi-\beta_2$	$\pi-\beta_1$	f_0	γ_2	γ_1		$*$		
$\pi-i$	$\pi-\beta$	$\pi-\beta_2$	$\pi-\beta_1$	f_0	γ_2	γ_1	$*$	$*$	$*$	
$\pi-i$	β	$\pi-\beta_2$	β_1	$\pi+f_0$	$\pi+\gamma_2$	γ_1			$*$	
$\pi-i$	β	β_2	$\pi-\beta_1$	$\pi+f_0$	γ_2	$\pi+\gamma_1$			$*$	
$\pi-i$	β	β_2	β_1	f_0	$\pi+\gamma_2$	$\pi+\gamma_1$		$*$		
$\pi-i$	$\pi-\beta$	β_2	β_1	f_0	$\pi+\gamma_2$	$\pi+\gamma_1$	$*$	$*$	$*$	

while $F^{(1)}$ and $F^{(2)}$ have the opposite sign. Sets $3\,\mathrm{b}_1$ and $3\,\mathrm{b}_2$ have the same meaning (Eqs. (1)-(2) show that $\boldsymbol{\mathcal{B}}(\boldsymbol{r})$ is reversed if either $\boldsymbol{u}$ and $\boldsymbol{u}_1$, or $\boldsymbol{u}$ and $\boldsymbol{u}_2$, are reversed): $F^{(3)}$ is unchanged, $F^{(1)}$ and $F^{(2)}$ have the opposite sign.

— Set $1\,\mathrm{c}$ describes a star with reversed magnetic field direction (like $1\,\mathrm{b}$), rotated through the angle π with respect to the reference star: $F^{(2)}$ is unchanged while $F^{(1)}$ and $F^{(3)}$ are in general different. Similarly, set $3\,\mathrm{c}$ represents the inversion and the phase shift through π of the dipole field, while the quadrupole field is the same as the reference star: again $F^{(2)}$ is unchanged while $F^{(1)}$ and $F^{(3)}$ are different.

— Set $1\,\mathrm{d}$ describes a star where the values of the angles i and β are exchanged: all the three functions are unchanged.

— Set $2\,\mathrm{e}$ describes the same star as the reference case. The transformation means that both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are reversed, thus the quadrupole field $\boldsymbol{\mathcal{B}}_{\mathrm{q}}(\boldsymbol{r})$ is identical (see Eq. (2)). The same holds for set $3\,\mathrm{e}$: both $\boldsymbol{\mathcal{B}}_{\mathrm{d}}(\boldsymbol{r})$ and $\boldsymbol{\mathcal{B}}_{\mathrm{q}}(\boldsymbol{r})$ are identical.

— Also sets $2\,\mathrm{f}$ and $3\,\mathrm{f}$ describe the same stars as the respective reference cases: Eq. (2) shows that $\boldsymbol{\mathcal{B}}_{\mathrm{q}}(\boldsymbol{r})$ is unaffected by the exchange of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

It can be proved that each of the remaining sets in the tables is a consequence of the transformations described above.

The preceding analysis shows that, provided observations of the three functions $F^{(k)}$ are used *together*, the only configurations that cannot be distinguished are:

- dipole field

$$\left.\begin{array}{ll}
(i,\,\beta,\,f_0) & (i)\\
(\pi-i,\,\pi-\beta,\,f_0) & (ii)\\
(\beta,\,i,\,f_0) & (iii)\\
(\pi-\beta,\,\pi-i,\,f_0) & (iv)
\end{array}\right\} \quad (9)$$

- quadrupole field

$$\left.\begin{array}{ll}
(i,\,\beta_1,\,\beta_2,\,\gamma_1,\,\gamma_2) & (i)\\
(i,\,\pi-\beta_1,\,\pi-\beta_2,\,\pi+\gamma_1,\,\pi+\gamma_2) & (ii)\\
(i,\,\beta_2,\,\beta_1,\,\gamma_2,\,\gamma_1) & (iii)\\
(i,\,\pi-\beta_2,\,\pi-\beta_1,\,\pi+\gamma_2,\,\pi+\gamma_1) & (iv)\\
(\pi-i,\,\pi-\beta_1,\,\pi-\beta_2,\,\gamma_1,\,\gamma_2) & (v)\\
(\pi-i,\,\beta_1,\,\beta_2,\,\pi+\gamma_1,\,\pi+\gamma_2) & (vi)\\
(\pi-i,\,\pi-\beta_2,\,\pi-\beta_1,\,\gamma_2,\,\gamma_1) & (vii)\\
(\pi-i,\,\beta_2,\,\beta_1,\,\pi+\gamma_2,\,\pi+\gamma_1) & (viii)
\end{array}\right\} \quad (10)$$

