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ASTRONOMY AND ASTROPHYSICS

The Hanle effect

The density matrix and scattering approaches to the $\sqrt{\epsilon}$-law

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Abstract. A $\sqrt{\epsilon}$-law was demonstrated by Landi Degl'Innocenti & Bommier (1994) for resonance polarization in a magnetic atmosphere where the primary source of photons is of thermal origin (isotropic and unpolarized). In this paper we propose a generalized form of this law by dropping the hypothesis on the primary source of photons. We restrict ourselves to the case of weak magnetic fields (Hanle effect).

For spectral lines formed with complete redistribution, it has been shown by Landi Degl'Innocenti et al (1990), using the density matrix theory in its irreducible tensorial operator version, that the Hanle effect can be reduced to an integral equation of the convolution type for a six-component source vector. As shown by Faurobert-Scholl (1991), a similar equation can be obtained by performing an azimuthal Fourier decomposition of the transfer equation for the Stokes parameters.

In the first part of the paper we recall the main steps of the two methods and establish the correspondence between the convolution equations that they provide. In the second part we use these equations to obtain a generalized $\sqrt{\epsilon}$-law. For the equation coming from the density matrix formalism, we essentially follow the original proof of Landi Degl'Innocenti & Bommier (1994). For the equation coming from the Fourier decomposition, because of a lack of symmetry in operator describing the action of the magnetic field, we use as intermediate step the *Hopf-Bronstein-Rybicki* relation established by Ivanov (1995) for transport operators which are not self-adjoint.

Key words: line: formation – magnetic fields – polarization – radiative transfer – scattering – methods: analytical

1. Introduction

The Hanle effect describes the action of a weak magnetic field on resonance polarization, i.e. on resonance polarization created by the scattering of an anisotropic radiation field. Two types of equations are needed to describe the Hanle effect : a set of statistical equilibrium equations for the populations and Zeeman coherences of the atomic levels, and a vector transfer equation for the Stokes parameters. These equations are coupled by a radiative excitation term appearing in the statistical equilibrium equations and depending on the polarized radiation field. For spectral lines formed with complete redistribution, these equations can be combined into a convolution type integral equation for a vector which depends only on the space variable. For the Hanle effect, the linear polarization, can be described by a six-component vector.

Two approaches, seemingly rather different, have been developed in the literature to recast the Hanle radiative transfer problem as a convolution equation. There is a scattering approach (see e.g. Faurobert-Scholl, 1991 or Nagendra et al. 1998, henceforth referred to as FS91 and NFFS98, respectively) which makes use of the analytic expression of the Hanle phase matrix calculated by Landi Degl'Innocenti and Landi Degl'Innocenti (1988) and of a Fourier decomposition of the radiation field and of this phase matrix. In this method the radiative transfer equation plays a pivotal role. In the density matrix approach developed earlier by Landi Degl'Innocenti et al. (1990, henceforth referred to as LBS90) and in all subsequent work relying on this technique (see Bommier 1996 for a complete list of references), the central role is played by the statistical equilibrium equations.

In the first part of the paper (Sect. 2), we recall the main steps which lead to the convolutions equations, first for the scattering formulation and then for the density matrix one. We then establish the correspondence between the equations obtained with these two methods. In the case of the density matrix method, we consider two different frames of reference, one defined with respect to the magnetic field and the other one with respect to the atmosphere.

There are very few exact analytical results for polarized transfer, even less than for non-polarized transfer, so any of them will be extremely precious for predicting the radiation field and testing numerical codes. Integral equations describing multiple scattering problems, without or with polarization, are often a convenient starting point for establishing such results. One of the best known, which is easy to apply, is the so-called $\sqrt{\epsilon}$-law which holds for spectral lines formed with complete redistribution and concerns the surface value of the source vector

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(function) in a semi-infinite uniform medium. It is a quadratic relation of the form

$$\sum_i \sum_j a_{i,j} S_i(0) S_j(0) = r, \tag{1}$$

where S_i are the components of a source vector. The coefficients $a_{i,j}$ and the quantity r depend on the problem at hand.

Such a law was first established for non-polarized transfer (Avrett and Hummer 1965). In this case it takes the simple form

$$S^2(0) = G^2/\epsilon, \tag{2}$$

where ϵ is a photon destruction probability per scattering ($1-\epsilon$ is the single scattering albedo) and G the inhomogeneous term in the integral equation for the source function. The name "$\sqrt{\epsilon}$-law" comes from this relation. We shall used it a generic name for relations of the type (1).

The generalization to resonance polarization in a non-magnetic atmosphere was proposed by Ivanov (1990) and the generalization to resonance polarization in a magnetic atmosphere, including the case of the Hanle effect, was proposed by Landi Degl'Innocenti and Bommier (1994, henceforth referred to as LB94) for a non-polarized inhomogoneous term.

The generalized $\sqrt{\epsilon}$-law given in this paper can be used for physical problems with a polarized primary source such as a semi-infinite atmosphere where atoms are excited by collisions with an anisotropic beam of particles (impact polarization). It provides also a direct way of evaluating the accuracy on the polarization in a numerical calculation. When the primary source of photons is isotropic and unpolarized the direct input of photons is only on the Stokes I parameter. Stokes Q and U, which describe the linear polarization, are excited only through their fairly weak coupling to I. Hence, in an expression such as (1) the contribution from Stokes I will typically be a hundred times larger than the contributions from Q and U and the accuracy on the polarization may be hard to evaluate. With a primary source that has a direct input on the polarization components, Q and U increase with respect to I, and the precision on these parameters can be checked with a formula such as (1). This remark was also made by V.V. Ivanov (private communication) some times back. It has already been verified by Bommier and Landi Degl'Innocenti (1998) that the generalized $\sqrt{\epsilon}$-law proposed in this paper can be used to assess the accuracy on the linear polarization reached by their numerical code.

In the second part of the paper (Sect. 3), we consider the integral equations established in the first part and for each of them we show how to construct the $\sqrt{\epsilon}$-law for an arbitrary primary source term. The law given in LB94 for the Hanle effect is recovered as a particular case. For the integral equations obtained with the density matrix formalism, we generalize the proof given in LB94. For the equation obtained with the scattering formulation, our proof makes use of the *Hopf-Bronstein-Rybicki* quadratic identity established by Ivanov (1995). We note here that the generalization of the $\sqrt{\epsilon}$-law to an arbitrary inhomogeneous term affects only the r.h.s. in Eq. (1).

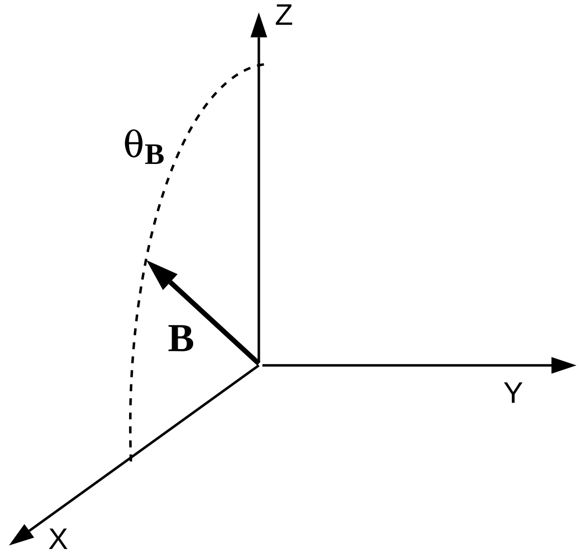

Fig. 1. Geometry specifying the direction of the magnetic field $\boldsymbol{B}$. Z is the outward normal to the atmosphere and $\theta_{\rm B}$ is the co-latitude of $\boldsymbol{B}$.

2. Integral equations for the Hanle effect

We consider a uniform 1D semi-infinite magnetic medium and assume a two-level atom model. There is no external incident radiation on the medium. In this section we also assume that the primary source of photons is isotropic and unpolarized. The case of a general primary source is considered only in the second part of the paper.

2.1. The scattering formulation

We choose a reference system $\Sigma = (X, Y, Z)$ with the Z-axis in the direction of the outward normal to the medium and the X-axis along the projection of the magnetic field vector on the horizontal plane. The magnetic field can then be characterized by its modulus B and the co-latitude angle $\theta_{\rm B}$ (see Fig. 1).

If the magnetic field is sufficiently weak for the Zeeman splitting to be negligible compared to the Doppler width, the radiative transfer equation for the Stokes vector $\boldsymbol{\mathcal{I}} = (I, Q, U)$ may be written as

$$\mu \frac{\partial \boldsymbol{\mathcal{I}}(\tau, x, \boldsymbol{n})}{\partial \tau} = \phi(x) \big[\boldsymbol{\mathcal{I}}(\tau, x, \boldsymbol{n}) - \boldsymbol{\mathcal{S}}(\tau, x, \boldsymbol{n}) \big], \tag{3}$$

where ϕ is the scalar absorption profile function (Landi Degl'Innocenti 1985). Stokes Q is defined as in Chandrasekhar (1960) ($Q = I_{\rm l} - I_{\rm r}$, with $I_{\rm l}$ and $I_{\rm r}$ the components of vibration of the electric vector which are perpendicular and parallel to the nearest solar limb, respectively). All the sign conventions, and the symbols for the physical quantities have the same meaning as in NFFS98 : τ is the frequency averaged line optical depth, x is the frequency separation from line center, measured in Doppler width units, $\boldsymbol{n}(\theta, \varphi)$ is the propagation direction of the ray having co-latitude θ ($\mu = \cos\theta$) and azimuth φ. The positive optical depth is measured in the direction opposite to the Z-axis.

