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The influence of turbulence on the solar p-mode frequencies

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Abstract. Calculations were carried out in order to probe the influence of turbulence on the solar oscillation frequencies as a possible explanation of the difference between observed and computed mode frequencies. As a first step the analysis is restricted to the radial oscillation modes. Corrections to the oscillation equations due to turbulence are determined with the help of perturbation theory, and the influence on the solar mixing length model is tested by recalculating the solar convection zone.

The first interesting result is that the first order corrections in the oscillation equations arising from the Reynolds stress do not simply produce a *blueshift*. For small frequencies, due to the form of the turbulent velocity field, we can observe a *redshift* of the frequencies as well. The zeroth order corrections seem to have only a small influence on the results.

For the turbulent modifications of the speed of sound (i.e., the correlation of density and temperature fluctuations), that were predicted as *redshifted* in a previous paper we find on the other hand, when we include the zeroth order corrections in our model as well, the difference turns into an overall *blueshift*.

Key words: turbulence – Sun: oscillations

1. Introduction

It has been well established that the computed frequencies for the solar p-mode oscillations systematically exceed the observed values (Brown 1984; Zhugzhda & Stix 1994; Gabriel & Carlier 1997; Rosenthal 1998; also see our Fig. 1). The observed frequencies are therefore often considered as *redshifted*. In reality, of course, the theoretical values are too high and an extra effect is necessary to lower them (Christensen-Dalsgaard & Thompson 1997). As shown in Fig. 2, the details of the phenomenon are more complicated. The redshift only exists for high frequencies. For frequencies smaller than 2.5–2.6 mHz there is a transition to *blueshift* with a maximum of a few μHz (see also Guzik & Swenson 1997). Obviously two opposite effects are influencing the eigenmodes and dominate in different domains of the frequency diagram.

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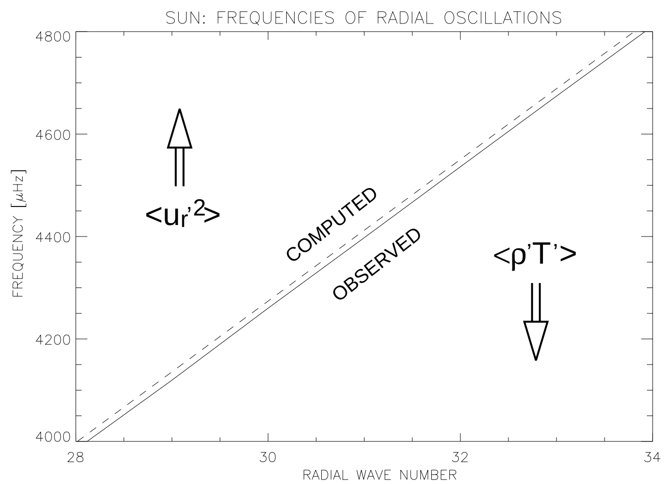

Fig. 1. Observed and computed eigenfrequencies for solar p-modes with $l = 0$. The observed data (solid) are from the GOLF-experiment on-board SOHO. The simulations (dotted) are computed by Stix (1997). The observed modes are *redshifted*. Also, the response of the computed frequencies to turbulence effects, after Eq. (1), are given

In this paper we probe the ability of the turbulence to affect the mode frequencies (Delache & Fossat 1988; Stein et al. 1988; Baturin et. al. 1991; Murawski & Roberts 1992; Antia & Basu 1998; Canuto & Christensen-Dalsgaard 1998). In Rüdiger et al. (1997) we presented the basic trends for the eigenfrequencies under turbulent influences using a very rough model. A free soundwave in a turbulent medium changes its frequency, $\omega^{\rm turb}$, in relation to the laminar case, $\omega^{\rm lam}$, in accordance to

$$\omega^{\rm turb} = \omega^{\rm lam} \left(1 - \frac{\zeta}{2} + \frac{\mathcal{M}^2}{2}\right). \qquad (1)$$

