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LETTER

Letter to the Editor

Remarks on the "Ω-limit"

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Abstract. The dependence on rotation of the Eddington luminosity and the condition of critical rotation for stars close to the Eddington limit are reexamined with respect to the "Ω-limit" introduced recently. The latter is an artifact due to disregarding gravity darkening. It is shown that the standard condition for critical rotation is not affected by the proximity to the Eddington limit, while rotation reduces the Eddington luminosity by at most ≈ 40 per cent in the critical case. As a consequence, the stellar mass loss rate will not diverge at the "Ω-limit" but rather weakly depend on rotation.

Key words: hydrodynamics – stars: interiors – stars: mass loss – stars: rotation

1. Introduction

In a series of papers (Langer 1997, 1998; Langer et al. 1998) the implications of rotation for stellar mass loss and the Eddington limit were investigated recently. In these studies the classical Eddington limit was claimed not to be applicable in rotating stars. Rather a new limit for "hydrostatic stability" was derived and denoted by "Ω-limit". It consists of the existence of a critical rotation speed for the object similar to the standard critical rotation speed. Contrary to the latter, however, in its derivation radiative acceleration is taken into account with the consequence that the critical rotation speed vanishes as the star approaches the Eddington limit. Thus, any star having a finite rotation rate will – irrespective of its magnitude – critically rotate (by virtue of the Ω-limit) before reaching the Eddington limit.

Subsequently, taking a study on radiation-driven stellar mass loss in the presence of rotation (Friend & Abbott 1986) and replacing the classical by the revised critical rotation rate a rotation-dependent mass loss rate was modelled and implemented in stellar evolution calculations. The mass loss rate adopted diverges at the – revised – critical rotation rate and therefore implies stellar evolution of massive stars to be controlled by rotation and to proceed at the Ω limit for a wide range of initial conditions and over a significant fraction of the object's lifetime.

None of the studies mentioned considers the effect of gravity darkening (von Zeipel 1924, Tassoul 1978). The latter reduces the radiative acceleration at the equator and is therefore expected to influence the occurrence of the Ω-limit significantly. This is the motivation for the present investigation dealing with the conditions for critical rotation and the Eddington luminosity when gravity darkening is taken into account (Sect. 2). In Sect. 3 implications for a rotation-dependent mass loss rate will be discussed. Our conclusions follow.

2. Critical rotation and Eddington luminosity

General assumptions and approximations are largely identical with those adopted by Langer (1997, 1998). In particular, we restrict our analysis to rigidly rotating configurations. Following Langer, the Ω-limit is derived by considering the momentum balance in the equatorial plane at the stellar surface. Ignoring the contribution of gas pressure and describing the radiative acceleration within the diffusion approximation we are left with:

$$\frac{\kappa F_r}{c} - \frac{GM}{R^2} + \Omega^2 R = 0. \tag{1}$$

In Eq. (1) which is almost identical with Eq. (5) of Langer (1997) G, c, κ denote the gravitational constant, the speed of light and the opacity, respectively. M, R, Ω and F_r are mass, radius, angular velocity and the radial component of the energy flux of the star.

Assuming the radiation field to be spherically symmetric ($F_r = L/(4\pi R^2)$ with constant luminosity L) Langer derives the critical rotation speed from Eq. (1) as:

$$(\Omega R)^2_{\mathrm{crit}} = \frac{GM}{R}(1-\Gamma) \tag{2}$$

with

$$\Gamma = \frac{\kappa L}{4\pi c G M} \tag{3}$$

being the Eddington factor. Langer emphasizes that the critical rotation velocity (2) vanishes for $\Gamma \to 1$.

Crucial in the derivation sketched is the assumption of a spherically symmetric radiation field by which the effect of gravity darkening (von Zeipel 1924) is discarded. The latter means that in any pseudo-barotrope, in particular in the rigidly rotating models considered, the energy flux on the stellar surface is proportional to the gradient of the effective potential (see, e.g., Tassoul 1978, Sect. 7.2). Thus – rather than by $L/(4\pi R^2)$ – F_r is given by:

$$F_r = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Phi_{\rm eff}}\frac{\partial\Phi_{\rm eff}}{\partial r} \tag{4}$$

r, ρ, T, a denote (cylindrical) radial coordinate, density, temperature and radiation pressure constant and the gradient of the effective potential $\Phi_{\rm eff}$ is in the equatorial plane of the stellar envelope to first approximation given by:

$$\frac{\partial\Phi_{\rm eff}}{\partial r} = -\frac{GM}{r^2} + \Omega^2 r \tag{5}$$