- dipole plus quadrupole field

$$\left.\begin{array}{ll}
(i,\,\beta,\,\beta_1,\,\beta_2,\,f_0,\,\gamma_1,\,\gamma_2) & (i)\\
(i,\,\beta,\,\pi-\beta_1,\,\pi-\beta_2,\,f_0,\,\pi+\gamma_1,\,\pi+\gamma_2) & (ii)\\
(i,\,\beta,\,\beta_2,\,\beta_1,\,f_0,\,\gamma_2,\,\gamma_1) & (iii)\\
(i,\,\beta,\,\pi-\beta_2,\,\pi-\beta_1,\,f_0,\,\pi+\gamma_2,\,\pi+\gamma_1) & (iv)\\
(\pi-i,\,\pi-\beta,\,\pi-\beta_1,\,\pi-\beta_2,\,f_0,\,\gamma_1,\,\gamma_2) & (v)\\
(\pi-i,\,\pi-\beta,\,\beta_1,\,\beta_2,\,f_0,\,\pi+\gamma_1,\,\pi+\gamma_2) & (vi)\\
(\pi-i,\,\pi-\beta,\,\pi-\beta_2,\,\pi-\beta_1,\,f_0,\,\gamma_2,\,\gamma_1) & (vii)\\
(\pi-i,\,\pi-\beta,\,\beta_2,\,\beta_1,\,f_0,\,\pi+\gamma_2,\,\pi+\gamma_1)\,. & (viii)
\end{array}\right\} \quad (11)$$

It should be borne in mind that Eqs. (9) (dipole field) identify only two truly different magnetic configurations. The sets of parameters in Eqs. (9 i) and (9 ii) represent the same star rotating in opposite directions, and the same is true for Eqs. (9 iii) and (9 iv).

The formal invariance properties of Eq. (2) for the quadrupole field show that Eqs. (10 i) – (10 iv) identify a unique magnetic configuration, which is different from that one, unique, identified by Eqs. (10 v) – (10 $viii$).

Similarly, for the case of a dipole plus quadrupole field, Eqs. (11 i) – (11 iv) identify exactly the same magnetic configuration, which is different from that one, unique, identified by Eqs. (11 v) – (11 $viii$).

We recall that the four sets in Eqs. (9) can in principle be discriminated by using BBLP measurements (Landolfi et al. 1997). Similarly, BBLP observations permit to discriminate between the two different magnetic configurations identified by the sets in Eqs. (10) for the case of simple quadrupole, as well as between the two different magnetic configurations identified by the sets in Eqs. (11) for the case of dipole plus quadrupole field. BBLP observations allow one also to recover the azimuth angle Θ (within multiples of π).

4. An inversion procedure

The problem of recovering the parameters of the functions $F^{(k)}$ from a set of observational data is a typical inversion problem that can be expressed as follows. Given a set of measurements $\{F_i^{(k)}\}$, with $k = 1, 2, 3$ and $i = 1, 2, \ldots, n_k$, performed at times $\{t_i^{(k)}\}$ and affected by errors $\{\sigma_i^{(k)}\}$, we wonder whether a set of parameters values exists such that the functions $F^{(k)}$ defined in Eqs. (4) are consistent with the measurements. A strictly related problem is of course to ascertain the uniqueness of the set.

The most direct method to treat the problem is to define an overall χ^2 of the form

$$\chi^2 = \sum_{k=1}^{3} \sum_{i=1}^{n_k} \frac{\left[F_i^{(k)} - F^{(k)}\left(t_i^{(k)}\right)\right]^2}{\left[\sigma_i^{(k)}\right]^2}, \tag{12}$$

and try to determine its minimum in the parameters space. In principle, this can be done with a "brute force" approach, i.e. by fixing a grid in the parameters space and by evaluating the χ^2 at each grid point. This procedure allows one to find the absolute minimum of the χ^2 hypersurface within the grid step, which can then be further refined to improve the location of the minimum; such method was adopted by Bagnulo et al. (1995) in a similar problem.