We assume that the phase matrix which describes the redistribution in frequency, direction and polarization is of the form

$$\phi(x)\phi(x')\hat{P}_{\rm H}(\boldsymbol{n},\boldsymbol{n}',\boldsymbol{B}), \tag{4}$$

the prime quantities denoting frequency and direction before scattering. This expression implies that there is complete frequency redistribution at each scattering and that frequency redistribution is decoupled from the polarization described by the phase matrix $\hat{P}_{\rm H}$. The precise physical conditions which justify this widely used complete redistribution approximation are still partly under debate (see e.g. Bommier 1997b).

For the phase matrix given in Eq. (4), the vector source function $\boldsymbol{\mathcal{S}}$ is independent of frequency and may be written as

$$\boldsymbol{\mathcal{S}}(\tau,\boldsymbol{n}) = \boldsymbol{\mathcal{S}}^* + \int_{-\infty}^{+\infty}\phi(x')\oint \hat{P}_{\rm H}(\boldsymbol{n},\boldsymbol{n}',\boldsymbol{B})\,\boldsymbol{\mathcal{I}}(\tau,x',\boldsymbol{n}')\frac{d\Omega'}{4\pi}\,dx'. \tag{5}$$

In this section we assume that the primary source term $\boldsymbol{\mathcal{S}}^*$ is of thermal origin, i.e. isotropic and unpolarized. Hence, it is of the form $\boldsymbol{\mathcal{S}}^* = G_{\rm I}\,\boldsymbol{e}$ with $\boldsymbol{e} = (1,0,0)$. For a two-level atom and no external radiation incident on the medium, $G_{\rm I}$ is proportional to the Planck function at the corresponding line frequency.

In this paper all matrices are denoted by italic letters accentuated with a hat and vectors with bold face characters.

As shown in LBS90, or more recently in Bommier (1997a,b),

$$\hat{P}_{\rm H}(\boldsymbol{n},\boldsymbol{n}',\boldsymbol{B}) = (1-\epsilon_o)\hat{P}^{(0)} + (1-\epsilon_p)W_2\,\hat{P}^{(2)}(\boldsymbol{n},\boldsymbol{n}',\boldsymbol{B}). \tag{6}$$

$\hat{P}^{(0)}$ is the isotropic scattering phase matrix (all its element are zero, except the element $\{1,1\}$ which is unity) and $\hat{P}^{(2)}(\boldsymbol{n},\boldsymbol{n}',\boldsymbol{B})$ describes the linearly polarized scattering. The sum $\hat{P}^{(0)}+\hat{P}^{(2)}$ is often referred to as the *Hanle phase matrix*. The parameter W_2 is the depolarization factor which depends on the quantum numbers J and J' of the lower and upper level of the atomic transition. Eq. (6) takes into account depolarizing collisions which were not included in NFFS98. The parameters ϵ_o and ϵ_p are destruction probabilities. They are defined in Eqs. (37) and (38). When the depolarizing collisions are neglected, these two parameters are equal and represent the destruction probability per scattering denoted by ϵ in NFFS98.

The transformation of Eqs. (3) and (5) into an integral equation for a six-component source vector $\boldsymbol{S}^{\rm sc}(\tau)$ is described in detail in FS91 and NFFS98. Here we present only the main steps.

First we take the real Fourier transform of $\boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{S}}$ with respect to the azimuth φ. The Hanle phase matrix has a 2D-Fourier expansion with respect to φ and φ', limited to terms of order 2. Therefore the Fourier expansion of $\boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{S}}$ are also limited to terms of order 2. We write the real Fourier transform of $\boldsymbol{\mathcal{S}}$ as

$$\boldsymbol{\mathcal{S}}(\tau,\mu,\varphi) = \overline{\boldsymbol{S}_0}(\tau,\mu) + \sum_{k=1}^{k=2}[\overline{\boldsymbol{S}_k}(\tau,\mu)\ \cos k\varphi + \overline{\boldsymbol{S}_{-k}}(\tau,\mu)\ \sin k\varphi]. \tag{7}$$

There is a similar expansion for $\boldsymbol{\mathcal{I}}$. The vectors $\overline{\boldsymbol{S}_k}$, $k=1,2$ are three-component vectors, but $\overline{\boldsymbol{S}_0}$ is a two-component vector because the azimuthal average of Stokes U vanishes for symmetry reasons.

We now consider

$$\boldsymbol{S}_{\rm F} = \left[\overline{\boldsymbol{S}_0},\overline{\boldsymbol{S}_1},\overline{\boldsymbol{S}_{-1}},\overline{\boldsymbol{S}_2},\overline{\boldsymbol{S}_{-2}}\right], \tag{8}$$

and

$$\boldsymbol{I}_{\rm F} = \left[\overline{\boldsymbol{I}_0},\overline{\boldsymbol{I}_1},\overline{\boldsymbol{I}_{-1}},\overline{\boldsymbol{I}_2},\overline{\boldsymbol{I}_{-2}}\right], \tag{9}$$

which are 14-component vectors containing all the Fourier expansion coefficients of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{I}}$. The field $\boldsymbol{I}_{\rm F}$ satisfies Eq. (3) with $\boldsymbol{S}_{\rm F}$ in place of $\boldsymbol{\mathcal{S}}$.

The vector $\boldsymbol{S}_{\rm F}$ can be factorized as

$$\boldsymbol{S}_{\rm F}(\tau,\mu) = \hat{B}(\mu)\,\boldsymbol{S}^{\rm sc}(\tau), \tag{10}$$

where $\hat{B}(\mu)$ is a (14×6) matrix, defined in NFFS98, and $\boldsymbol{S}^{\rm sc} = (S_{\rm I}^{\rm sc}, S_{\rm Q}^{\rm sc}, S_{+1}^{\rm sc}, S_{-1}^{\rm sc}, S_{+2}^{\rm sc}, S_{-2}^{\rm sc})$ is a six-component vector. The component $S_{\rm I}^{\rm sc}$ will be referred to as the intensity component and the five others as the polarization components. This factorization is made possible by the structure of the phase matrix $\hat{P}^{(2)}$, which, in Fourier space, can be represented as a sum of diadic products of the form $\boldsymbol{Z}_i(\mu)\boldsymbol{Z}_j(\mu')^{\rm T}$, $(i,j=1,5)$. The $\boldsymbol{Z}_i(\mu)$ are three-component vectors. The notation T stands for transpose.

Introducing the six-dimension field $\boldsymbol{I}^{\rm sc}$, defined by

$$\boldsymbol{I}_{\rm F}(\tau,x,\mu) = \hat{B}(\mu)\,\boldsymbol{I}^{\rm sc}(\tau,x,\mu), \tag{11}$$

and the unit vector $\boldsymbol{e}_1 = (1,0,0,0,0,0)$, the vector $\boldsymbol{S}^{\rm sc}$ can be written as

$$\boldsymbol{S}^{\rm sc}(\tau) = (\hat{I}-\hat{\mathcal{E}})\,\hat{M}_{\rm B}\,\boldsymbol{J}^{\rm sc}(\tau) + G_{\rm I}\,\boldsymbol{e}_1, \tag{12}$$

where

$$\boldsymbol{J}^{\rm sc}(\tau) = \frac{1}{2}\int_{-\infty}^{+\infty}\phi(x)\int_{-1}^{+1}\hat{\Psi}(\mu)\boldsymbol{I}^{\rm sc}(\tau,x,\mu)\,d\mu\,dx. \tag{13}$$

The matrix $\hat{\Psi}(\mu)$ is defined by

$$\hat{\Psi}(\mu) = \hat{B}^{\rm T}(\mu)\hat{B}(\mu). \tag{14}$$

$\hat{I}$ is the (6×6) identity matrix and

$$\hat{\mathcal{E}} = {\rm diag}\{\epsilon_o,\epsilon_p,\epsilon_p,\epsilon_p,\epsilon_p,\epsilon_p\}. \tag{15}$$

The notation ${\rm diag}\{\ \}$ stands for diagonal matrix.

$\hat{M}_{\rm B}$ describes the effect of the magnetic field. It has a block diagonal structure :

$$\hat{M}_{\rm B} = \begin{pmatrix} 1 & 0 \\ 0 & \hat{M}_{\rm B}^p \end{pmatrix}. \tag{16}$$

$\hat{M}_{\rm B}^p$ is a (5×5) matrix which couples together the polarization components. Two different factorizations of $\hat{M}_{\rm B}^p$ as a product of matrices, sufficient for our present purpose, are given in Eqs. (62) and (70). The elements of $\hat{M}^p_{\rm B}$ depend on the colatitude $\theta_{\rm B}$ and the strength of the magnetic field. Explicit expressions may be found in FS91 or NFFS98.

Several matrices with the same block structure are introduced in this paper. The (5×5) block acting in the sub-space of the polarization components has the same name as the full (6×6) matrix but carries a superscript p.