Moreover, in a pseudo-barotropic star the meridional circulation – developing due to von Zeipel's paradox – transports no net energy over a level surface defined by constant $\Phi_{\rm eff}$ (Schwarzschild 1958, Roxburgh et al. 1965, Tassoul 1978). Therefore a luminosity L constant in the envelope may be defined as the integral of the energy flux over a surface determined by constant $\Phi_{\rm eff}$ which implies the relation:

$$L = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Phi_{\rm eff}}\int_{\Phi_{\rm eff}=\rm constant}\nabla\Phi_{\rm eff}\,d\boldsymbol{o} \tag{6}$$

Defining a dimensionless quantity f (of order unity; $f = 1$ for $\Omega = 0$) by

$$f = -\frac{1}{4\pi GM}\int_{\Phi_{\rm eff}=\rm constant}\nabla\Phi_{\rm eff}\,d\boldsymbol{o} \tag{7}$$

we are left with:

$$-\frac{L}{4\pi GMf} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Phi_{\rm eff}} \tag{8}$$

By Eqs. (4,5) the flux F_r at the stellar surface and in the equatorial plane is then given by:

$$F_r = -\frac{L}{4\pi GMf}\left(-\frac{GM}{R^2} + \Omega^2 R\right) \tag{9}$$

Inserting (9) into (1) and using (3) we obtain

$$\left(-\frac{GM}{R^2} + \Omega^2 R\right)\left(-\frac{\Gamma}{f} + 1\right) = 0 \tag{10}$$

Eq. (10) implies two critical conditions one of which is identical with the classical condition for critical rotation:

$$(\Omega R)^2_{\rm crit} = \frac{GM}{R} \tag{11}$$

rather than with the critical rotation speed (2) ("Ω-limit") postulated by Langer. Thus the existence of an "Ω-limit" has to be considered as an artifact based on disregarding gravity darkening. *In a correct treatment the Eddington factor has no effect on the condition for critical rotation.*

The second critical condition

$$\Gamma = f \tag{12}$$

where f varies between 1 (zero rotation) and ≈ 0.6 (critical rotation) obviously corresponds to the Eddington limit for rotating pseudo-barotropes.

In principle, Eq. (12) may also be interpreted as condition for a "second" critical rotation rate depending on the Eddington factor. Thus, the only basis for the definition of an "Ω-limit" would be the solution of Eq. (12) for the Eddington-limit of rotating configurations. An analysis using the condition for critical rotation modified by radiative forces without taking into account gravity darkening is, however, not selfconsistent and therefore not admissible.

Rather than using Eq. (12) for the definition of an Ω-limit – which would then be conceptionally different from that derived by Langer – we adopt the more conventional point of view and interpret (12) as the dependence on rotation of the critical Eddington factor. Compared to the case without rotation it can be reduced by up to ≈ 40 per cent for critical rotation. However, depending on the rotation law, $f > 1$ and super-Eddington luminosities are also possible. Such cases have been considered in connection with accretion tori by Abramowicz et al. (1980). This paper also contains a generally valid derivation of the Eddington limit of rotating objects which yields for the critical Eddington factor (where a flux-weighted average over the stellar surface is to be used for the opacity):

$$\Gamma = 1 - \frac{1}{4\pi GM}\int dV\,\frac{1}{r}\frac{\partial}{\partial r}(\Omega^2 r^2) \tag{13}$$

The r.h.s. of Eq. (13) can be shown to be identical with the factor f occurring in (12). We emphasize that – contrary to the Ω-limit (2) – both these criteria for the critical Eddington factor rely on globally defined quantities. In particular, the dependence on rotation involves an integral of a function of the angular velocity over the entire configuration. While the interpretation of (13) and (12) as conditions for the critical Eddington factor with given angular velocity Ω is straightforward their global nature in general excludes the opposite procedure: Solving, e.g., (13) for a critical rotation rate with given Eddington factor, i.e., inverting the integral involving the rotation rate, is ambiguous. This can be done only by adopting further restrictive assumptions on the rotation law thus supporting the point of view that (13) and (12) have to be understood as critical conditions for the Eddington factor with given rotation rate but not vice versa.