Here $\mathcal{M}$ is the Mach number of the turbulence

$$\mathcal{M} = \frac{\sqrt{\langle u_r'^2 \rangle}}{c}, \qquad (2)$$

with c as the sound velocity. The ζ [1] in Eq. (1) takes into account that for an ideal gas the pressure is a nonlinear combination of

[1] Note ζ is identical with κ in Rüdiger et al. (1997)

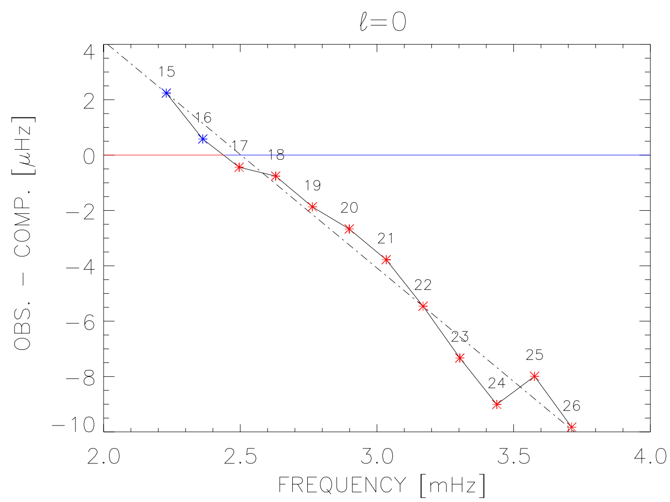

Fig. 2. Observed minus computed ('O-C') eigenfrequencies for low orders n taken from Rüdiger et al. (1997) . There is a crossing of the O-C data close to 2.5–3.0 mHz. The *redshift* of Fig. 1 only appears for the high-frequency oscillations. Lower frequencies are *blueshifted*. Similar results were found by Guzik & Swenson (1997)

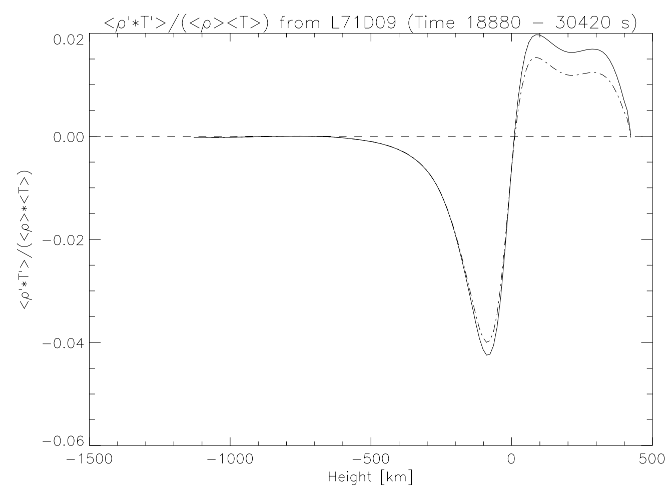

Fig. 3. The run of the turbulence-correction $-\zeta$ with the depth of the convection zone. Increasing depth is towards the left. Courtesy M. Steffen

density and temperature so that under the presence of turbulence we find

$$\bar{p} = \frac{\mathcal{R}}{\mu} \left(\bar{\rho}\bar{T} + \langle \rho' T' \rangle\right). \tag{3}$$

In convection zones the correlation of density and temperature fluctuations is negative and we formally write

$$\langle \rho' T' \rangle = -\zeta \bar{\rho}\bar{T} \tag{4}$$

with positive ζ, hence

$$\bar{p} = \frac{\mathcal{R}}{\mu}(1-\zeta)\bar{\rho}\bar{T} \tag{5}$$

should give a relation between mean pressure, density and temperature. Indeed, from Eq. (1) one can see that ζ clearly reduces the oscillation frequency. While ζ basically lowers the effective pressure, the Reynolds stress $\langle u_r'^2 \rangle$ increases it. Therefore an increase of the eigenfrequency by the turbulence pressure is the immediate consequence.

2. The laminar fluid

In a laminar fluid the conservation laws for mass and density are

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0, \tag{6}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} = -\rho g_i, \tag{7}$$

which are considered in a spherical coordinate system with the coordinates (r, θ, ϕ). Without turbulence, treating the oscillation as a small perturbation and assuming no motion in the unperturbed state ($\boldsymbol{u}^{(0)} = 0$), we get for radial oscillations ($l = 0$) from Eqs. (6) and (7)

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \xi^{\mathrm{lam}}) - \frac{g}{c^2}\xi^{\mathrm{lam}} + \frac{1}{\rho_0 c^2}\tilde{p}^{\mathrm{lam}} = 0, \tag{8}$$

$$\frac{\partial}{\partial r}\tilde{p}^{\mathrm{lam}} + \frac{g}{c^2}\tilde{p}^{\mathrm{lam}} + \rho_0(N^2 - \omega^2)\xi^{\mathrm{lam}} = 0. \tag{9}$$

The unperturbed stratification is described by ρ_0 and p_0 while quantities with a tilde describe the oscillating hydrodynamic field, the radial displacement is ξ^{lam} with

$$i\omega\xi^{\mathrm{lam}} = \tilde{u}_r^{\mathrm{lam}}. \tag{10}$$

Throughout the current paper the Cowling approximation is adopted, i.e. $\tilde{g} \equiv 0$. N^2 is the Brunt-Väisälä frequency

$$N^2 = g\left(\frac{1}{\gamma p_0}\frac{\partial p_0}{\partial r} - \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial r}\right). \tag{11}$$

Substituting Eq. (9) in (8) leads to a Sturm-Liouville-type differential equation for ξ^{lam}

$$\frac{\partial}{\partial r}\left[\gamma p_0 r^4 \frac{\partial}{\partial r}\left(\frac{\xi^{\mathrm{lam}}}{r}\right)\right] + \\ + \left\{\omega^2 \rho_0 r^4 + r^3\left[4g\rho_0 + \frac{\partial}{\partial r}(3\gamma p_0)\right]\right\}\frac{\xi^{\mathrm{lam}}}{r} = 0. \tag{12}$$

The radial displacements ξ_n^{lam} for different order n can therefore be scaled to form a complete and orthonormal set of eigenvectors with

$$\int_0^R \rho_0 \xi_i^{\mathrm{lam}} \xi_j^{\mathrm{lam}} r^2 dr = \delta_{ij}. \tag{13}$$

They can thus be used to develop the solutions in a weakly disturbed physical configuration, e.g. the oscillations under the influence of turbulence.

3. Including turbulence

Density, velocity and pressure are split into their mean and fluctuating parts, i.e.

$$\rho = \bar\rho + \rho', \qquad \boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}', \qquad p = \bar p + p', \tag{14}$$

where the mean parts contain the oscillation denoted again by a tilde:

$$\bar\rho = \rho_0 + \tilde\rho, \qquad \bar{\boldsymbol{u}} = \boldsymbol{u}^{(0)} + \tilde{\boldsymbol{u}}, \qquad p = p_0 + \tilde p. \tag{15}$$

In the first order of the perturbations one can then formulate the following linear theory.

3.1. Mass conservation

In *zeroth order* of the oscillation it is

$$\mathrm{div}(\rho_0(\boldsymbol{u}^{(0)} + \boldsymbol{U}^{(0)})) = 0, \tag{16}$$

where $\boldsymbol{U}$ is the turbulence-originated effective velocity

$$\langle \rho' \boldsymbol{u}' \rangle = \rho_0 \boldsymbol{U} \tag{17}$$

(cf. Rüdiger et al. 1997). The solution of (16) is simply

$$\frac{1}{\rho_0}\,\mathrm{rot}\,\boldsymbol{A} = \boldsymbol{u}^{(0)} + \boldsymbol{U}^{(0)} \tag{18}$$

with $\boldsymbol{A}$ considered as given. In *first order* of the (weak) oscillation it is therefore

$$i\omega\tilde\rho + i\omega\,\mathrm{div}\,\rho_0\boldsymbol{\xi} = -\,\mathrm{rot}\,\boldsymbol{A}\cdot\mathrm{grad}\,\frac{\tilde\rho}{\rho_0}, \tag{19}$$

where in addition a separation of time ($\partial/\partial t \Rightarrow i\omega$) was performed and $\tilde{\boldsymbol{U}}$ was neglected.