It is easy to verify that the field $\boldsymbol{I}^{\rm sc}$, defined in Eq. (11), satisfies Eq. (3) with $\boldsymbol{S}^{\rm sc}$ as source function. Solving the transfer equation formally for $\boldsymbol{I}^{\rm sc}$ (assuming that $\boldsymbol{S}^{\rm sc}$ is known) and inserting the result into Eq. (12), we get

$$\boldsymbol{S}^{\rm sc}(\tau) = \int_0^\infty \hat{K}_{\rm sc}(\tau - \tau')\boldsymbol{S}^{\rm sc}(\tau')\, d\tau' + G_{\rm I}\, \boldsymbol{e}_1, \tag{17}$$

where

$$\hat{K}_{\rm sc}(\tau) = (\hat{I} - \hat{\mathcal{E}})\, \hat{M}_{\rm B}\, \hat{K}^{\rm sc}(\tau). \tag{18}$$

The (6×6) matrix $\hat{K}^{\rm sc}$ is of the form

$$\begin{pmatrix} K_{1,1} & K_{1,2} & 0 & 0 & 0 & 0 \\ K_{1,2} & K_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{4,4} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{4,4} \end{pmatrix}. \tag{19}$$

The $K_{i,j}$, defined by

$$K_{i,j}(\tau) = \frac{1}{2}\int_0^1 \Psi_{i,j}(\mu) \int_{-\infty}^{+\infty} \phi^2(x) {\rm e}^{-|\tau|\phi(x)/\mu}\, dx\, d\mu, \tag{20}$$

are even functions of τ. The kernel $K_{1,1}$ is normalized to unity and the kernel $K_{1,2}$ to zero. All the other kernels have the same normalization, namely $7\,W_2/10$. By normalization we mean the integral from minus infinity to plus infinity. Explicit expressions for the $K_{i,j}$ may be found in LBS90 or NFFS98.

We stress that the matrix $\hat{K}^{\rm sc}$ which describes the scattering of photons is symmetrical, but the matrix $\hat{K}_{\rm sc}$, is not. This property matters when it comes to prove the $\sqrt{\epsilon}$-law for the Hanle effect with Eq. (17).

2.2. The density matrix formalism

A clear and concise presentation of the density matrix formalism is given in Bommier (1996). We repeat here the main points. The density matrix formalism has been introduced to describe an atomic system in which all the atoms are not in the same state (from a quantum-mechanical point of view). The system is described by a so-called statistical or density operator. The matrix representation of this operator depends on the choice of the expansion basis. A standard representation is that of the diadic operators $|l><k|$ associated to the Hamiltonian eigenstates. For the Hanle effect, the quantization reference frame (Z axis parallel to the magnetic field) will in general be different from the natural reference frame of the medium (Z axis normal to the atmosphere) or from the observer reference frame (Z axis parallel to the line of sight). A change of reference frame is easy to implement when the density operator is represented in terms of the irreducible tensorial operators (ITO) because the latter form an orthonormal basis of the Hilbert space of operators, irreducible with respect to the rotation group. These operators are defined in Brink and Satchler (1968) or any other book devoted to angular momentum theory (see e.g. Blum, 1981). The elements of the density matrix depend on three indices, J characterizing the angular momemtum, K a positive or null integer ranging from 0 to $2J$ and Q a relative integer ranging from $-K$ to $+K$. In a rotation of the reference frame, K is conserved and the linear expansion runs over Q only.

The component of the density matrix corresponding to the index $K = 0$ is called the population and is denoted by ρ^0_0. The three elements characterized by $K = 1$, $Q = 0, \pm 1$ are called the orientation components because they define a vector proportional to the net angular momentum $< \mathbf{J} >$ of the system. Circular polarization appears only if this net angular momentum is not zero. The five components characterized by $K = 2$, $Q = 0, \pm 1, \pm 2$ are called the alignment components. They describe the anisotropy of the medium and are the components responsible for linear polarization.

Another important property of the density matrix is its hermiticity, which implies that

$$[\rho^K_Q]^* = (-1)^Q \rho^K_{-Q}, \tag{21}$$

where $*$ denotes complex conjugation.

We concentrate now on the Hanle effect for a two-level atom with unpolarized ground level. This assumption implies that the population ρ^0_0 is the only non-vanishing element of the ground-level statistical tensor. When the radiation field is not circularly polarized, the upper level statistical tensor is fully described by the components $K = 0$ and $K = 2$. Circular polarization cannot be created by the Hanle effect itself since the kinetic angular momentum is not modified. The Hanle effect produces a precessional motion of the angular momentum vector around the direction of the magnetic field, only. Thus, the statistical tensor of the upper level can be fully described by six complex numbers, which, because of the conjugation property, can be defined in terms of six real quantities. It is clear that these six real quantities correspond to the six components of the source vector $\boldsymbol{S}^{\rm sc}$ introduced in Sect. 2.1.

2.2.1. Magnetic reference frame

For the Hanle effect, if one chooses the quantization axis in the direction of the magnetic field, the operator describing the action of the magnetic field on the system of atoms becomes diagonal in K and Q. This is clearly an advantage of the ITO representation. It also simplifies the description of depolarizing (elastic) collisions, which, if they are isotropic, will depend only on K. These collisions mix the Zeeman sublevels.

For the atomic model considered above, the statistical equilibrium equations for the density matrix elements have been given in LBS90. With the notation of Bommier (1997a,b) the density matrix elements $\tilde{\rho}^K_Q$ of the upper level, renormalized by the population of the lower level, may be written as :

$$(i\omega_{\rm L} g_{J'} Q + A_{ba} + C_{ba} + D_b^{(K)})\tilde{\rho}_Q^K(\tau) = \\ B_{ab}\, w_{J'J}^{(K)} (-1)^Q \bar{J}_{-Q}^K(\tau) + C_{ab}\,\delta_{K,0}\,\delta_{Q,0}. \quad (22)$$

Here the index a refers to the lower level, of angular momentum J, and the index b to the upper level, of angular momentum J'. The renormalized density matrix elements are defined by

$$\tilde{\rho}_Q^K(\tau) = \left(\frac{2J'+1}{2J+1}\right)^{\frac{1}{2}} \frac{{}^b\rho_Q^K(\tau)}{{}^a\rho_0^0(\tau)}. \quad (23)$$

In case of complete redistribution, the ρ_Q^K depend only on the space variable τ.

In Eq. (22), the l.h.s. gives the rate of destruction of the $\tilde{\rho}_Q^K$ and the r.h.s., the rate of creation. In the l.h.s., the first term describes the action of the magnetic field, the second term spontaneous deexcitation, the third term, de-excitation by inelastic collisions and the fourth term depolarizing collisions. In the r.h.s., the first term describes radiative excitations and the second one excitation by inelastic collisions. The latter term yields the creation term for the primary source of photons. Note that it appears only in the equation for $\tilde{\rho}_0^0$. In Eq. (22), $\omega_{\rm L} = 2\pi\nu_{\rm L} g_{J'}$, with $\nu_{\rm L}$ the Larmor frequency, proportional to the strength of the magnetic field, $g_{J'}$ is the Landé factor of the upper level and $i = \sqrt{-1}$). The coefficients A_{ba} and B_{ab} are the Einstein coefficients for spontaneous emission and absorption (stimulated emission is neglected). The coefficient $w_{J'J}^{(2)}$ obeys

$$[w_{J'J}^{(2)}]^2 = W_2, \quad (24)$$

where W_2 is the depolarization factor already introduced in Sect. 2.1. Tables for $w_{J'J}^{(K)}$, $K = 1, 2$, may be found in Landi Degl'Innocenti (1984). For a normal Zeeman triplet ($J = 0$, $J' = 1$), $w_{J'J}^{(2)} = 1$. All the collision rates are proportional to the density of the perturbers, usually electrons for inelastic collisions and hydrogen atoms for elastic collisions. We note also that $w_{J'J}^{(0)} = 1$ and $D_b^{(0)} = 0$.