Eq. (13) clearly demonstrates that the critical Eddington factor sensitively depends on the rotation law in a global rather than a local way. For suitably chosen Ω the critical Eddington factor may even exceed unity.

The discussion presented demonstrates that in any case (also for zero rotation rate) the Eddington factor is to be interpreted in a global way. Considering Γ as a local quantity does not imply any limit for the object: Should Γ exceed unity in some region of the envelope this only means (at least for a nonrotating star) that a positive gas pressure gradient is necessary to guarantee equilibrium. As a consequence, the density gradient is also positive there and the region is convectively unstable. In an attempt to rephrase earlier comments on the connection between convection, density inversions and super Eddington luminosities by Glatzel & Kiriakidis (1993) this point was discussed by Langer (1997), however, by erroneously inverting the implication. Langer argues that strongly nonadiabatic convection may lead to density inversions which imply a positive gas pressure gradient. (The latter is equivalent to $\Gamma(') > 1$.) This line of reasoning is not correct: Even if convection is sufficiently inefficient to provide a density inversion this does not necessarily imply gas pressure inversion and thus –locally– super Eddington luminosities. E.g., common Cepheid models having small Eddington factors exhibit density inversions. The inverse implication is valid: If locally $\Gamma(') > 1$ prevails, a gas pressure inversion (implying a density inversion) is necessary for equilibrium, which leads to negative entropy gradients indicating convective instability. If convection is already taken into account in the model, this means that convective transport of energy is inefficient and corresponds to a strongly nonadiabatic environment. For a thorough discussion of this issue we refer the reader to Glatzel & Kiriakidis (1993).

Throughout any of Langer's papers mentioned both the Eddington limit and the condition for critical rotation (also the Ω-limit) are referred to as "stability" limits. For clarification, we would like to point out that none of the limits and conditions considered here are based on a stability analysis. Rather they are derived by considering equilibria and correspond to a limit for the existence of an equilibrium solution.

3. Rotation and mass loss

We have shown in the previous Section that the concept of the Ω limit is a consequence of omitting the effect of gravity darkening. As soon as the latter is taken into account the classical condition for critical rotation remains unaffected. Thus, any rotation dependent mass loss rate based on the Ω limit (Langer 1997, 1998) is inconclusive. In particular, the mass loss rate does not diverge at the Ω limit, which therefore has no meaning for stellar evolution. Accordingly, if the mass loss rate depends on rotation, it should rather be determined by the difference of the actual and the *classical* critical rotation rate (Friend & Abbott 1986). If at all, a divergence is expected to occur at most at the classical critical rotation rate.

The dependence on rotation of line-driven mass flux was studied by Friend & Abbott (1986). They ignored gravity darkening and the analysis was restricted to considering the equatorial plane. As a result, the mass flux was found to increase with decreasing effective gravity. Taking gravity darkening into account, however, the opposite dependence on effective gravity is obtained which was pointed out by Owocki et al. (1996) and Owocki & Gayley (1997) on the basis of the scaling laws for the mass flux in the standard CAK theory (Castor et al. 1975).

On the same level of approximation as Owocki et al. (1996) we find for the dependence on rotation of the mass flux $\dot{m}$ by using Eqs. (4) and (8):

$$\dot{m} \propto f^{-1/\alpha} \nabla \Phi_{\rm eff} \quad (14)$$

α being the usual CAK exponent. We note that in this estimate the role of centrifugal support is even overestimated. Eq. (14) is identical to the result of Owocki et al. (1996) apart from the factor containing f which is not of interest in their analysis. For the mass loss rate $\dot{M}$ and the asymptotic wind speed v_∞ we obtain by integration:

$$\frac{\dot{M}(\Omega)}{\dot{M}(\Omega = 0)} = f^{1-1/\alpha} \quad (15)$$

$$\frac{v_\infty(\Omega)}{v_\infty(\Omega = 0)} = f^{1/2}. \quad (16)$$

f decreases from unity ($\Omega = 0$) to at most 0.6 for critical rotation. Thus, according to our estimate the mass loss rate is slightly increased ($\alpha < 1$) and the asymptotic wind speed is somewhat decreased by rotation. We emphasize that the mass loss rate (15) never diverges. Rather it exhibits a very weak dependence on the rotation rate.

4. Discussion of basic assumptions

The analysis presented, in particular the concept of gravity darkening, is largely based on two assumptions: (i) