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Alfvén wave phase mixing in two-dimensional open magnetic configurations

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Abstract. The phase mixing of Alfvén waves in planar two-dimensional open magnetic plasma configurations is considered. It is assumed that the characteristic vertical spatial scale of the configuration is much larger than the horizontal scale, and that the latter is of the order of a wavelength. The WKB method is used to derive the governing equation for the wave amplitude, which in appropriate coordinates is the diffusion equation with the diffusion coefficient being spatially dependent. The dependency of the energy flux on the vertical coordinate is obtained for monochromatic waves, and illustrated for three particular cases. In all three cases, at low heights phase-mixed Alfvén waves damp at the same rate as in a one-dimensional configuration. However, in the first and third cases phase mixing operates only at low and intermediate heights and practically stops at heights larger than a few characteristic vertical length scales. Only a part of the energy flux is damped due to phase mixing. In the second case the situation is reversed: the damping of the energy flux with height is much faster that in one-dimensional configurations. The rate of damping of the energy flux with height due to phase mixing in two-dimensional configurations thus depends strongly on the particular form of the configuration. The theory is applied to Alfvén wave damping in coronal holes.

Key words: MHD – Sun: corona – Sun: oscillations – Sun: magnetic fields – waves – methods: analytical

1. Introduction

The problem of solar coronal heating remains one of the most challenging problems for solar physicists. Heyvaerts & Priest (1983) proposed Alfvén wave damping due to phase mixing as a possible source of coronal heating. Since this original paper phase mixing of Alfvén waves has been considered for both open and closed magnetic plasma configurations in the solar corona (see reviews in Browning 1991 and Narain & Ulmschneider 1990, 1996; Parker 1991). In particular, Rytova & Habbal (1995) studied the effect on phase mixing of plasma flows along the magnetic field lines in one-dimensional plasma configurations. Hood et al. (1997a) and Hood et al. (1997b) have found an analytical self-similar solutions describing Alfvén wave phase mixing in open and closed one-dimensional configurations. Taking into account the effects of finite amplitude and compressibility of the plasma, Nakariakov et al. (1997) have shown that phase mixing can dramatically increase the nonlinear coupling of Alfvén and fast magnetosonic waves. A mechanism of indirect heating of the plasma by Alfvén wave phase mixing due to nonlinear generation of obliquely propagating fast magnetosonic waves has been suggested. This work has been further developed for one-dimensional open magnetic configurations with inhomogeneous steady flows by Nakariakov et al. (1998). Possible observational evidence of coronal plasma heating by phase mixing is discussed by Ireland (1996).

Up to now phase mixing of Alfvén waves was analytically studied in one-dimensional magnetic plasma configurations where the equilibrium magnetic field is unidirectional and the equilibrium state inhomogeneous only in the direction perpendicular to the magnetic field lines. However, an important property of the solar corona is its stratification in the vertical direction. This stratification results in plasma inhomogeneity along the magnetic field lines. In addition, in typical coronal structures the magnetic field lines are curved.

Propagation of Alfvén waves in stratified atmospheres has been intensively studied as a source for the acceleration of stellar winds (see, e.g., An et al. 1990; Moor et al. 1991, 1992; Lou & Rosner (1994)). The acceleration is associated with the reflection of Alfvèn waves from the longitudinal inhomogeneity in the Alfvèn speed due to stratification and the radial divergence of magnetic field lines. A model for such considerations is to consider a one-dimensional inhomogeneity in the radial direction. However, it is important to take into account the inhomogeneity of the plasma across the magnetic field, in accordance with observations of open regions of the solar corona (boundaries of coronal holes and streamers, coronal plumes, etc.).

In this work, we consider the case when the ratio of the Alfvèn wave length and a characteristic spatial scale of the inhomogeneity along the magnetic field lines is supposed small. We restrict ourself to a consideration of travelling waves, propagating outward from the Sun without reflection. This assumption allows us to use the WKB approximation to describe the

Send offprint requests to: M.S. Ruderman effect of Alfvèn wave phase mixing in a smooth longitudinal inhomogeneity of the plasma.

The WKB method is a powerful tool for studying wave propagation in inhomogeneous media. We use the WKB method to study phase mixing of Alfvén waves in a plasma structure inhomogeneous in both horizontal and vertical directions, under the assumption that the characteristic spatial scale in the vertical direction is much greater than the horizontal spatial scale. The inhomogeneity of the plasma in the longitudinal direction has important consequencies for phase mixing.

The paper is organized as follows. In the next section we describe the general properties of the two-dimensional magnetic plasma configurations considered. In Sect. 3 we use the WKB method to derive the diffusion equation (with the diffusion coefficient depending on spatial coordinates) that describes the wave behaviour. In Sect. 4, three examples of two-dimensional magnetic plasma configurations are considered. Sect. 5 contains our general conclusions and an illustration of the theory applied specifically to coronal holes.