For the purpose of comparison with the scattering approach, it is convenient to rewrite this equation with new unknowns and dimensionless parameters. Following Bommier (1997a, see also LB94), we introduce

$$S_Q^K(\tau) = \frac{2h\nu_o^3}{c^2}\frac{2J+1}{2J'+1}\tilde{\rho}_Q^K(\tau), \quad (25)$$

where ν_o is the line center frequency. We also introduce the notation $\Gamma_{\rm R} = A_{ba}$ and $\Gamma_{\rm I} = C_{ba}$ and define the dimensionless parameters,

$$\epsilon = \frac{\Gamma_{\rm I}}{\Gamma_{\rm R}}, \quad (26)$$

$$\delta^{(K)} = \frac{D_b^{(K)}}{\Gamma_{\rm R}}, \quad (27)$$

$$\Gamma = \frac{\omega_{\rm L}}{\Gamma_{\rm R}}. \quad (28)$$

We can now rewrite Eq. (22) as

$$(1 + \epsilon + \delta^{(K)} + i\Gamma Q)\, S_Q^K(\tau) = \\ w_{J'J}^{(K)}(-1)^Q \bar{J}_{-Q}^K(\tau) + \epsilon B_{\nu_o}\,\delta_{K,0}\,\delta_{Q,0}, \quad (29)$$

where B_{ν_o} is the Planck function at line center frequency. To obtain Eq. (29) one has to make use of the relations

$$C_{ab} = \frac{2J'+1}{2J+1} C_{ba} \exp(-h\nu_o/kT), \quad (30)$$

and

$$\frac{A_{ba}}{B_{ab}} = \frac{2h\nu_o^3}{c^2}\frac{2J+1}{2J'+1}. \quad (31)$$

We now turn to the radiative excitation term which couples the statistical equilibrium equations and the transfer equation. For complete redistribution,

$$\bar{J}_Q^K(\tau) = \int_{-\infty}^{+\infty} \phi(x) J_Q^K(\tau, x)\, dx. \quad (32)$$

The profile ϕ and the dimensionless frequency x have been introduced in Eq. (3). The J_Q^K are the angle averaged statistical tensors of the radiation field. They are related to the Stokes vector $\boldsymbol{\mathcal{I}} = (I, Q, U)$ by

$$J_Q^K(\tau, x) = \oint \sum_{i=0}^{i=2} \mathcal{T}_Q^K(i, \boldsymbol{n})\, \mathcal{I}_i(\tau, x, \boldsymbol{n})\, \frac{d\Omega}{4\pi}. \quad (33)$$

The $\mathcal{T}_Q^K(i, \boldsymbol{n})$ are the irreducible spherical tensors for polarimetry defined by Landi Degl'Innocenti (1984). The vector $\boldsymbol{n}$ is the direction of the ray. The indices $i = 0$, $i = 1$ and $i = 2$ correspond to Stokes I, Stokes Q and Stokes U, respectively. Explicit expressions of the $\mathcal{T}_Q^K(i, \boldsymbol{n})$ can be found in Bommier (1997b, Table 1). The angle γ in this Table should be set to zero for consistency with the definition of Stokes Q used in Sect. 2.1 and in LB94. Fig. 2 in Landi Degl'Innocenti (1983) shows the definition of γ.

The components of the Stokes source vector $\boldsymbol{\mathcal{S}}$, introduced in Eq. (5), are related to the statistical tensors S_Q^K by

$$\mathcal{S}_i(\tau, \boldsymbol{n}) = \sum_K \sum_Q w_{J'J}^{(K)} \mathcal{T}_Q^K(i, \boldsymbol{n})\, S_Q^K(\tau). \quad (34)$$

Assuming that the S_Q^K are known quantities, one solves Eq. (3) for the Stokes vector $\boldsymbol{\mathcal{I}}$ and then, using Eqs. (32) and (33), obtains $\bar{J}_{-Q}^K$ in terms of the S_Q^K. To make contact with the scattering formulation more easily, we regroup the S_Q^K into a six-component vector

$$\boldsymbol{S}^{\rm dm} = (S_0^0, S_{-2}^2, S_{-1}^2, S_0^2, S_{+1}^2, S_{+2}^2), \quad (35)$$

where the superscript "dm" stands for density matrix formulation. We also introduce the parameters

$$\gamma_{\rm B} = \frac{\Gamma}{1 + \epsilon + \delta^{(2)}}, \quad (36)$$

$$\epsilon_o = \frac{\epsilon}{1 + \epsilon}, \quad (37)$$

and

$$\epsilon_p = \frac{\epsilon + \delta^{(2)}}{1+\epsilon+\delta^{(2)}}. \tag{38}$$

The vector $\boldsymbol{S}^{\rm dm}$ satisfies the integral equation

$$[\hat{I} + i\gamma_{\rm B}\,\hat{\mathcal{Q}}]\boldsymbol{S}^{\rm dm}(\tau) = G_{\rm I}\,\boldsymbol{e}_1 + (\hat{I} - \hat{\mathcal{E}}) \int_0^\infty \hat{\mathcal{D}}(-\theta_{\rm B})\hat{K}^{\rm dm}(\tau-\tau')\hat{\mathcal{D}}(\theta_{\rm B})\boldsymbol{S}^{\rm dm}(\tau')\,d\tau'. \tag{39}$$

$\hat{\mathcal{Q}}$ is a diagonal matrix which contains only the lower indices of the components of $\boldsymbol{S}^{\rm dm}$:

$$\hat{\mathcal{Q}} = {\rm diag}\{0,-2,-1,0,1,2\}. \tag{40}$$

The matrices $\hat{I}$ and $\hat{\mathcal{E}}$ have already been introduced (see Eqs. (12), (15), (37) and (38)). The matrix $\hat{K}^{\rm dm}$ describes the scattering of photons in the reference frame of the atmosphere. It is given in Eq. (55). The matrix $\hat{\mathcal{D}}(\theta_{\rm B})$ takes care of the rotation from the atmospheric to the magnetic field reference frame. It is defined in Eqs. (43) to (49).

2.2.2. Atmospheric reference frame

We now rewrite the integral equation for the density matrix elements in the reference system of the atmosphere. We introduce a vector

$$\boldsymbol{R}^{\rm dm} = (R^0_0, R^2_{-2}, R^2_{-1}, R^2_0, R^2_1, R^2_2), \tag{41}$$

related to $\boldsymbol{S}^{\rm dm}$ by the transformation

$$\boldsymbol{R}^{\rm dm} = \hat{\mathcal{D}}(\theta_{\rm B})\boldsymbol{S}^{\rm dm}. \tag{42}$$

The rotation matrix $\hat{\mathcal{D}}(\theta_{\rm B})$ acts on the polarization components while leaving the intensity component unchanged. It has thus the block diagonal structure shown in Eq. (16). The element $\{1,1\}$ is unity. The (5×5) lower block $\hat{\mathcal{D}}^p(\theta_{\rm B})$ can be written in terms of the reduced rotation matrices $d_{Q,Q'}$ as :

$$\begin{pmatrix} d_{-2,-2} & d_{-2,-1} & d_{-2,0} & d_{-2,+1} & d_{-2,+2} \\ -d_{-2,-1} & d_{-1,-1} & d_{-1,0} & d_{-1,+1} & d_{-2,+1} \\ d_{-2,0} & -d_{-1,0} & d_{0,0} & d_{-1,0} & d_{-2,0} \\ -d_{-2,+1} & d_{-1,+1} & -d_{-1,0} & d_{-1,-1} & d_{-2,-1} \\ d_{-2,+2} & -d_{-2,+1} & d_{-2,0} & -d_{-2,-1} & d_{-2,-2} \end{pmatrix}. \tag{43}$$

The elements of $\hat{\mathcal{D}}^p(\theta_{\rm B})$ satisfy the symmetry relation

$$\hat{\mathcal{D}}^p(\theta_{\rm B})(Q,Q') = (-1)^{Q-Q'}\hat{\mathcal{D}}^p(\theta_{\rm B})(-Q,-Q'), \tag{44}$$

where the indices Q and Q' vary from -2 to +2. If one ignores the changes of sign, this is a symmetry with respect to the pair of indices (0,0). Thus nine elements are sufficient to construct $\hat{\mathcal{D}}^p(\theta_{\rm B})$. They can be found in Brink and Satchler (1968; Table 1, J=2). We reproduce them below for the benefit of the reader :

$$d_{-2,-2} = \frac{1}{4}(1+C_{\rm B})^2, \quad d_{-2,-1} = \frac{1}{2}S_{\rm B}(1+C_{\rm B}), \tag{45}$$

$$d_{-2,0} = \sqrt{\frac{3}{8}}S^2_{\rm B}, \quad d_{-2,+1} = \frac{1}{2}S_{\rm B}(1-C_{\rm B}), \tag{46}$$

$$d_{-2,+2} = \frac{1}{4}(1-C_{\rm B})^2, \quad d_{-1,-1} = \frac{1}{2}(1+C_{\rm B})(2C_{\rm B}-1), \tag{47}$$

$$d_{-1,0} = \sqrt{\frac{3}{2}}C_{\rm B}S_{\rm B}, \quad d_{-1,+1} = \frac{1}{2}(1-C_{\rm B})(2C_{\rm B}+1), \tag{48}$$

$$d_{0,0} = \frac{1}{2}(3C^2_{\rm B}-1). \tag{49}$$

Here,

$$C_{\rm B} = \cos\theta_{\rm B}, \quad S_{\rm B} = \sin\theta_{\rm B}. \tag{50}$$

We also note that $\hat{\mathcal{D}}(\theta_{\rm B})$ is a unitary matrix and that the inverse of $\hat{\mathcal{D}}(\theta_{\rm B})$ is simply obtained by changing $\theta_{\rm B}$ into $-\theta_{\rm B}$. Thus

$$\hat{\mathcal{D}}(-\theta_{\rm B}) = \hat{\mathcal{D}}^{-1}(\theta_{\rm B}) = \hat{\mathcal{D}}^{\rm T}(\theta_{\rm B}). \tag{51}$$