In our case, however, this technique would require a very long computation time, because of the large number of parameters involved. For this reason we developed a code to search the minimum of the χ^2 defined in Eq. (12) based on the Marquardt algorithm described in Bevington (1969). The algorithm combines the gradient-search method (which follows the maximum slope direction on the χ^2 hypersurface) with the method of linearisation of the fitting function; on the whole it is a fast and reliable algorithm.

In general, we are interested in the interpretation of the longitudinal field, crossover, and square field measurements both in terms of the pure dipole model and of the dipole plus quadrupole model. Thus we developed two distinct codes: the first one (code1) uses expressions (4) limited to the terms that do not contain $B_{\rm q}$, the other one (code2) uses the full expressions (4). In the first case the free parameters are five

$$i,\, \beta,\, f_0,\, B_{\rm d},\, v_{\rm e}\,, \tag{13}$$

in the second case they are ten

$$i,\, \beta,\, \beta_1,\, \beta_2,\, f_0,\, \gamma_1,\, \gamma_2,\, B_{\rm d}\,, B_{\rm q},\, v_{\rm e}\,. \tag{14}$$

It should be noticed that expressions (4) depend also on the stellar rotation period P and on the limb-darkening coefficient u, which however were considered constant in our codes. The rotation period of the stars we are interested in is usually well known from photometric observations, and it can be checked through an analysis of the magnetic data based on a simple Fourier expansion. On the contrary, the value of u is poorly known, but the dependence of expressions (4) on u is rather weak. As illustrated below, the values of the parameters (13) or (14) recovered by the fits are slightly affected by the value adopted for u.

Given a set of measurements $\{F_i^{(k)}\}$, the search of the χ^2 minimum is performed through an iteration process, which starts at a certain point in the parameters space and ends when some condition is met: e.g., when the value of χ^2 becomes smaller than a given number, or when the variations of the parameters in two successive iterations become smaller than a given amount. Since in this process no restrictions are set to the parameters values, the final set of parameters might be "unphysical": in other words, the parameters values could be outside the range of Eqs. (8). The set can usually be transformed into an equivalent "physical" set by application of suitable rules; for example, in the dipole plus quadrupole case, it can be easily seen that expressions (4) remain unchanged by substituting

$$B_{\rm q} \to -B_{\rm q}, \quad \beta \to -\beta,$$
$$\beta_1 \to -\beta_1, \quad \beta_2 \to \pi - \beta_2, \quad f_0 \to \pi + f_0,$$

which enables a set with a negative ("unphysical") value for $B_{\rm q}$ to be changed into a set with positive $B_{\rm q}$. These "pseudosymmetry" properties will not be described in detail; we just point out that they do not allow *any* set of parameters to be transformed into a "physical" set: certain sets are intrinsically "unphysical" and must be rejected. As for the "physical" sets, two sets related as in Eqs. (9) or (11) – for the dipole and dipole plus quadrupole cases, respectively – should of course be considered identical.

Another problem that must be dealt with is that the χ^2 minimum found by the algorithm when a specific set of parameters was adopted as starting point could be a relative minimum rather than the absolute minimum. Obviously one should repeat the iteration process using different sets of parameters values as starting points and compare the results. In principle no grid of initial sets, though large, can assure one that the absolute minimum will be found; in fact, in all our numerical simulations we found that a small grid (say ten starting sets) was always sufficient to locate the absolute minimum. This suggests that the χ^2 hypersurface, although dependent on a considerable number of parameters, is rather regular and smooth.

In order to check our codes, we carried out a number of numerical simulations. We generated the functions $F^{(k)}(f)$ from Eqs. (4) at assigned phase-points $f_i^{(k)}$ for given values of the parameters, and applied the inversion codes to these data. We considered both the dipole and dipole plus quadrupole configurations, and in each case we applied both code1 and code2. Owing to the symmetry properties of Eqs. (9) and (11), the angles of the input configuration can be chosen from a range smaller than the definition range of Eqs. (8), e.g., from the range

$$0 \le i \le \pi/2\,, \quad i \le \beta \le \pi - i\,, \quad 0 \le f_0 < 2\pi$$

for the dipole case, and

$$0 \le i \le \pi/2\,, \qquad 0 \le \beta \le \pi\,,$$
$$0 \le \beta_1 \le \pi/2\,, \qquad \beta_1 \le \beta_2 \le \pi - \beta_1\,,$$
$$0 \le f_0,\, \gamma_1,\, \gamma_2 < 2\pi$$