2. Basic equations and equilibrium state

We use the following set of viscous MHD equations for an infinitely conducting plasma:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{1}$$

$$\rho\left[\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \rho\boldsymbol{g} - \nabla\times(\rho\nu\nabla\times\boldsymbol{v}) + \nabla(\rho\nu\nabla\cdot\boldsymbol{v}), \tag{2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{3}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{4}$$

Here ρ is the density, p the pressure, $\boldsymbol{v}$ the velocity, and ν the kinematic viscosity of the plasma; $\boldsymbol{B}$ is the magnetic field and μ is the magnetic permiability. The gravitational acceleration $\boldsymbol{g}$ is assumed to be constant. In the momentum equation we take only the shear viscosity into account and neglect the compressional viscosity which does not effect Alfvén waves. The energy equation plays no role it in what follows.

We adopt the Cartesian coordinates x, y, z with the z-axis anti-parallel to the gravitational acceleration and consider a two-dimensional static equilibrium in which all quantities depend on x and z only, and the y-component of the magnetic field is zero. The equilibrium field $\boldsymbol{B}_0$, the plasma pressure p_0, and the density ρ_0 satisfy

$$\nabla p_0 - \frac{1}{\mu}(\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_0 + \boldsymbol{g}\rho_0 = 0, \tag{5}$$

$$\nabla\cdot\boldsymbol{B}_0 = 0. \tag{6}$$

The field $\mathbf{B}_0 = (B_{0x}, 0, B_{0z})$ can be expressed in terms of a magnetic flux function ψ through

$$B_{0x} = -B_{00}\frac{\partial\psi}{\partial z}, \quad B_{0z} = B_{00}\frac{\partial\psi}{\partial x}, \tag{7}$$

where $B_{00}(>0)$ is the field strength at $x = z = 0$. In open magnetic plasma configurations with $B_{0z} > 0$ everywhere we have $\partial\psi/\partial x > 0$. Magnetic surfaces are given by the equation $\psi(x,z) = \mathrm{const}$, and the equilibrium pressure and density are related to ψ by the equation

$$\nabla p_0 = \frac{B_{00}^2}{\mu}\nabla\psi\nabla^2\psi + \boldsymbol{g}\rho_0. \tag{8}$$

In order to find the quantities p_0, ρ_0, and ψ we need an additional equation. One possible choice is to assume that the plasma is isothermal, so that p_0 is proportional to ρ_0. Then 0(8) gives a closed set of two scalar equations for ρ_0 and ψ.

In what follows we consider open magnetic configurations with characteristic scale in the z-direction much larger than the characteristic scale in the x-direction. Such a model can be applied to a variety of open magnetic structures in the corona. In such configurations the variation of the total pressure across the magnetic tube defined by the magnetic field lines with footpoints at $x = \pm x_0$ is small. However, in the case where the plasma beta is small as in the solar corona, the plasma pressure and density can vary strongly across the magnetic tube. The x-component of the magnetic field is much smaller than the z-component, and the latter can be represented by the sum of a large term independent of x and a small term that varies strongly in the x-direction. If, in addition, we assume that the configuration is symmetric with respect to the z-axis, then we can write ψ in the form

$$\psi = \psi_0(Z)x + \epsilon\psi_1(x,Z), \tag{9}$$

where $Z = \epsilon z$, with $\epsilon = x_0/H \ll 1$ being the ratio of the horizontal scale x_0 to the vertical scale H. Then the x-component of (8) gives the approximate equation

$$\frac{\partial p_0}{\partial x} = \epsilon\frac{B_{00}^2}{\mu}\psi_0(Z)\frac{\partial^2\psi_1}{\partial x^2}, \tag{10}$$

so that

$$p_0 = \epsilon\rho_{00}V_{A0}^2\psi_0\frac{\partial\psi_1}{\partial x} + \bar{p}_0(Z), \tag{11}$$

where $\bar{p}_0(Z)$ is an arbitrary function. Here ρ_{00} is the density at $x = z = 0$ and $V_{A0} = (B_{00}^2/\mu\rho_{00})^{1/2}$ is the Alfvén speed at that location. If now we assume that the plasma beta is of order ϵ, then the characteristic scale of variation of p_0 is x_0. Under our assumption that ρ_0 has the same characteristic scale in the x-direction as p_0 we arrive at the conclusion that the Alfvén velocity $V_A(x,z) = |\boldsymbol{B}_0|/\sqrt{\mu\rho_0}$ also has x_0 as a characteristic scale of variation in the x-direction. This fact is important for phase mixing of Alfvén waves where the characteristic scale of phase mixing is inversely proportional to the characteristic scale of variation of V_A in the x-direction to the $\frac{2}{3}$ power.