Combining Eq. (42) with Eq. (39) and multiplying on the left by $\hat{\mathcal{D}}$, we obtain

$$[\hat{I} + i\gamma_{\rm B}\,\hat{\mathcal{L}}]\boldsymbol{R}^{\rm dm}(\tau) = (\hat{I} - \hat{\mathcal{E}}) \int_0^\infty \hat{K}^{\rm dm}(\tau-\tau')\boldsymbol{R}^{\rm dm}(\tau')\,d\tau' + G_{\rm I}\,\boldsymbol{e}_1, \tag{52}$$

where

$$\hat{\mathcal{L}} = \hat{\mathcal{D}}(\theta_{\rm B})\hat{\mathcal{Q}}\,\hat{\mathcal{D}}^{-1}(\theta_{\rm B}). \tag{53}$$

The effect of the magnetic field is described by the matrix $\hat{\mathcal{L}}$. It acts only on the polarization components R^2_Q, $Q = 0, \pm1, \pm2$. Therefore it has the block diagonal structure (16) with $\hat{\mathcal{L}}(1,1) = 0$. The elements of $\hat{\mathcal{L}}$ are given in LBS90 (Table 1), where they are denoted $\mathcal{K}_{i,j}$. They are also easily calculated from Eq. (53). The (5×5) lower block $\hat{\mathcal{L}}^p$ is

$$\begin{pmatrix} -2C_{\rm B} & S_{\rm B} & 0 & 0 & 0 \\ S_{\rm B} & -C_{\rm B} & \sqrt{\frac{3}{2}}S_{\rm B} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}S_{\rm B} & 0 & \sqrt{\frac{3}{2}}S_{\rm B} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}S_{\rm B} & C_{\rm B} & S_{\rm B} \\ 0 & 0 & 0 & S_{\rm B} & 2C_{\rm B} \end{pmatrix}. \tag{54}$$

It is important ot note that $\hat{\mathcal{L}}$ is a symmetric matrix.

The (6×6) kernel matrix $\hat{K}^{\rm dm}$ is given by

$$\begin{pmatrix} K_{1,1} & 0 & 0 & -K_{1,2} & 0 & 0 \\ 0 & K_{4,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{3,3} & 0 & 0 & 0 \\ -K_{1,2} & 0 & 0 & K_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{3,3} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{4,4} \end{pmatrix}. \tag{55}$$

The $K_{i,j}$ are the same elements as in Eq. (19). The corresponding matrix in LBS90 is slightly different (there is no minus sign in front of $K_{1,2}$) because the $K_{1,2}$ have a different definition in LBS90 and NFFS98. To be precise, $K^{LBS}_{1,2}/K^{NFFS}_{1,2} = -w^{(2)}_{J'J}/\sqrt{W_2}$. For a normal Zeeman triplet $w^{(2)}_{J'J} = \sqrt{W_2} = 1$, hence $K^{LBS}_{1,2}/K^{NFFS}_{1,2} = -1$. The definition of $K_{1,2}$ given in NFFS98 is restricted to the case of a positive $w^{(2)}_{J'J}$. This restriction can be removed if $\sqrt{W_2}$ is replaced by $\sigma\sqrt{W_2}$ with $\sigma = {\rm sign}(w^{(2)}_{J'J})$. Note that $\hat{K}^{\rm dm}$ is a symmetric matrix.

2.3. Relation with the scattering formulation

We are now ready to compare Eqs. (17) and (52) which describe the Hanle effect in the atmospheric reference frame, with the "density matrix" and "scattering" formulations, respectively.

First we need the transformation between the source vectors $\boldsymbol{R}^{\rm dm}$ and $\boldsymbol{S}^{\rm sc}$. Expressions of the Stokes parameters at the surface are given in LB94 (Eq. (31)) and in NFFS98 (Eqs. (81)-(83)). Replacing in NFFS98 $\sqrt{W}_2$ by $w^{(2)}_{J'J}$, we have found

$$(R^0_0, -R^2_0, -2\Re(R^2_1), 2\Im(R^2_1), 2\Re(R^2_2), 2\Im(R^2_2)) = (S^{\rm sc}_{\rm I}, S^{\rm sc}_{\rm Q}, S^{\rm sc}_{+1} S^{\rm sc}_{-1}, S^{\rm sc}_{+2}, S^{\rm sc}_{-2}), \tag{56}$$

where $\Re$ stands for real part and $\Im$ for imaginary part. When the conjugation property (21) is taken into account, Eq. (56) can be written as

$$\boldsymbol{R}^{\rm dm} = \hat{T}\,\boldsymbol{S}^{\rm sc}, \tag{57}$$

with

$$\hat{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} \end{pmatrix}. \tag{58}$$

One easily finds that

$$\hat{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -i & 0 & -i & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & i & 0 & 0 & 0 & -i \end{pmatrix}. \tag{59}$$

The matrix $\hat{T}$ describes the transformation from a vector with real components to a vector with complex components. It also takes care of a different ordering of the components.

We now observe that in Eq. (52) the action of the magnetic field appears in the l.h.s. whereas in Eq. (17) it is in the r.h.s.. This suggests to multiply Eq. (17) on the left by $\hat{M}^{-1}_{\rm B}$. This multiplication does not modify the primary source term $G_{\rm I}\,\boldsymbol{e}_1$ because of the block diagonal structure of $\hat{M}_{\rm B}$. Expressing then $\boldsymbol{S}^{\rm sc}$ in terms of $\boldsymbol{R}^{\rm dm}$, we see that the two equations are identical if

$$\hat{T}\,\hat{K}^{\rm sc}\,\hat{T}^{-1} = \hat{K}^{\rm dm}, \tag{60}$$

and

$$\hat{T}\,\hat{M}^{-1}_{\rm B}\,\hat{T}^{-1} = [\hat{I} + i\gamma_{\rm B}\,\hat{\mathcal{L}}]. \tag{61}$$

It is easy to verify the equality in Eq. (60). The verification of Eq. (61) is not as straightforward, because it requires to calculate the inverse of a full (5×5) matrix.

An examination of the $\sqrt{\epsilon}$-law given in LB94, of the symmetries of $\hat{M}^p_{\rm B}$ and of the relations between its elements has allowed us to find the factorization

$$\hat{M}^p_{\rm B} = \hat{B}^p\,\hat{U}^p\,\hat{D}^p_{\rm B}\,\hat{U}^p\,\hat{C}^p. \tag{62}$$

$\hat{D}^p_{\rm B}$ is a diagonal matrix given by

$$\hat{D}^p_{\rm B} = {\rm diag}\{1, 1, \frac{1}{1+\gamma^2_B}, 1, \frac{1}{1+4\gamma^2_B}\}. \tag{63}$$

$\hat{U}^p$ is a real symmetric unitary matrix. Hence it satisfies $\hat{U}^p = [\hat{U}^p]^T = [\hat{U}^p]^{-1}$. It is given by

$$\hat{U}^p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -C_{\rm B} & 0 & S_{\rm B} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & S_{\rm B} & 0 & C_{\rm B} \end{pmatrix}. \tag{64}$$

$\hat{B}^p$ and $\hat{C}^p$ satisfy $\hat{B}^p = [\hat{B}^p]^{-1}$ and $\hat{C}^p = [\hat{C}^p]^{-1}$. They are given by

$$\hat{B}^p = \begin{pmatrix} 1 & 0 & \frac{\sqrt{6}}{2}\gamma_{\rm B} S_{\rm B} & 0 & 0 \\ 0 & 1 & \gamma_{\rm B} C_{\rm B} & 0 & \gamma_{\rm B} S_{\rm B} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\gamma_{\rm B} S_{\rm B} & 1 & -2\gamma_{\rm B} C_{\rm B} \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{65}$$

and

$$\hat{C}^p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sqrt{6}\gamma_{\rm B} S_{\rm B} & -\gamma_{\rm B} C_{\rm B} & -1 & \gamma_{\rm B} S_{\rm B} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\gamma_{\rm B} S_{\rm B} & 0 & 2\gamma_{\rm B} C_{\rm B} & -1 \end{pmatrix}. \tag{66}$$

The factorization (62) holds for $\hat{M}_{\rm B}$ provided one adds to each of the matrices in the product (62), a first row and a first column with unity for the element $\{1,1\}$ and zeroes elsewhere. With Eq. (62), the calculation of $[\hat{M}^p_{\rm B}]^{-1}$ becomes quite easy and leads to

$$[\hat{M}^p_{\rm B}]^{-1} = \hat{I}^p + \gamma_{\rm B}\,\hat{L}^p, \tag{67}$$

where $\hat{I}^p$ is the (5×5) unit matrix and $\hat{L}^p$ is given by

$$\begin{pmatrix} 0 & 0 & \frac{\sqrt{6}}{2} S_{\rm B} & 0 & 0 \\ 0 & 0 & C_{\rm B} & 0 & S_{\rm B} \\ -\sqrt{6} S_{\rm B} & -C_{\rm B} & 0 & S_{\rm B} & 0 \\ 0 & 0 & -S_{\rm B} & 0 & -2\,C_{\rm B} \\ 0 & -S_{\rm B} & 0 & 2\,C_{\rm B} & 0 \end{pmatrix}. \tag{68}$$

It is now straightforward to verify that

$$\hat{T}\,\hat{L}\,\hat{T}^{-1} = i\,\hat{\mathcal{L}}. \tag{69}$$

We stress that the factorization (62) has no special physical content as opposed to the decomposition

$$\hat{M}_{\rm B} = \hat{T}^{-1}\,\hat{\mathcal{D}}(\theta_{\rm B})[\hat{I} + i\gamma_{\rm B}\hat{\mathcal{Q}}]^{-1}\,\hat{\mathcal{D}}(-\theta_{\rm B})\,\hat{T}, \tag{70}$$

which is easily deduced from Eqs. (53) and (61). The matrix $\hat{M}^{-1}_{\rm B}$ is given by Eq. (70) with the square bracketed term to the power 0. The imaginary part in the r.h.s. of Eq. (70) is zero since $\hat{M}_{\rm B}$ is a matrix with real coefficients.