3.2. Momentum conservation

In the same way the Reynolds equation can be written as

$$\frac{\partial p_0}{\partial x_i} = -g_i\rho_0 + \\ + \frac{\partial}{\partial x_j}(\rho_0 U_i^{(0)} U_j^{(0)} - \rho_0 Q_{ij} - \mathrm{rot}_i\,\boldsymbol{A}\,\mathrm{rot}_j\,\boldsymbol{A}) \tag{20}$$

in *zeroth order* and as

$$\frac{\partial\tilde p}{\partial x_i} + g_i\tilde\rho - \rho_0\omega^2\xi_i = \\ -i\omega\left(\frac{\tilde\rho}{\rho_0}\mathrm{rot}_i\,\boldsymbol{A} + \frac{\partial}{\partial x_j}(\xi_i\,\mathrm{rot}_j\,\boldsymbol{A} + \xi_j\,\mathrm{rot}_i\,\boldsymbol{A})\right) + \\ + \frac{\partial}{\partial x_j}(\tilde\rho U_i^{(0)} U_j^{(0)} - \tilde\rho Q_{ij}) \tag{21}$$

in *first order* of the oscillation, with the influence of the oscillation on the turbulent quantities, i.e., $\tilde Q_{ij}$ and $\tilde{\boldsymbol{U}}$, again neglected.

Here

$$Q_{ij} = \langle u_i'(\boldsymbol{x},t) u_j'(\boldsymbol{x},t)\rangle \tag{22}$$

is the correlation tensor of the turbulence.

As we are interested in radial oscillations, $\boldsymbol{\xi}$ has only a radial component and will be written as a scalar. Also we assume that the Sun is non-rotating. The Reynolds stress tensor without viscosity then has only diagonal elements, and we set $Q_{\theta\theta} = Q_{\phi\phi}$. We henceforth assume $\boldsymbol{A} = 0$, and we neglect a possible meridional flow.

The next step is to eliminate the remaining quantities $\tilde p, \tilde\rho$. In the Lagrangian formulation – valid in the comoving system – we have

$$\frac{\delta\rho}{\rho_0} = \frac{1}{\Gamma}\frac{\delta p}{p_0}. \tag{23}$$

But one has to be careful that this relation is no longer valid with $\Gamma = 5/3$ as in the laminar fluid. An adiabatic analysis using (5) gives

$$\Gamma = \gamma + \zeta(1-\gamma), \tag{24}$$

so that the influence of ζ is included in the definition of the polytropic index. Using the relation between a Lagrangian (δf) and Eulerian ($\tilde f$) perturbation of a quantity f

$$\delta f = \tilde f + \boldsymbol{\xi}\cdot\mathrm{grad}\,f_0, \tag{25}$$

we get

$$\tilde p = -\frac{\Gamma p_0}{r^2}\frac{\partial}{\partial r}(r^2\xi) - \xi\frac{\partial p_0}{\partial r}. \tag{26}$$

Now (19) simply reads as

$$\tilde\rho = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho_0\xi\right). \tag{27}$$

Using the radial component of (21) and replacing $\tilde p$ and $\tilde\rho$ gives

$$H_0\xi + \omega^2\rho_0 r^3\xi = \rho_0 r^3 V\xi, \tag{28}$$

where the left-hand side equals the left-hand side of Eq. (12) for ξ and Γ with

$$H_0\xi = \frac{\partial}{\partial r}\left[\gamma p_0 r^4\frac{\partial}{\partial r}\left(\frac{\xi}{r}\right)\right] + \\ + \left\{r^3\left[4g\rho_0 + \frac{\partial}{\partial r}(3\gamma p_0)\right]\right\}\frac{\xi}{r} \tag{29}$$

and the operator V is

$$V\xi = \frac{1}{\rho_0 r^2}\frac{\partial}{\partial r}\left(\left(U_r^{(0)2} - Q_{rr}\right)\frac{\partial}{\partial r}(r^2\rho_0\xi)\right) + \\ + \frac{2Q_{\phi\phi}}{r^3\rho_0}\frac{\partial}{\partial r}\left(r^2\rho_0\xi\right) + \\ + \frac{1}{\rho_0}\frac{\partial}{\partial r}\left(\xi\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_0(Q_{rr} - U_r^{(0)2})) - \frac{2Q_{\phi\phi}}{r}\rho_0\right)\right). \tag{30}$$