for the dipole plus quadrupole case. Actually, we considered 16 input models for the dipole case, corresponding to the values

$$\begin{aligned} &i = 30°/60°, \quad \beta = 70°/110°, \quad f_0 = 120°/270°, \\ &B_\mathrm{d} = 10\,\mathrm{kG}\,, \quad v_\mathrm{e} = 5\,/30\,\mathrm{km\,s^{-1}}\,, \end{aligned}$$

and 16 input models for the dipole plus quadrupole case, corresponding to the values

$$\begin{aligned} &i = 25°/65°, \quad \beta = 45°/135°, \\ &\beta_1 = 45°, \quad \beta_2 = 80°, \\ &f_0 = 150°, \quad \gamma_1 = 150°, \quad \gamma_2 = 300°, \\ &B_\mathrm{d} = 10\,\mathrm{kG}\,, \quad B_\mathrm{q} = 5\,/30\,\mathrm{kG}\,, \quad v_\mathrm{e} = 5\,/30\,\mathrm{km\,s^{-1}}\,. \end{aligned}$$

The functions $F^{(k)}$ were evaluated at 20 equidistant phase-points.

The inversion codes proved to work properly in all the cases considered. More precisely:

(A) *Dipole configuration:* code1 recovered the "true" set of parameters, corresponding to $\chi^2 = 0$; code2 gave the correct values for i, β, f_0, B_d, v_e plus $B_\mathrm{q}^\mathrm{fit} = 0$, again corresponding to $\chi^2 = 0$ (obviously the values of the other angles are meaningless);

(B) *Dipole plus quadrupole configuration:* code2 recovered the true set of parameters with $\chi^2 = 0$; code1 found a minimum with a non-zero χ^2 value corresponding to $f_0^\mathrm{fit} \simeq f_0$, $B_\mathrm{d}^\mathrm{fit} \simeq \max(B_\mathrm{d}, B_\mathrm{q})$, while v_e was somewhat overestimated and the values of i and β were considerably different from the true values.

Obviously, such numerical simulations are of little use to check the *actual* invertibility of Eqs. (4), because exact values (unaffected by errors) were adopted for the quantities $F^{(k)}$; however they show that, at least in this idealized case, the inversion procedure is always able to recover the true set of parameters, and in particular to distinguish the dipole configuration from the dipole plus quadrupole configuration, which was not obvious because of the large number of free parameters involved.

In order to simulate more realistic conditions, we then repeated the calculations introducing an "error" on the input data. For a given set of model parameters, we replaced the exact values $F^{(k)}(f_i^{(k)})$ by $F^{(k)}(f_i^{(k)})+\delta F^{(k)}(f_i^{(k)})$, where the "errors" $\delta F^{(k)}(f_i^{(k)})$ were deduced from a Gaussian distribution of half-width $\sigma^{(k)}$ using a random number generator. For the dipole configuration we considered the 16 input models described above, while for the dipole plus quadrupole configuration we considered a larger sample, i.e. the 256 models corresponding to all combinations of the values

$$\begin{aligned} &i = 25°/65°, \quad \beta = 45°/135°, \\ &\beta_1 = 30°/60°, \quad \beta_2 = 70°/110°, \\ &f_0 = 150°, \quad \gamma_1 = 100°/300°, \quad \gamma_2 = 120°/270°, \\ &B_\mathrm{d} = 10\,\mathrm{kG}\,, \quad B_\mathrm{q} = 5\,/30\,\mathrm{kG}\,, \quad v_\mathrm{e} = 5\,/30\,\mathrm{km\,s^{-1}}\,. \end{aligned} \qquad (15)$$

For the half-widths we adopted the values

$$\sigma^{(1)} = 0.2\,\mathrm{kG}\,, \quad \sigma^{(2)} = 3\,\mathrm{kG\,km\,s^{-1}}\,, \quad \sigma^{(3)} = 1\,\mathrm{kG} \qquad (16)$$

(the third one refers to the quadratic field, $\sqrt{F^{(3)}}$); such values are typical of real measurements, though the precise values depend on the specific stellar spectrum and instrumentation.

As expected, the results were substantially different from the "error-free" case. When the input magnetic configuration was purely dipolar, both code1 and code2 were able to find a (formally) good fit – i.e., a fit with reduced χ^2 of the order of unity – but the two sets of parameters recovered were usually *very different* from each other. When the input configuration was dipole plus quadrupole, code2 could always find a good fit – in the sense above – but in several cases the same was true for code1 *as well*.