3. WKB solution for Alfvén wave phase mixing

We start the analysis from the derivation of the governing equation for Alfvén waves. This derivation parallels that given by Ruderman et al. (1997) but differs from the later in that here

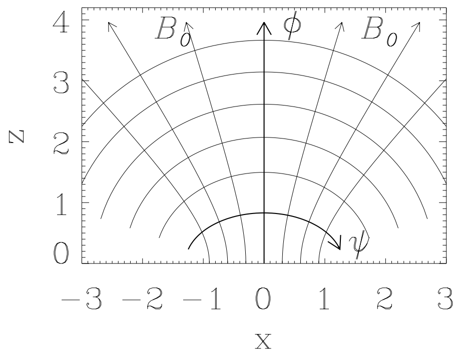

Fig. 1. A sketch of the locally orthogonal reference frame $\psi(x,z)$, $\phi(x,z)$. The unit vector $\boldsymbol{e}_\phi$ is directed along the magnetic field lines.

we do not take plasma resistivity into account. We linearise Eqs. (1)–(3) writing $\boldsymbol{v} = (0, v, 0)$ and $\boldsymbol{B} = \boldsymbol{B}_0 + (0, b, 0)$; this describes Alfvén waves. The y-components of the momentum and induction equations then yield

$$\rho_0 \frac{\partial v}{\partial t} = \frac{1}{\mu}(\boldsymbol{B}_0 \cdot \nabla) b + \nabla \cdot (\rho_0 \nu \nabla v), \qquad (12)$$

$$\frac{\partial b}{\partial t} = (\boldsymbol{B}_0 \cdot \nabla) v. \qquad (13)$$

Eliminating b from this set of equations gives

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \frac{1}{\mu}(\boldsymbol{B}_0 \cdot \nabla)^2 v + \nabla \cdot \left(\rho_0 \nu \nabla \frac{\partial v}{\partial t}\right). \qquad (14)$$

Eq. (14) describes Alfvén wave propagation in two-dimensional planar magnetic configurations.

Introduce a function $\phi(x,y)$ satisfying the equation

$$\frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial z} = 0. \qquad (15)$$

This equation implies that the level lines of the function ϕ given by the equation $\phi(x,z) = \mathrm{const}$ are perpendicular to the magnetic field lines. Since the function $\phi(x,z)$ is determined by (15) up to multiplication by a constant factor, we choose ϕ in such a way that it increases along a magnetic field line. At fixed ψ, the quantity ϕ is a coordinate along a field line.

The two functions ψ and ϕ constitute an orthogonal curvilinear coordinate system in the xz-plane, shown in Fig. 1. In this coordinate system the operator $\boldsymbol{B}_0 \cdot \nabla$ takes the simple form

$$\boldsymbol{B}_0 \cdot \nabla = B_{00} J \frac{\partial}{\partial \phi}, \qquad (16)$$

where J, the Jacobian of the coordinate transformation, is given by

$$J = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x}. \qquad (17)$$

Note that due to the particular choice of the signs of the functions ψ and ϕ we have $J > 0$. Now, with the use of Eq. (16), we rewrite Eq. (14) as

$$\sigma \frac{\partial^2 v}{\partial t^2} = V_{A0}^2 J \frac{\partial}{\partial \phi} J \frac{\partial v}{\partial \phi} + J \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \psi}\left(\nu \sigma h_\psi \frac{\partial v}{\partial \psi}\right) + \frac{\partial}{\partial \phi}\left(\nu \sigma h_\phi \frac{\partial v}{\partial \phi}\right)\right], \qquad (18)$$

where $\sigma = \rho_0/\rho_{00}$ is the dimensionless density. The scale factors h_ψ and h_ϕ are given by

$$h_\psi = J\left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2\right], \qquad (19)$$

$$h_\phi = J\left[\left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2\right]. \qquad (20)$$

We now make the following assumptions:

i) the ratio of the characteristic scales in the x- and z-direction is small, $x_0/H = \epsilon \ll 1$;

ii) the wavelength of v is of order x_0;

iii) the characteristic scale of wave damping is H.

In our curvilinear coordinates the coordinate ϕ is the analogue of the coordinate z in the Cartesian coordinates, while ψ is the analogue of coordinate x. Assumption i) enables us to introduce the stretched coordinate $\Phi = \epsilon\phi$, similar to the stretched coordinate $Z = \epsilon z$ introduced earlier. Phase mixing in a plasma that is homogeneous in the z-direction produces a characteristic scale of wave damping of order $x_0 R^{1/3}$, where the Reynolds number $R = x_0 V_{A0}/\nu$ (see Heyvaerts and Priest 1983). In order to have $x_0 R^{1/3}$ of the order H we take $R = \mathcal{O}(\epsilon^{-3})$ and introduce the scaled coefficient of viscosity $\bar{\nu} = \epsilon^{-3}\nu$.