The first terms in Eq. (30) arise from the corrections in the first order equations (in the following first-order corrections). The last term arises from the corrections in the zeroth-order equations (in the following zeroth-order corrections) given in Sect. 4.

3.3. Perturbation theory

The eigenfunctions ξ will be developed after the eigenfunctions $\xi^{\rm lam}$ of the equations without turbulence, i.e.,

$$\xi_n = \sum_n a_{mn} \xi_m^{\rm lam} \tag{31}$$

with $a_{mn} = 1$ for $m = n$. Along this way Eq. (28) turns into

$$\sum_m a_{mn} \left(H_0 \xi_m^{\rm lam} + \omega^2 \rho_0 r^3 \xi_m^{\rm lam} - \rho_0 r^3 \sum_l \xi_l^{\rm lam} V_{lm} \right) = 0 \tag{32}$$

with

$$V_{lm} = \int_0^R \rho_0 r^2 \xi_l^{\rm lam}(r) V(r) \xi_m^{\rm lam}(r) dr. \tag{33}$$

Now the frequency ω may be written as

$$\omega_n = \omega_n^{\rm lam} + \Delta\omega_n \tag{34}$$

with $\omega_n^{\rm lam}$ being the frequency of the unperturbed state. Eq. (32) then reads as

$$\sum_m 2 a_{mn} \omega_n^{\rm lam} \Delta\omega_n \rho_0 r^3 \xi_m^{\rm lam} = \sum_m a_{mn} \rho_0 r^3 \sum_l \xi_l^{\rm lam} V_{lm}. \tag{35}$$

Integration after multiplication with $\xi_n^{\rm lam}/r$ leads to the first-order correction

$$\Delta\omega_n = \frac{V_{nn}}{2\omega_n^{\rm lam}} \tag{36}$$

(no summation over n) for the frequencies.

4. The solar model

To solve the oscillation and perturbation equations in Sect. 3 one requires a solar model that provides the input data p_0, ρ_0, etc. In the calculation of the model the influence of turbulence in zeroth order has to be considered. Although the comparison between the frequencies obtained from different solar models, calculated with and without turbulence included, is somewhat ambiguous (cf. Christensen-Dalsgaard 1998), we nevertheless tried to get first insights.

For the core, where there is no influence of turbulence, we used an already existing table for a solar model by Stix & Skaley (1990). In the convection zone, where turbulence-pressure modifications have to be considered, a standard mixing-length theory (Böhm-Vitense 1958) was used for calculation of the model.

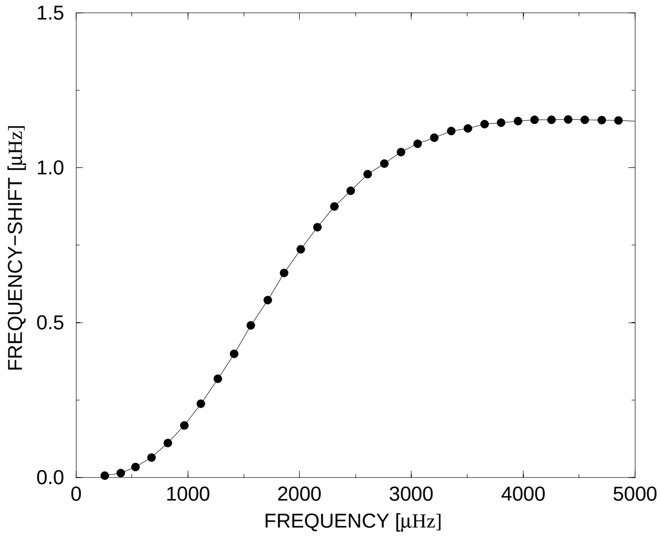

Fig. 4. Frequency shift $\Delta\omega = \omega^{\rm turb} - \omega^{\rm lam}$ (or O-C) due to the influence of the Reynolds stress