Let us consider the situation in greater detail. We define χ^2_d as the minimum χ^2 of the best model recovered with code1, and χ^2_dq as the minimum χ^2 of the best model recovered with code2. We fix a threshold value of approximately 2.2 for the reduced χ^2 and denote by $(++)$ the cases where both code1 and code2 yield a good fit ($\chi^2_\mathrm{d} < 2.2$, $\chi^2_\mathrm{dq} < 2.2$), and by $(-+)$ the cases where only code2 yields a good fit ($\chi^2_\mathrm{d} > 2.2$, $\chi^2_\mathrm{dq} < 2.2$). All the cases considered fall within either class. Furthermore:

(A) *Dipole configuration:* all the fits belong to class $(++)$. The set of parameters recovered by code1 is correct (with typical differences of a few degrees on i, β, f_0 and of a few percent on B_d and v_e), while the set recovered by code2 is unreliable (only the angle f_0 is correctly recovered): usually, a spurious quadrupole of strength B_q comparable to B_d is introduced, and the values recovered for the angles i and β are correct in some cases, wrong in other cases.

(B) *Dipole plus quadrupole configuration:* some fits belong to class $(-+)$, some to class $(++)$. The former class can be divided in two subclasses, depending on the topology of the χ^2 hypersurface in the ten-dimensional space of the parameters of Eq. (14):

— B1: the χ^2 hypersurface has only one minimum (or a few minima corresponding to very close values both for χ^2 and for the parameters): in all such cases the parameters recovered by code2 are the correct ones. The errors on the angles are somewhat larger than in case (A) but seldom larger than $20°$; the errors on B_d, B_q, v_e are usually of order 10%; when the quadrupole field is "weak" ($5\,\mathrm{kG}$, cf. Eqs. (15)) the error on B_q is sometimes larger ($\simeq 40\%$).

— B2: besides the absolute minimum there are other minima with similar χ^2 values, corresponding to sets of parameters very different from each other: in these cases, one of the sets recovered by code2 is usually, but not always, the correct one.

— As far as the fits of class $(++)$ are concerned, the sets of parameters recovered by code2 are correct in some cases, wrong in other cases: in practice all these cases cannot be distinguished from those described under point (A) above: if both code1 and code2 are able to find a good fit, one cannot rule out the possibility that the magnetic configuration is indeed purely dipolar.

Occurrence of conditions $(-+;\mathrm{B1})$, $(-+;\mathrm{B2})$, or $(++)$ is not random, rather it is related to quite definite regions in the parameters space. If both the quadrupole field magnitude and the inclination angle are "large" ($B_\mathrm{q} = 30\,\mathrm{kG}$, $i = 65°$) the best fit always belongs to class $(-+;\mathrm{B1})$. If both are "small" ($B_\mathrm{q} = 5\,\mathrm{kG}$, $i = 25°$) the fit is always of class $(++)$. The intermediate cases (large B_q and small i, or vice versa) yield either $(-+)$ or $(++)$, and this is strongly correlated to the value of the velocity v_e : nearly all cases with "large" v_e ($30\,\mathrm{km\,s^{-1}}$) yield a $(-+)$ fit, while nearly all cases with "small" v_e ($5\,\mathrm{km\,s^{-1}}$) yield a $(++)$ fit.

Such characteristics are easily understood. The importance of the quadrupolar contribution increases with increasing B_q, i, and v_e, so that, owing to the observational errors, the quadrupole field can be identified and "measured" only if these quantities are sufficiently large. In fact, Eqs. (4)-(7) show that in the extreme case $i = 0°$ the three functions $F^{(k)}$ become constant (independent of the rotational phase; $F^{(2)}$ is zero), thus a quadrupole field could not be distinguished from a dipole field even if there were no errors. As for v_e, the contribution of the function $F^{(2)}$ – which is strongly sensitive to the presence of a quadrupole field, see Fig. 3 – to the overall χ^2 in Eq. (12) increases when v_e is increased.