We look for a solution that locally has the form of a propagating wave, so that

$$v = v(\theta, \Phi) \qquad (21)$$

where the phase θ is given by

$$\theta = \omega t - \epsilon^{-1}\Theta(\Phi, \psi). \qquad (22)$$

Substitution of Eqs. (21) and (22) into Eq. (18) yields

$$\left[\sigma\omega^2 - V_{A0}^2 J^2 \left(\frac{\partial \Theta}{\partial \Phi}\right)^2\right]\frac{\partial^2 v}{\partial \theta^2} = -\epsilon V_{A0}^2 J\left[\frac{\partial v}{\partial \theta}\frac{\partial}{\partial \Phi}\left(J\frac{\partial \Theta}{\partial \Phi}\right) + 2J\frac{\partial \Theta}{\partial \Phi}\frac{\partial^2 v}{\partial \theta \partial \Phi}\right] + J\epsilon\bar{\nu}\sigma\omega h_\psi \left(\frac{\partial \Theta}{\partial \psi}\right)^2 \frac{\partial^3 v}{\partial \theta^3} + \mathcal{O}(\epsilon^2). \qquad (23)$$

We look for the solution of Eq. (23) in the form of the expansion

$$v = v_1(\theta, \Phi) + \epsilon v_2(\theta, \Phi) + \ldots . \qquad (24)$$

Substitution of this expansion into Eq. (23) gives, in the first order approximation (as a condition that v_1 is non-zero), the result

$$\left(\frac{\partial \Theta}{\partial \Phi}\right)^2 = \frac{\omega^2 \sigma}{V_{A0}^2 J^2}. \qquad (25)$$

For upwards ($\Theta > 0$) propagation we obtain

$$\Theta = \int_0^\Phi \frac{\omega\sigma^{1/2}}{V_{A0}J}\,d\Phi + \Theta_0(\Phi,\psi). \tag{26}$$

In what follows we take $\phi = 0$ at $x = x_0$, $z = 0$ and we assume that θ is constant at $\phi = 0$, so that $\Theta_0 = 0$.

In the second order approximation, we obtain

$$2J\frac{\partial\Theta}{\partial\Phi}\frac{\partial^2 v_1}{\partial\theta\partial\Phi} + \frac{\partial v_1}{\partial\theta}\frac{\partial}{\partial\Phi}\left(J\frac{\partial\Theta}{\partial\Phi}\right) - \frac{\omega\sigma\bar{\nu}h_\psi}{V_{A0}^2}\left(\frac{\partial\Theta}{\partial\psi}\right)^2\frac{\partial^3 v_1}{\partial\theta^3} = 0. \tag{27}$$

Using Eq. (26) we reduce this equation to

$$\frac{\partial w}{\partial\Phi} = \lambda\frac{\partial^2 w}{\partial\theta^2}, \tag{28}$$

where

$$w = \sigma^{1/4}v_1, \tag{29}$$

$$\lambda(\Phi,\psi) = \frac{\bar{\nu}\sigma^{1/2}h_\psi}{2V_{A0}}\left(\frac{\partial\Theta}{\partial\psi}\right)^2. \tag{30}$$

Eq. (28) is the diffusion equation in coordinates Φ and θ with coefficient of diffusion λ spatially dependent on Φ and ψ. The variable ψ is present in this equation only as a parameter.

Cosider a monochromatic wave and take w to be proportional to $e^{i\omega t}$. Then Eq. (28) reduces to

$$\frac{\partial w}{\partial\Phi} = -\lambda w, \tag{31}$$

which is integrated to

$$w = W(\psi)\exp\left(-\int_0^\Phi \lambda(\Phi',\psi)\,d\Phi'\right) \tag{32}$$

where the function $W(\psi)$ is determined by the boundary condition at $\phi = 0$. In the next section we use this expression to study the effect of inhomogeneity in the z-direction on phase mixing of Alfvén waves.

4. Alfvén wave damping

The rate of damping of Alfvén waves is determined by the integral term in the exponent in Eq. (32). In this section we study this integral term in different particular cases. However first of all we rewrite it in a form that makes clear the physical meaning of its components.