Here the mean molecular weight is kept constant at the value for fully ionized particles, which is a good approximation for most of the convection zone except the outer part. An adiabatic layer is presumed by using $\nabla = d\log T/d\log r = \nabla_{\rm ad}$. Starting from the outer boundary, so that pressure and temperature at the surface as well as the solar radius are the same for both models with and without turbulence, an integration is performed over pressure and temperature to achieve the quantities of interest. The integration is stopped when all energy is transported by radiation. At this point the quantities are fitted to those of the core model.

The turbulent correction terms can be implemented directly as a correction to the pressure. They have the following form for the Reynolds stress

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} = -g - \frac{1}{r^2\rho_0}\frac{\partial}{\partial r}(r^2\rho_0 Q_{rr}) + \frac{2Q_{\phi\phi}}{r}. \tag{37}$$

For the anisotropic turbulence field considered, it turns into

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} = -g - \frac{1}{r^2\rho_0}\frac{\partial}{\partial r}(r^2\rho_0 Q_{rr}). \tag{38}$$

5. Results

Only the influences of ζ and the radial part of the Reynolds stress on the oscillation frequencies are examined because they are dominant. The turbulence field is assumed to be extremely anisotropic. This means in particular that in Eq. (30) $U_r^{(0)}$, $Q_{\theta\theta}$ and $Q_{\phi\phi}$ are set to zero.

Following Stix (1989), we used homogeneous boundary conditions for the radial displacement ξ at the center, i.e., $\xi(0) = 0$, and for the Lagrangian perturbation of the pressure at the surface, i.e., $\delta P = 0$. We separately examined the influences arising from the two effects.

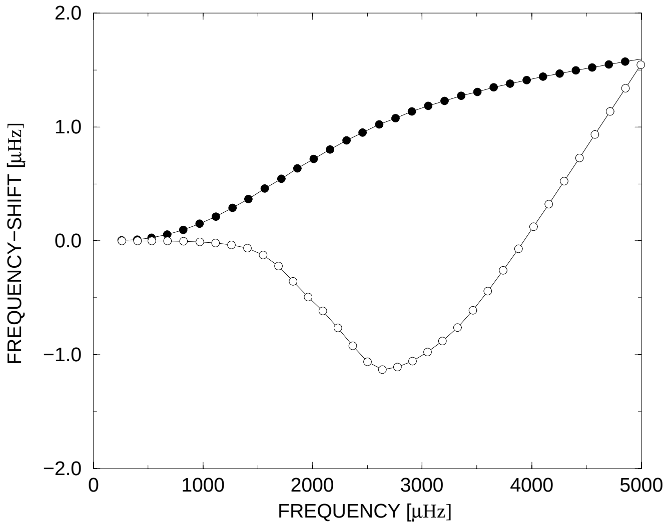

Fig. 5. First order corrections of p-mode frequencies due to the Reynolds stress in our model (filled circles) and with the complete model of Stix & Skaley (1990) (open circles)

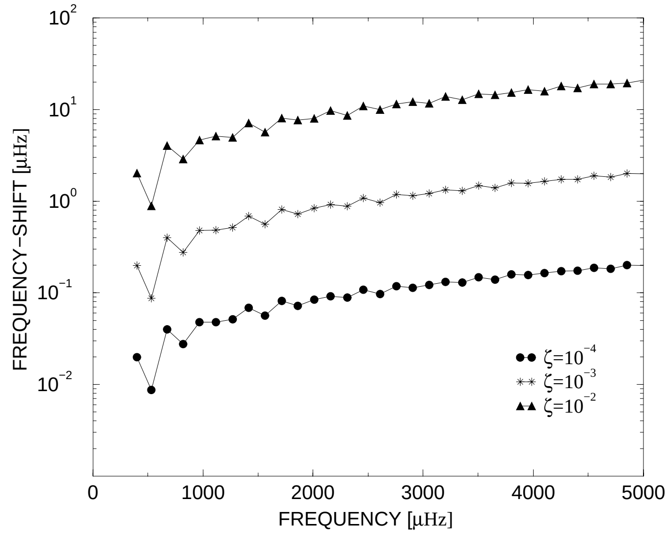

Fig. 6. Frequency shift due to the influence of ζ for $\zeta = 10^{-4}$, $\zeta = 10^{-3}$ and $\zeta = 10^{-2}$