3. The $\sqrt{\epsilon}$-law for the Hanle effect

The integral equations established in Sect. 2 are now used to demonstrate the $\sqrt{\epsilon}$-law for an arbitrary primary source term. We first consider the equations obtained with the density matrix formalism and closely follow the proof given in LB94. Obtaining the $\sqrt{\epsilon}$-law with the scattering formulation requires some more algebraic work, but is interesting in the sense that it shows how one can handle the case of integral operators which are not self-adjoint.

3.1. Density matrix approach

In symbolic notation, Eqs. (39) and (52) may be written as

$$\hat{\mathcal{M}}\,\boldsymbol{X} = (\hat{I} - \hat{\mathcal{E}})\Lambda_{\rm dm}\boldsymbol{X} + \boldsymbol{\mathcal{G}}, \tag{71}$$

where $\Lambda_{\rm dm}$ is the integral operator, $\boldsymbol{\mathcal{G}}$ a given inhomogeneous term that is supposed from now on independent of optical depth, $\hat{\mathcal{E}}$ the diagonal matrix defined in Eqs. (15), (37), (38) and $\hat{\mathcal{M}}$ the matrix which describes the action of the magnetic field. It has the block diagonal structure shown in Eq. (16) and the (5×5) lower block is a symmetric matrix. In Eq. (39),

$$\hat{\mathcal{M}} = [\hat{I} + i\gamma_{\rm B}\hat{\mathcal{Q}}], \tag{72}$$

and in Eq. (52),

$$\hat{\mathcal{M}} = [\hat{I} + i\gamma_{\rm B}\hat{\mathcal{L}}]. \tag{73}$$

The operator $\Lambda_{\rm dm}$ is self-adjoint because the kernels $\hat{K}^{\rm dm}$ and $\hat{\mathcal{D}}(-\theta_{\rm B})\hat{K}^{\rm dm}\hat{\mathcal{D}}(\theta_{\rm B})$ are symmetrical with elements which are even functions of τ. To simplify the notation, it is convenient to rewrite Eq. (71) as

$$\hat{\mathcal{M}}_\epsilon\,\boldsymbol{X} = \Lambda_{\rm dm}\boldsymbol{X} + \boldsymbol{\mathcal{G}}_\epsilon, \tag{74}$$

with

$$\hat{\mathcal{M}}_\epsilon = (\hat{I} - \hat{\mathcal{E}})^{-1}\,\hat{\mathcal{M}}, \quad \boldsymbol{\mathcal{G}}_\epsilon = (\hat{I} - \hat{\mathcal{E}})^{-1}\,\boldsymbol{\mathcal{G}}. \tag{75}$$

Note that $\hat{\mathcal{M}}_\epsilon$ is also a symmetric matrix.

Following LB94, we introduce the integral

$$F = \int_0^\infty \hat{\mathcal{M}}_\epsilon\,\boldsymbol{X}(\tau) \cdot \frac{d\boldsymbol{X}(\tau)}{d\tau}\,d\tau. \tag{76}$$

The operator under the integral is a scalar product. This integral is a vector generalization of a similar quantity,

$$F = \int_0^\infty S(\tau)\,\frac{dS(\tau)}{d\tau}\,d\tau, \tag{77}$$

introduced in the scalar case (Frisch and Frisch 1975). Other quadratic quantities, more or less of the same type, but involving the radiation field instead of the source function, have been introduced by Rybicki (1977) to demonstrate the $\sqrt{\epsilon}$–law and the Hopf-Bronstein relation which holds for monochromatic conservative scattering. They were used by Ivanov (1990) to prove the $\sqrt{\epsilon}$-law for resonance polarization.

Taking the derivative with respect to τ in Eq. (74), we obtain

$$\hat{\mathcal{M}}_\epsilon\frac{d\boldsymbol{X}(\tau)}{d\tau} = \Lambda_{\rm dm}\boldsymbol{X}' + \hat{K}_{\rm dm}(\tau)\boldsymbol{X}(0), \tag{78}$$

where the prime means derivation with respect to τ and $\hat{K}_{\rm dm}$ is the kernel of the operator $\Lambda_{\rm dm}$. It stands for either $\hat{K}^{\rm dm}$ or $\hat{\mathcal{D}}(-\theta_{\rm B})\hat{K}^{\rm dm}\hat{\mathcal{D}}(\theta_{\rm B})$. Eq. (78) is easily established by calculating $\boldsymbol{X}'$ as $\lim_{h\to 0}[(\boldsymbol{X}(\tau+h) - \boldsymbol{X}(\tau))/h]$ (Ivanov 1994). Because the kernel is symmetric with elements which are even function of τ, we have the identities

$$(\boldsymbol{X}_2, \Lambda_{\rm dm}\boldsymbol{X}_1) = (\Lambda_{\rm dm}\boldsymbol{X}_2, \boldsymbol{X}_1), \tag{79}$$

and

$$(\boldsymbol{X}_2, \hat{K}_{\rm dm}\boldsymbol{X}_1) = (\hat{K}_{\rm dm}\boldsymbol{X}_2, \boldsymbol{X}_1). \tag{80}$$

The symbol $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ denotes the scalar product of $\boldsymbol{X}_1$ by $\boldsymbol{X}_2$ integrated over optical depth.

Introducing Eq. (78) into Eq. (76) and making use of Eq. (74) and of the identities (79) and (80), we obtain

$$F = \int_0^\infty \frac{d\boldsymbol{X}(\tau)}{d\tau} \cdot [\hat{\mathcal{M}}_\epsilon\,\boldsymbol{X}(\tau) - \boldsymbol{\mathcal{G}}_\epsilon]\,d\tau + \boldsymbol{X}(0) \cdot [\hat{\mathcal{M}}_\epsilon\,\boldsymbol{X}(0) - \boldsymbol{\mathcal{G}}_\epsilon]. \tag{81}$$

Using now the property that $\hat{\mathcal{M}}_\epsilon$ is a symmetric matrix and performing the integration over τ, we find

$$\hat{\mathcal{M}}_\epsilon\,\boldsymbol{X}(0) \cdot \boldsymbol{X}(0) = \boldsymbol{X}(\infty) \cdot \boldsymbol{\mathcal{G}}_\epsilon. \tag{82}$$

This is the $\sqrt{\epsilon}$-law in the density matrix formulation. The explicit expression of the l.h.s. is given in LB94, Eq. (16), for the magnetic reference frame. For the atmospheric reference frame, it is given in LB94, Eq. (26) and in Eq. (106) of the present paper, but in terms of the components of $\boldsymbol{S}^{\rm sc}$. Eq. (26) of LB94 is recovered if the components of $\boldsymbol{S}^{\rm sc}$ are replaced by the components of $\boldsymbol{R}^{\rm dm}$, according to Eq. (56).

Eq. (82) implies that $\boldsymbol{X}(\tau)$ goes to a finite limit as $\tau \to \infty$. To calculate $\boldsymbol{X}(\infty)$ we take the limit $\tau \to \infty$ in Eqs. (39) and (52). In both equations the kernel goes to the diagonal matrix

$$\hat{K}_\infty = {\rm diag}\{1, \frac{7}{10}W_2, \frac{7}{10}W_2, \frac{7}{10}W_2, \frac{7}{10}W_2, \frac{7}{10}W_2\}. \tag{83}$$

Hence,

$$\boldsymbol{X}(\infty) = \hat{\mathcal{O}}_\infty\,\boldsymbol{\mathcal{G}}, \tag{84}$$

with

$$\hat{\mathcal{O}}_\infty = [\hat{\mathcal{M}} - (\hat{I} - \hat{\mathcal{E}})\hat{K}_\infty]^{-1}. \tag{85}$$

$\hat{\mathcal{O}}_\infty$ has a block diagonal structure with $\hat{\mathcal{O}}_\infty(1,1) = 1/\epsilon_o$. In the atmospheric reference frame,

$$\hat{\mathcal{O}}^p_\infty = \hat{\mathcal{D}}(\theta_{\rm B})[c\hat{I}^p + i\gamma_{\rm B}\,\hat{\mathcal{Q}}^p]^{-1}\,\hat{\mathcal{D}}(-\theta_{\rm B}), \tag{86}$$

with

$$c = 1 - (1 - \epsilon_p)\frac{7}{10}W_2. \tag{87}$$

In the magnetic reference frame, $\hat{\mathcal{O}}^p_\infty$ is given by Eq. (86) with the rotation matrices set to unity. It is a diagonal matrix. We recall that the index p refers to the five-dimension polarization subspace.