The preceding analysis suggests that, in the presence of errors of the order of Eqs. (16), *a dipole plus quadrupole configuration can be reliably identified only if the best fit belongs to class* $(-+;\mathrm{B1})$, *while all the other cases should be considered ambiguous.* A fit of class $(++)$ is compatible with a purely dipolar field (though a quadrupolar contribution cannot be ruled out); we may only add that *if* the stellar magnetic configuration is actually dipolar, the parameters recovered by our fitting procedure are most likely correct (see point (A) above).

It should be pointed out that the preceding results are slightly affected by the limb-darkening coefficient u. In our numerical simulations, the input values $F^{(k)}(f_i^{(k)})$ were generated using the value $u = 0.5$, and the same value was used in the fitting codes code1 and code2. However, if a different value (within the range $0 \le u \le 1$) is adopted in these codes, one finds the same overall picture and the same values for the parameters within their errors. The effect of u seems to be negligible when dealing with errors as large as in Eqs. (16).

5. Further remarks on the inversion procedure

Two further points related to the inversion of Eqs. (4) are worth noticing. The first one concerns the combined use of longitudinal field and square field measurements, and provides a qualitative method to distinguish a pure dipole from a dipole plus quadrupole configuration. It follows from Eqs. (4)-(7) that for the dipole configuration the functions $F^{(1)}$ and $F^{(3)}$ satisfy the relations

$$F^{(1)}(2f_0 - f) = F^{(1)}(f), \qquad F^{(3)}(2f_0 - f) = F^{(3)}(f),$$

which do *not* hold for the dipole plus quadrupole configuration (unless $\gamma_1 = \gamma_2 = 0°$). Therefore, if a quadrupole contribution is present, the representative point in the plane $(F^{(1)}, F^{(3)})$ describes during a star's rotation cycle a closed curve enclosing a non-zero area; if, on the contrary, the field is purely dipolar, such area reduces to zero and the curve (whose end points correspond

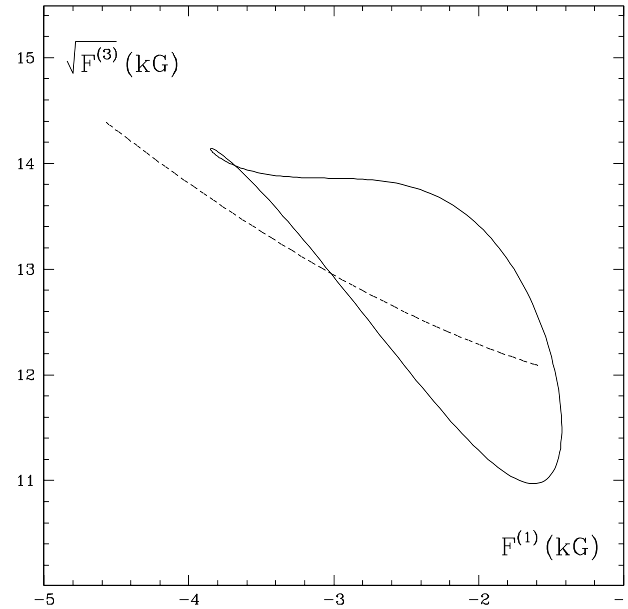

Fig. 5. Curves in the plane $(F^{(1)}, \sqrt{F^{(3)}}\,)$ corresponding to one stellar rotation, for a dipole configuration (dashed line) and a dipole plus quadrupole configuration (solid line). The values of the parameters are: $i = 30°$, $\beta = 140°$, $f_0 = 0°$, $B_\mathrm{d} = 15\,\mathrm{kG}$, $u = 0.5$ (both models); $\beta_1 = 20°$, $\beta_2 = 60°$,$\gamma_1 = 0°$, $\gamma_2 = 140°$, $B_\mathrm{q} = 25\,\mathrm{kG}$ (quadrupole)

to phases $f = f_0$ and $f = f_0 + \pi$) is alternately described in either direction. This behaviour is illustrated in Fig. 5 (again, $\sqrt{F^{(3)}}$ rather than $F^{(3)}$ is plotted). Even in the presence of observational errors, a curve enclosing a "large" area should be considered a strong indication of the presence of a quadrupole contribution (see also Bagnulo 1998).

The second point concerns the equatorial velocity v_e. In principle, the diagnostic method proposed in this paper (the combined use of observations of the three quantities $F^{(k)}$) allows one to determine v_e. The knowledge of v_e and of the rotation period P enables the star's radius R_* to be determined,

$$R_* = \frac{P\,v_\mathrm{e}}{2\pi}\ , \tag{17}$$

or

$$R_* = 0.0198\ P\,v_\mathrm{e}\,, \tag{18}$$

where R_* is expressed in solar radii, P in days and v_e in $\mathrm{km\,s^{-1}}$. At the same time, the method allows one to determine the inclination angle i, whence the projected equatorial velocity $v_\mathrm{e}\sin i$.