Since we consider a magnetic plasma configuration with a vertical characteristic scale larger than its horizontal characteristic scale, the magnetic field is almost in the vertical direction. The function ψ can be represented by the approximate formula (9) and $|\partial\psi/\partial x| \gg |\partial\psi/\partial z|$. Then it follows from the orthogonality condition (15) that $|\partial\phi/\partial z| \gg |\partial\phi/\partial x|$, so that the function ϕ is almost independent of x. Taking into account these two inequalities and the fact that $B_{0z} \approx B_0$, together with the formulae

$$\frac{\partial x}{\partial\phi} = -\frac{1}{J}\frac{\partial\psi}{\partial z}, \quad \frac{\partial z}{\partial\phi} = \frac{1}{J}\frac{\partial\psi}{\partial x}, \tag{33}$$

we obtain the expressions

$$J \approx \frac{B_0}{B_{00}}\frac{\partial\phi}{\partial z}, \quad h_\psi \approx \frac{B_0}{B_{00}}\left(\frac{\partial\phi}{\partial z}\right)^{-1}. \tag{34}$$

With the aid of (34) we rewrite expression (26) for Θ as

$$\Theta \approx \epsilon\omega\int_0^z \frac{dz'}{V_A(x,z')}. \tag{35}$$

Substitution of (34) and (35) into expression (30) for λ yields

$$\lambda \approx \epsilon^{-1}\frac{\nu\omega^2 B_0^2}{2V_A B_{00}^2}\frac{\partial z}{\partial\phi}I^2, \tag{36}$$

where

$$I(x,z) = \int_0^z \frac{1}{V_A^2}\frac{\partial V_A}{\partial x}dz'. \tag{37}$$

Finally we arrive at

$$\Lambda \equiv \int_0^\Phi \lambda(\Phi',\psi)\,d\Phi' \approx \frac{\omega^2}{2B_{00}^2(x,z)}\int_0^z \frac{\nu(x,z')B_0^2(x,z')I^2(x,z')}{V_A(x,z')}\,dz'. \tag{38}$$

The quantity Λ determines the damping rate of the variable w. However, of greater interest is the energy flux in an elemental magnetic tube between the magnetic field lines ψ and $\psi + \Delta\psi$. This flux is proportional to $\rho_0 V_A(v^{(1)})^2\Delta\psi$, and $\Delta\psi$ is proportional to B_0, so the energy flux per unit length in the y-direction is

$$S\Delta\psi = e^{-2\Lambda}S_0(x)\Delta x, \tag{39}$$

where $S_0(x)\Delta x$ is the energy flux at $z = 0$ where $\psi = x$. It is straightforward to check that in the case where the unperturbed state is independent of z our results coincide with those obtained by Heyvaerts & Priest (1983).

We consider now three particular cases, assuming in each that ν is a constant.

4.1. *Uniform magnetic field and exponentially decreasing density*

We take $B_{0x} = 0$, $B_{0z} = B_{00}$, and $\rho_0(x,z) = \hat{\rho}_0(x)e^{-z/H}$. This case corresponds to a uniform vertical magnetic field in an isothermal atmosphere. We have to restrict our analysis to heights z not larger than a few H. This is because the exponential growth of the wavelength with height means that the WKB approximation is inappropriate for $z \gg H$. Since ρ_0 depends on x, so also does p_0. However, as explained in Sect. 2, we can neglect the x-dependence of the total pressure because of the fact that the plasma beta is small.

Eq. (38) yields

$$\Lambda = \overline{\Lambda}(x)\left(1 - e^{-z/2H}\right)^3, \tag{40}$$

where

$$\overline{\Lambda}(x) = \frac{\nu\omega^2 H^3}{3V_{A0}^3 \rho_{00}^{3/2} \hat{\rho}_0^{1/2}} \left(\frac{d\hat{\rho}_0}{dx}\right)^2. \tag{41}$$

When $z \ll H$ the quantity Λ is proportional to $z^3/6H^3$ as for the case when the unperturbed state is uniform in z. However, for values of z that are of the order of a few H we have $\Lambda \approx \overline{\Lambda}(x)$, so that there is then almost no damping of the energy flux. Hence, only a part of the energy is dissipated due to phase mixing. The ratio of the dissipated energy flux to the energy flux at the bottom of the magnetic flux tube is

$$K(x) = 1 - e^{-2\overline{\Lambda}(x)}. \tag{42}$$

4.2. *Density uniform in the vertical direction and exponentially diverging magnetic field*