When the primary source is of thermal origin,

$$\boldsymbol{\mathcal{G}} = \epsilon_o \, B_{\nu_o} \, \boldsymbol{e}_1, \tag{88}$$

with $\boldsymbol{e}_1 = (1, 0, 0, 0, 0, 0)$, in both Eqs. (39) and (52). Hence, for both reference frames,

$$\boldsymbol{X}(\infty) \cdot \boldsymbol{\mathcal{G}}_\epsilon = \frac{\epsilon_o}{1 - \epsilon_o} \, B^2_{\nu_o}. \tag{89}$$

For checking numerical codes, the vector $\boldsymbol{\mathcal{G}}$ can be chosen in an arbitrary way provided the solutions of Eqs. (39) and (52) satisfy the conjugation property (21). The case of an arbitrary inhomogeneous term is treated in more detail in the next section.

In the case of a magnetic field of arbitrary strength (LB94), the $\sqrt{\epsilon}$-law is also given by Eq. (82). Although the operator Λ is more complicated because the absorption coefficient becomes a matrix, the corresponding kernel has the same symmetry properties as the kernel of the Hanle effect and also goes to the diagonal matrix $\hat{K}_\infty$ defined in Eq. (83). As for the action of the magnetic field, it is described by the same matrix $\hat{\mathcal{M}}$.

3.2. Scattering approach

The proof given above for the the $\sqrt{\epsilon}$-law breaks down because the integral equation for $\boldsymbol{S}^{\rm sc}$ cannot be written as in (71) with $\hat{\mathcal{M}}$ a symmetric matrix. So we start from Eq. (17) which we write in the symbolic form

$$\boldsymbol{S} = \Lambda_{\rm sc} \boldsymbol{S} + \boldsymbol{G}. \tag{90}$$

The inhomogeneous term $\boldsymbol{G}$ is assumed to be independent of optical depth and is treated as a free parameter. To simplify the notation we drop the index "sc" on $\boldsymbol{S}$. First we establish a relation of the *Hopf-Bronstein-Rybicki* type for the scalar product $\boldsymbol{S}(0) \cdot \boldsymbol{S}^{\rm A}(0)$, where $\boldsymbol{S}^{\rm A}$ is the solution of an equation associated to Eq. (90). Then we find the relation between $\boldsymbol{S}$ and $\boldsymbol{S}^{\rm A}$.

Following Ivanov (1995), we introduce the integral equation

$$\boldsymbol{S}^{\rm A} = \Lambda^{\rm T}_{\rm sc} \boldsymbol{S}^{\rm A} + \boldsymbol{G}^{\rm A}, \tag{91}$$

where $\Lambda^{\rm T}_{\rm sc}$ is the adjoint of $\Lambda_{\rm sc}$ and $\boldsymbol{G}^{\rm A}$ a uniform arbitrary source term. To construct $\Lambda^{\rm T}_{\rm sc}$ we simply replace the kernel $\hat{K}_{\rm sc}$, defined in Eq. (18), by its transpose. We then introduce the integral

$$F = \int_0^\infty \boldsymbol{S}^{\rm A}(\tau) \cdot \frac{d\boldsymbol{S}(\tau)}{d\tau} \, d\tau. \tag{92}$$

The operator under the integral is a scalar product. The proof then goes exactly as in the case of the density matrix approach, except that the identities (79) and (80) are replaced by

$$(\boldsymbol{X}_2, \Lambda_{\rm sc} \boldsymbol{X}_1) = (\Lambda^{\rm T}_{\rm sc} \boldsymbol{X}_2, \boldsymbol{X}_1), \tag{93}$$

$$(\boldsymbol{X}_2, \hat{K}_{\rm sc} \boldsymbol{X}_1) = (\hat{K}^{\rm T}_{\rm sc} \boldsymbol{X}_2, \boldsymbol{X}_1). \tag{94}$$

We thus obtain

$$\boldsymbol{S}^{\rm A}(0) \cdot \boldsymbol{S}(0) = \boldsymbol{G}^{\rm A} \cdot \boldsymbol{S}(\infty). \tag{95}$$

The *Hopf-Bronstein-Rybicki* relation proved in Ivanov (1995) for matrices is somewhat more general since it accounts for the case where $\boldsymbol{G}$ and $\boldsymbol{G}^{\rm A}$ are depth-dependent.

We now establish the relation between $\boldsymbol{S}$ and $\boldsymbol{S}^{\rm A}$. First we note that

$$\hat{M}^{\rm T}_{\rm B} = \hat{R}^{-1} \hat{M}_{\rm B} \hat{R}, \tag{96}$$

with

$$\hat{R} = {\rm diag}\{1, \frac{1}{2}, 1, -1, 1, -1\}. \tag{97}$$

The relation (96) can be obtained by examining the symmetries of the matrix $\hat{M}_{\rm B}$ (see Eq. (38) in NFFS98). The integral equation for $\boldsymbol{S}^{\rm A}$ may thus be written as

$$\boldsymbol{S}^{\rm A}(\tau) = \int_0^\infty \hat{K}(\tau - \tau') \hat{R}^{-1} \hat{M}_{\rm B} \hat{R} \, (\hat{I} - \hat{\mathcal{E}}) \, \boldsymbol{S}^{\rm A}(\tau') \, d\tau' + \boldsymbol{G}^{\rm A}. \tag{98}$$

Introducing the vector

$$\boldsymbol{S}^{\rm a}(\tau) = \hat{M}_{\rm B} \, \hat{R} \, (\hat{I} - \hat{\mathcal{E}}) \, \boldsymbol{S}^{A}(\tau), \tag{99}$$

we have found, by comparing the integral equations for $\boldsymbol{S}^{\rm a}$ and $\boldsymbol{S}$, that there exists a matrix $\hat{A}$ such that

$$\boldsymbol{S}^{\rm A}(\tau) = (\hat{I} - \hat{\mathcal{E}})^{-1} \, \hat{A} \, \boldsymbol{S}(\tau), \tag{100}$$

provided

$$\boldsymbol{G}^{A} = (\hat{I} - \hat{\mathcal{E}})^{-1} \, \hat{A} \, \boldsymbol{G}. \tag{101}$$

The matrix $\hat{A}$ can be written as

$$\hat{A} = \hat{E} \, \hat{M}^{-1}_{\rm B} \tag{102}$$

with

$$\hat{E} = {\rm diag}\{1, 1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}. \tag{103}$$

$\hat{A}$ is a block diagonal matrix of the type (16) with the (5×5) lower block $\hat{A}^p$ given by

$$\begin{pmatrix} 1 & 0 & \frac{\sqrt{6}}{2}\gamma_{\rm B} S_{\rm B} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}\gamma_{\rm B} C_{\rm B} & 0 & \frac{1}{2}\gamma_{\rm B} S_{\rm B} \\ \frac{\sqrt{6}}{2}\gamma_{\rm B} S_{\rm B} & \frac{1}{2}\gamma_{\rm B} C_{\rm B} & -\frac{1}{2} & -\frac{1}{2}\gamma_{\rm B} S_{\rm B} & 0 \\ 0 & 0 & -\frac{1}{2}\gamma_{\rm B} S_{\rm B} & \frac{1}{2} & -\gamma_{\rm B} C_{\rm B} \\ 0 & \frac{1}{2}\gamma_{\rm B} S_{\rm B} & 0 & -\gamma_{\rm B} C_{\rm B} & -\frac{1}{2} \end{pmatrix} \tag{104}$$

$\hat{A}^p$ is a symmetric matrix. To calculate $\hat{A}^p$ we have used the decomposition of $\hat{M}^{-1}_{\rm B}$ given in Eqs. (67) and (68) and shown that $\boldsymbol{S}^{\rm a}(\tau) = \hat{D} \boldsymbol{S}(\tau)$, with $\hat{D} = {\rm diag}\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. We have also used the fact that $(\hat{I} - \hat{\mathcal{E}})$ and $\hat{D}$ commute with matrices which have the block diagonal structure (16).

Combining Eqs. (100) and (101) with Eq. (95), and reintroducing the superscript "sc", we find that $\boldsymbol{S}^{\rm sc}(0)$ satisfies the $\sqrt{\epsilon}$-law

$$(\hat{I} - \hat{\mathcal{E}})^{-1} \, \hat{A} \, \boldsymbol{S}^{\rm sc}(0) \cdot \boldsymbol{S}^{\rm sc}(0) = (\hat{I} - \hat{\mathcal{E}})^{-1} \, \hat{A} \, \boldsymbol{G} \cdot \boldsymbol{S}^{\rm sc}(\infty). \tag{105}$$

The l.h.s. is given by

$$\frac{1}{1-\epsilon_o}[S^{\rm sc}_{\rm I}]^2 + \frac{1}{1-\epsilon_p}[S^{\rm sc}_{\rm Q}]^2 + \frac{1}{1-\epsilon_p}\Bigg[\frac{1}{2}([S^{\rm sc}_{+1}]^2 - [S^{\rm sc}_{-1}]^2 + [S^{\rm sc}_{+2}]^2 - [S^{\rm sc}_{-2}]^2) + \gamma_{\rm B}\sin\theta_{\rm B}(\sqrt{6}\,S^{\rm sc}_{\rm Q}S^{\rm sc}_{-1} + S^{\rm sc}_{+1}S^{\rm sc}_{-2} - S^{\rm sc}_{-1}S^{\rm sc}_{+2}) + \gamma_{\rm B}\cos\theta_{\rm B}(S^{\rm sc}_{+1}S^{\rm sc}_{-1} - 2\,S^{\rm sc}_{+2}S^{\rm sc}_{-2})\Bigg]. \quad (106)$$

The notation $[\,]^2$ indicates a squared quantity.

To calculate the r.h.s. we take the limit $\tau \to \infty$ in Eq. (90). This yields

$$\boldsymbol{S}^{\rm sc}(\infty) = \hat{O}_\infty\,\boldsymbol{G}, \quad (107)$$

with

$$\hat{O}_\infty = [\hat{I} - (\hat{I} - \hat{\mathcal{E}})\hat{K}_\infty \hat{M}_{\rm B}]^{-1}. \quad (108)$$

$\hat{O}_\infty$ has the block diagonal structure of Eq. (16) with $\hat{O}_\infty(1,1) = 1/\epsilon_o$. Using the decomposition (67), the (5×5) lower block may be written as

$$\hat{O}^p_\infty = [c\hat{I}^p + \gamma_{\rm B}\,\hat{L}^p]^{-1}\,[\hat{M}^p_{\rm B}]^{-1}. \quad (109)$$

$\hat{O}_\infty$ has the same symmetries as $\hat{M}_{\rm B}$, i.e. it satisfies Eq. (96).

Since $\hat{A}$ is a symmetric matrix, we finally obtain for the r.h.s.,

$$(\hat{I} - \hat{\mathcal{E}})^{-1}\,\hat{A}\,\boldsymbol{G}\cdot\boldsymbol{S}^{\rm sc}(\infty) = \boldsymbol{G}\cdot(\hat{I} - \hat{\mathcal{E}})^{-1}\,\hat{P}_\infty\,\boldsymbol{G}, \quad (110)$$

with

$$\hat{P}_\infty = \hat{A}\,\hat{O}_\infty. \quad (111)$$

Using Eqs. (96), (102) and (108), one can show that $\hat{P}_\infty$ is a symmetric matrix.

For a realistic polarized primary source term, the full matrix $\hat{P}_\infty$ will be needed. The elements of $\hat{P}_\infty$ and of $\hat{O}^p_\infty$ have somewhat complicated expressions. They may be obtained from the author upon request, as Fortran or Latex files. However, if $\boldsymbol{G}$ is a vector with only one non-zero component, say the ith one, the components of $\boldsymbol{S}^{\rm sc}(\infty)$ are given by the ith column of $\hat{O}_\infty$ and the r.h.s. in Eq. (110) by the diagonal elements of $(\hat{I}-\hat{\mathcal{E}})^{-1}\,\hat{P}_\infty$ which are :

$$\hat{P}_\infty(1,1) = \epsilon_o^{-1}, \quad (112)$$

$$\hat{P}_\infty(2,2) = [\alpha_1\alpha_2 - 3a^2\gamma^2_{\rm B}S^2_{\rm B}(\alpha_1 + 3\gamma^2_{\rm B}C^2_{\rm B})]/(c\alpha_1\alpha_2), \quad (113)$$

$$\hat{P}_\infty(3,3) = [c\alpha_1\alpha_3 + 12a^2\gamma^4_{\rm B}S^2_{\rm B}C^2_{\rm B}]/(2c\alpha_1\alpha_2), \quad (114)$$

$$\hat{P}_\infty(4,4) = -[\alpha_2\alpha_3 - 3a^2\,c\gamma^2_{\rm B}S^2_{\rm B}]/(2\alpha_1\alpha_2), \quad (115)$$

$$\hat{P}_\infty(5,5) = [c\alpha_1\alpha_4 + 3a^2\,\gamma^2_{\rm B}S^2_{\rm B}(c^2 + \gamma^2_{\rm B}S^2_{\rm B})]/(2c\alpha_1\alpha_2), \quad (116)$$

$$\hat{P}_\infty(6,6) = -[\alpha_2\alpha_3 - 3a^2\,c\gamma^2_{\rm B}C^2_{\rm B}]/(2\alpha_1\alpha_2), \quad (117)$$

with

$$\begin{aligned} a &= (1-\epsilon_p)\frac{7}{10}W_2 = 1 - c,\\ \alpha_1 &= c^2 + \gamma^2_{\rm B},\\ \alpha_2 &= c^2 + 4\gamma^2_{\rm B},\\ \alpha_3 &= c + \gamma^2_{\rm B} + a\gamma^2_{\rm B},\\ \alpha_4 &= c + 4\gamma^2_{\rm B} + 4a\gamma^2_{\rm B}. \end{aligned} \quad (118)$$

We have carried out some numerical calculations for the case where $\boldsymbol{G}$ has only one non-zero component, say the ith one. For $i = 1$, i.e. for a primary source of thermal origin, we recover the standard result that $S^{\rm sc}_{\rm I}$ goes to $G_{\rm I}/\epsilon_o$ and the other components go to zero. The cases $i \neq 1$ are not physically meaningful, but interesting for numerical tests. When $i \neq 1$, the largest component is $S^{\rm sc}_i$. The intensity component $S^{\rm sc}_{\rm I}$ goes to zero at infinity and the polarization components go to finite limits which are given by the elements of $\hat{O}_\infty$. Numerical calculations show that these asymptotic limits are reached at an optical depth around ten. This can be explained by the fairly low effective value of the single scattering albedo for the polarization components which is $(1-\epsilon_p)\,0.7\,W_2$ and hence less than 0.7.

It is clear that Eq. (105) could have been deduced from Eq. (82) with much less algebraic work than was needed for the direct proof. The comparison between Eqs. (82) and (105) shows in particular that $\hat{E} = \hat{T}^{\rm T}\,\hat{T}$, where $\hat{E}$ is the matrix defined in Eq. (103) and $\hat{T}$ the matrix defined in Eq. (58). However we think that it is nice to have a proof which stays in the framework of the scattering formulation of the Hanle effect, an approach which has also shown its usefulness for the analysis of spectral line polarization (see Faurobert-Scholl 1996 for references).

As final remark we would like to stress that the $\sqrt{\epsilon}$-law may be used irrespectively of whether the transfer problem for the Stokes parameters is transformed by the density matrix formalism or Fourier decomposition. In the latter case, one deduces from Eqs. (7) and (10) that the three components of the Stokes source vector $\boldsymbol{\mathcal{S}} = (\mathcal{S}_{\rm I}, \mathcal{S}_{\rm Q}, \mathcal{S}_{\rm U})$ are related to the six components of $\boldsymbol{S}^{\rm sc}$ by

$$\mathcal{S}_{\rm I} = S^{\rm sc}_{\rm I} + \sigma\sqrt{\frac{W_2}{8}}\,(1-3\mu^2)\,S^{\rm sc}_{\rm Q} + \sigma\frac{\sqrt{3W_2}}{2}\,\mu\sqrt{1-\mu^2}\,[S^{\rm sc}_{+1}\cos\varphi + S^{\rm sc}_{-1}\sin\varphi] + \sigma\frac{\sqrt{3W_2}}{4}\,(1-\mu^2)\,[S^{\rm sc}_{+2}\cos 2\varphi - S^{\rm sc}_{-2}\sin 2\varphi], \quad (119)$$

$$\mathcal{S}_{\rm Q} = \sigma\sqrt{\frac{W_2}{8}}\,3(1-\mu^2)\,S^{\rm sc}_{\rm Q} + \sigma\frac{\sqrt{3W_2}}{2}\,\mu\sqrt{1-\mu^2}\,[S^{\rm sc}_{+1}\cos\varphi + S^{\rm sc}_{-1}\sin\varphi] - \sigma\frac{\sqrt{3W_2}}{4}\,(1+\mu^2)\,[S^{\rm sc}_{+2}\cos 2\varphi - S^{\rm sc}_{-2}\sin 2\varphi], \quad (120)$$

$$\mathcal{S}_{\rm U} = \sigma\frac{\sqrt{3W_2}}{2}\,\sqrt{1-\mu^2}\,[S^{\rm sc}_{-1}\cos\varphi - S^{\rm sc}_{+1}\sin\varphi] + \sigma\frac{\sqrt{3W_2}}{2}\,\mu\,[S^{\rm sc}_{-2}\cos 2\varphi + S^{\rm sc}_{+2}\sin 2\varphi]. \quad (121)$$

We recall that $\sigma = \text{sign}(w^{(2)}_{J'J})$. The components S^{sc}_{α}, $\alpha = \text{I}, \text{Q}, \pm 1, \pm 2$ can be deduced from $\mathcal{S}_{\text{I}}$, $\mathcal{S}_{\text{Q}}$, $\mathcal{S}_{\text{U}}$ by conveniently choosing μ and φ and then inserted in the $\sqrt{\epsilon}$-law. Of course, what is also needed is $\boldsymbol{G}$ in terms of the primary source term $\boldsymbol{\mathcal{S}}^*$. In the most frequent case where $\boldsymbol{\mathcal{S}}^*$ is of thermal origin, the first component of $\boldsymbol{G}$ is equal to the first component of $\boldsymbol{\mathcal{S}}^*$ and all the others are zero.

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