Fig. 4 shows the total frequency shift due to the influence of the Reynolds stress. The Reynolds stress increases the eigenfrequencies and therefore produces a *blueshift* of the frequencies. The relative shift is of order 10^{-4}.

As the influence of the Reynolds stress depends on the derivative of Q_{rr}, the form of the turbulence velocity field is of importance. In our simple model, and therefore in Fig. 4, the turbulence velocity increases towards the outer boundary and has its maximum at the solar radius. To have an idea of how a more realistic velocity field, decreasing again towards the outer boundary (Stix & Skaley 1990; Lydon et. al. 1992), might affect the frequencies we calculated the shift arising from the first-order corrections (no change in the solar model!) using the complete model of Stix & Skaley (1990). These were then compared with the same corrections in our model, as can be seen in Fig. 5. For the smaller frequencies the *blueshift* is turned into a *redshift*.

Comparing the lines with the filled circles in Fig. 4 and Fig. 5 one can see that the zeroth-order corrections, that correspond to the difference between both curves, are smaller than the first order corrections. Now we can assume that this connection is still true when calculating the zeroth order corrections with the more realistic velocity field, which produce in the first order the lineshift plotted with the open circles in Fig. 5. Then for this velocity field the total lineshift due to the Reynolds stress is negative for small frequencies.

In a first step towards examining the influence of ζ we assumed a constant ζ over the outermost 7 % of the solar radius, everywhere else ζ is zero. We did this for three values of ζ. Numerical simulations (see Fig. 3) predict ζ to dominate in the very outer part of the sun with a maximum of the order 10^{-2}, therefore we chose the three values to be $\zeta = 10^{-4}, 10^{-3}$ and 10^{-2}.

In Fig. 6, the frequency shift due to ζ is plotted for these three values. As predicted by Eq. (1) the effect is linear in ζ, as can be seen with the logarithmic scale, and increasing with frequency. The relative shift is 0.4ζ. However in contrast to the first order corrections it is no longer negative. The zeroth-order corrections are dominant and force the shift in the other direction resulting in an overall *blueshift*. The first order corrections are not explicitly calculated in our model as mentioned before, but if this is done they confirm the analytically predicted results, i.e. a *redshift* of the frequencies.

Here the question arises of how the zeroth-order corrections from Reynolds stress and ζ are affected by the model which is used, and whether this change of sign due to the zeroth-order corrections in ζ is an artifact of the model. Since we chose the solar radius to remain constant, the depth of the convection zone remains variable and increases with increasing ζ. A better method would be, to vary other parameters such as the mixing-length, which are less well known (Paternò et. al. 1993; Gabriel 1995; Abbett et. al. 1997; Basu & Antia 1997). On the other hand the quantities of the integrated convection zone were attached to the given core values. This implies a discontinuity that surely has an influence on the frequencies. Both problems may be cured by using a modified but complete model for the Sun, that allows a smooth transition between the stratified and convective regions and a better fit to the parameters.

Focusing only on the first-order corrections (see also Rüdiger et al. 1997) the Reynolds stress, as well as the effect due to ζ, contribute to the observed *redshift*. Both contributions are the right order of magnitude and therefore not negligible. Further calculations will show, whether the total lineshift, including the contributions in zeroth and first order, will be negative or whether there is a competition between both effects. The latter could lead to an explanation of the *blueshift* for the small frequencies. In case the *blueshift* due to ζ survives for even larger frequencies an additional effect will be necessary to explain the *redshift* observed in this frequency range.

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