The possibility of determining a fundamental parameter such as the stellar radius is remarkable in itself; in favourable conditions (i.e. if the observational errors are sufficiently small and the results of the fit are not ambiguous) one might obtain an estimate of the radius more accurate than by other methods. On the other hand, in several cases the quantity $v_\mathrm{e}\sin i$ is directly derived (though with some uncertainty) from Doppler broadening measurements in spectral lines. Thus we have an independent way to check the consistency of the fits of the quantities $F^{(k)}$

to real data: a reliable fit should yield $v_{\rm e}$ and i values consistent with the known value of $v_{\rm e} \sin i$, as well as a reasonable value for the star's radius.

6. Conclusions

We have presented a new framework for modelling the magnetic field of CP stars via observations of I and V Stokes parameters, by considering a dipole plus quadrupole field. The results obtained in this paper show that the observable quantities derived via Stokes I and V analysis, i.e. the mean longitudinal field, crossover, and mean square field, when used together, are a valuable tool to study the magnetic field of CP stars. The dependence of these observational quantities on the rotational phase predicted by the oblique rotator model with a dipole plus quadrupole magnetic configuration can be noticeably different from that of the purely dipolar configuration: this is of great importance for the interpretation of the numerous cases where the dipole model is found to be inadequate.

Both for the dipole and quadrupole fields, particular magnetic geometries exist which cannot be distinguished from each other because of intrinsic symmetry properties of the expressions of the three quantities. Apart from this (minor) limitation, the possibility of recovering the true magnetic configuration is hampered by the observational errors, which up to now may be too large to allow one to derive certain results from the data analysis. However, in favourable conditions (a large value for the angle between the rotation axis and the line of sight, and a quadrupole field significantly stronger than the dipole field), the actual magnetic configuration can be unambiguously recovered.

Acknowledgements. Stefano Bagnulo has been supported by the Austrian *Fonds zur Förderung der Wissenschaftlichen Forschung*, project P12101-AST.

References

Babcock H.W., 1947, ApJ 105, 105
Babcock H.W., 1949, Observatory 69, 191
Babcock H.W., 1960, ApJ 132, 521
Bagnulo S., 1998, Broadband linear polarisation and modelling of magnetic fields of CP stars. In: Proceedings of the 26th meeting and workshop of the European Working Group on CP stars, North P., Schnell A., Žižňovský J. (eds.) p. 431
Bagnulo S., Landi Degl'Innocenti E., Landolfi M., Leroy J.-L., 1995, A&A 295, 459
Bagnulo S., Landi Degl'Innocenti M., Landi Degl'Innocenti E., 1996, A&A 308, 115
Bevington P.R., 1969, Data reduction and error analysis for the physical sciences, McGraw-Hill, New York
Landi Degl'Innocenti M., Calamai G., Landi Degl'Innocenti E., Patriarchi P., 1981, ApJ 249, 228
Landolfi M., Bagnulo S., Landi Degl'Innocenti M., Landi Degl'Innocenti E., Leroy J.-L., 1997, A&A 322, 197
Leroy J.-L., 1995, A&AS 114, 79
Leroy J.-L., Landstreet J.D., Bagnulo S., 1994, A&A 284, 491
Leroy J.-L., Landolfi M., Landi Degl'Innocenti M., Landi Degl'Innocenti E., Bagnulo S., Laporte P., 1995, A&A 301, 797
Leroy J.-L., Landolfi M., Landi Degl'Innocenti E., 1996, A&A 311, 513
Mathys G., 1993, Magnetic Field Diagnosis through Spectropolarimetry. In: Peculiar versus normal phenomena in A–type and related stars, M.M. Dworetsky, F. Castelli and R. Faraggiana (eds.) I.A.U. Colloquium No. 138, ASP Conferences Series 44, 232
Mathys G., 1994, A&AS 108, 547
Mathys G., 1995a, A&A 293, 733
Mathys G., 1995b, A&A 293, 746
Mathys G., Hubrig S., 1997, A&AS 124, 475
Stift M.J.S., 1975, MNRAS 172, 133