Let us now take

$$\rho_0 = \rho_0(x), \quad \psi = H\exp(-z/H)\sin(x/H) \tag{43}$$

with equilibrium magnetic field

$$B_x = B_{00}e^{-z/H}\sin\frac{x}{H}, \quad B_z = B_{00}e^{-z/H}\cos\frac{x}{H}; \tag{44}$$

then the magnetic field strength $B_0 = B_{00}e^{-z/H}$, declining exponentially. The density does not change with height. The vertical dependence of the unperturbed state is due to the divergence of the magnetic field lines only. A sketch of this magnetic configuration is given in Fig. 1. Recall that we are considering a thin magnetic tube restricted by the magnetic field lines with footpoints $x = \pm x_0$ at $z = 0$, where $x_0 \ll H$. The thickness of this magnetic tube grows with height as $\exp(z/H)$, and once again we have to restrict our analysis to $z \le H$. The damping decrement Λ is now given by

$$\Lambda = \overline{\Lambda}(x)\left(\sinh\frac{z}{H} - \frac{z}{H}\right), \tag{45}$$

where now

$$\overline{\Lambda}(x) = \frac{\nu\omega^2 H^3}{4V_{A0}^3 \rho_{00}^{3/2} \rho_0^{1/2}} \left(\frac{d\rho_0}{dx}\right)^2. \tag{46}$$

Once again Λ is proportional to z^3, as in the one-dimensional case, when $z \ll H$. However, for values of z of the order of H the energy flux is proportional to

$$\exp\left(-\overline{\Lambda}e^{z/H}\right),$$

and so the wave damping due to phase mixing is much faster than in the one-dimensional case. Note, that now $K(x) \equiv 1$.

4.3. *Constant Alfvén velocity*

For our third example we take $\rho_0 = \hat{\rho}_0(x)e^{-2z/H}$ with the magnetic field given by expressions (44). In this case the Alfvén velocity V_A depends on x only. The damping decrement is

$$\Lambda = \overline{\Lambda}\left[1 - e^{-z/H}\left(1 + \frac{z}{H} + \frac{z^2}{2H^2}\right)\right], \tag{47}$$

where

$$\overline{\Lambda}(x) = \frac{\nu\omega^2 H^3}{V_A^5}\left(\frac{dV_A}{dx}\right)^2. \tag{48}$$

As in the previous examples, Λ coincides with the damping decrement for the one-dimensional case when $z \ll H$. However, for z of the order of a few H we have $\Lambda \approx \overline{\Lambda}$ and again we see that only a part of wave energy is dissipated due to phase mixing. Again, the ratio of the dissipated energy to the energy flux at the bottom is given by Eq. (42), with $\overline{\Lambda}(x)$ given by Eq. (48).

5. Discussion and conclusions

We have studied phase mixing in two-dimensional magnetic plasma configurations assuming that the characteristic vertical spatial scale H of the configurations is much larger than the horizontal spatial scale which, in turn, was taken to be of the order of a wave length. Viscosity was scaled in such a way that the characteristic damping length due to phase mixing was of order H. The WKB method was used to derive the dependence of wave amplitude and energy flux on height.

Three particular examples were considered as illustrations of the general formula for the energy flux. In the first example the magnetic field was vertical and uniform, while the equilibrium density decreased exponentially with height. At a height z of a few H, phase mixing practically stops and then the Alfvén wave propagates undamped.

In the second example we considered the opposite situation, where the density is independent of height while the magnetic field diverges with height. In this example the energy flux damps much faster than in the one-dimensional configurations considered by Heyvaerts & Priest (1983). The damping decrement is proportional to the exponential of the height, instead of the height cubed as in one-dimensional configurations.

In our third example, both the density and the magnetic field decrease exponentially with height but in such a way that the Alfvén velocity is constant. Again, phase mixing stops at a distance of the order of a few H and then the Alfvén wave propagates undamped.

The dependences of the damping rates of the phase mixed Alfvén waves on the height for our three examples of open magnetic structures are shown in Fig. 2. The dependences are calculated according to formulae (40), (45) and (47) and compared with the damping rate in the one-dimensional model of Heyvaerts & Priest (1983), for which $\Lambda/\overline{\Lambda} = z^3/6H^3$.

The results obtained here may be applied to conditions typical of the solar corona. As an example, we consider phase mixing

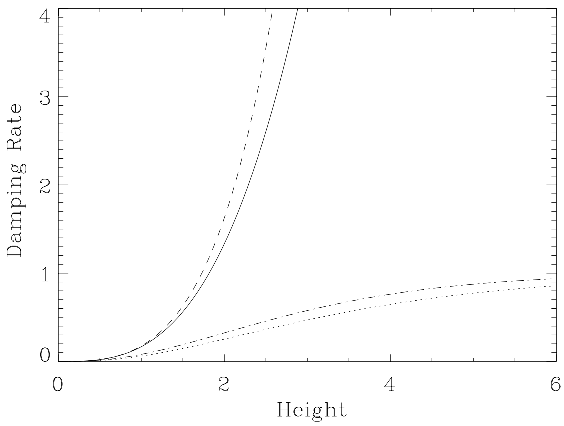

Fig. 2. The dependence of the normalised damping rate $\Lambda/\overline{\Lambda}$ of an Alfvén wave on the normalised height z/H for different models of the structure. The dotted curve corresponds to the case of a uniform magnetic field and exponentially decreasing density; the dashed curve applies to the case of uniform density and exponentially diverging magnetic field; the dash-dotted curve is for the case of constant Alfvén speed; the solid line corresponds to the case of a structure homogeneous in the vertical direction.

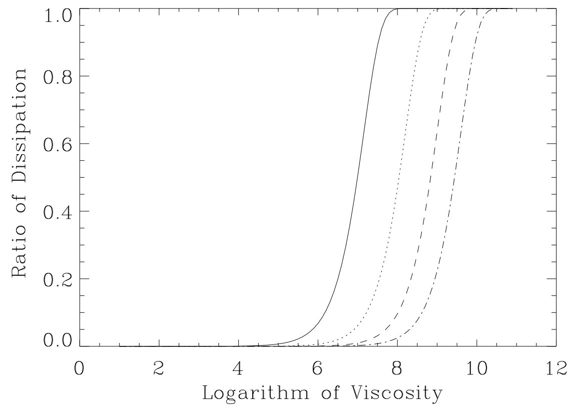

Fig. 3. The dependence on the kinematic viscosity coefficient ν of the ratio Q of the energy of the Alfvén wave dissipated per second in the inhomogeneous region of the hole between the levels $z = 0$ and $z = H$ to the wave energy flux at $z = 0$. The solid, dotted, dashed, and dashed-dotted curves correspond to the waves with periods 6 s, 20 s, 50 s, and 100 s respectively.

of Alfvén waves in a coronal hole. The plasma in a coronal hole is strongly inhomogeneous due to presence of plumes. We note that MHD waves in plumes have been registered in recent EUV observations (DeForest et al. 1998). The typical size of a plume at its base is $5\ \mathrm{Mm}$ (see, e.g., DeForest et al. 1997). Typically the plasma density in a plume is one order of magnitude larger than that in the surrounding plasma. The transition from the rarefied surrounding plasma to the dense plume plasma may occur in plume boundary layers with thickness as much smaller than the plume horizontal size.

At low heights the magnetic field in coronal holes is strongly divergent. However, at larger heights it is almost radial. We assume that Alfvén waves are generated at the base of the upper part of the hole, where the equilibrium magnetic field is approximately radial. Then the characteristic scale of the equilibrium magnetic field variation is of the order of the solar radius, i.e. 700 Mm. On the other hand, the characteristic scale of the density variation in the vertical direction is $H \simeq 100\ \mathrm{Mm}$. Hence the density varies with the height much faster than the magnetic field. This fact enables us to take the magnetic field approximately constant and use the results obtained in Sect. 4.1 to describe the phase mixing of Alfvén waves propagating in a coronal hole.

In what follows we take the magnetic field in the hole to be 10 G and the temperature of the plasma as 10^6 K. We assume that there is sharp density gradient in the horizontal direction with the density varying from $10^{14}\ \mathrm{m}^{-3}$ to $10^{15}\ \mathrm{m}^{-3}$ within a distance of 1 Mm. We choose such a density profile that the quantity $\overline{\Lambda}$ given by Eq. (41) is independent of x (recall that ν is assumed constant). This results in the following expression for the electron density n_e at $z = 0$:

$$n_e(x) = n_0(1 + x/x_0)^{2/3}, \tag{49}$$

with $n_0 \approx 6.43\times10^{14}\ \mathrm{m}^{-3}$ and $x_0 \approx 0.532\ \mathrm{Mm}$; x is measured in Mm. Note that $x = 0$ is the center of the inhomogeneous layer where the transition from the low to the high density plasma occurs. Then $V_{A0} \simeq 10^6\ \mathrm{m/s}$ at $x = 0$, and using $H \simeq 100\ \mathrm{Mm}$ we arrive at

$$\overline{\Lambda} \approx 5.24 \times 10^{-7}\,\nu\omega^2, \tag{50}$$

where ω is measured in s^{-1} and ν in $\mathrm{m}^2\,\mathrm{s}^{-1}$.

It is straightforward to obtain that for the parameters choosen the mean ion collisional time is $\tau_i \simeq 1$ s and the gyrofrequency $\omega_i \simeq 10^5\ \mathrm{s}^{-1}$. The WKB approximation used in the present paper is valid only when the wavelength is much smaller than H. This results in the restriction that the wave period has to be smaller than $H/V_{A0} \simeq 100$ s. Using Eqs. (39) and (40), we obtain that the ratio Q of the energy of the Alfvén wave dissipated in the inhomogeneous region of the hole per second between the levels $z = 0$ and $z = H$ to the wave energy flux at $z = 0$ is given by

$$Q \approx 1 - \exp(-0.122\overline{\Lambda}) = 1 - \exp(-6.4 \times 10^{-8}\,\nu\omega^2), \tag{51}$$

where ω is in s^{-1} and ν in $\mathrm{m}^2\,\mathrm{s}^{-1}$.

The value of kinematic viscosity coefficients, ν, is one of the most uncertain parameters in the solar corona. Using the formula for dynamical viscosity coefficient given by Spitzer (1962) (see also Priest 1982) we obtain $\nu \approx 10^{10}\ \mathrm{m}^2\,\mathrm{s}^{-1}$. However, the viscosity tensor in the solar corona is strongly anisotropic due to the presence of strong magnetic field. In accordance with Braginskii (1965) it is characterized by five coefficients of viscosity. The value, given by Spitzer (1962) corresponds to the first Braginskii coefficient of viscosity, the coefficient of compressional viscosity. The viscosity tensor used in the present paper describes the shear viscosity, which is the only part of the full Braginskii's viscosity tensor that provides damping of Alfvén waves. According to Braginskii, the coefficient of the shear viscosity is smaller than the coefficient of the compressional viscosity by the factor $(\tau_i\omega_i)^{-2}$, so that in our case we

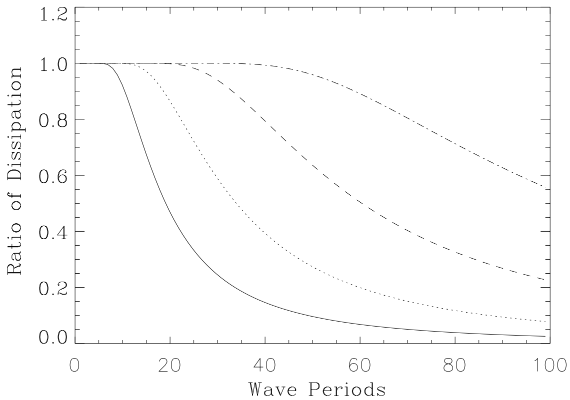

Fig. 4. The dependence on the wave period ω of the ratio Q of the energy of the Alfvén wave dissipated per second in the inhomogeneous region of the hole between the levels $z = 0$ and $z = H$ to the wave energy flux at $z = 0$. The solid, dotted, dashed, and dashed-dotted curves correspond to $\nu = 10^8$ $\mathrm{m}^2\,\mathrm{s}^{-1}$, $10^{8.5}$ $\mathrm{m}^2\,\mathrm{s}^{-1}$, 10^9 $\mathrm{m}^2\,\mathrm{s}^{-1}$, and $10^{9.5}$ $\mathrm{m}^2\,\mathrm{s}^{-1}$ respectively.

take $\nu \approx 10^{10}\,(\tau_i\omega_i)^{-2} \approx 1\ \mathrm{m}^2\mathrm{s}^{-1}$. However, it is quite possible that in reality the coefficient of shear viscosity is determined not by the momentum transfer due to ion diffusion, which is the only process accounted in Braginskii's theory, but by micro-turbulence present in the coronal plasma. In this case it can be many orders of magnitude larger than that given by Braginskii's theory. This fact inspired us to consider ν as a free parameter.

Braginskii's expression for the viscosity tensor is only valid for collisional plasmas, so that we only consider waves with periods larger than $\tau_i \simeq 1$ s. Since at present observational information about waves in coronal holes with periods shorter than 100 s is hardly available, we consider ω as a free parameter as well. The dependences of the quantity Q on ν for different fixed values of ω, and on ω for different fixed values of ν are shown in Fig. 3 and Fig. 4 respectively. Fig. 3 shows that waves with periods between 6 s and 100 s are practically undamped in the part of the plume boundary layer between $z = 0$ and $z = H$ when $\nu \lesssim 10^5\ \mathrm{m}^2\,\mathrm{s}^{-1}$, and they are almost completely damped when $\nu \simeq 10^{10}\ \mathrm{m}^2\,\mathrm{s}^{-1}$.

Our main conclusion is that the rate of wave damping due to phase mixing in two-dimensional magnetic configurations depends strongly on the particular geometry of the configuration and can be either weaker or stronger than that in one-dimensional configurations. The other very important quantities determining the wave damping rate are kinematic coefficient of the shear viscosity and the Alfvén wave frequency.

Our conclusions are of cause based on the simplified model used in the present paper. Other effects, e.g. nonlinear generation of fast MHD waves (Nakariakov et al. 1997, 1998) and the stationary flow along magnetic field lines (Rytova & Habbal 1995; Nakariakov et al. 1998), can strongly affect our results.

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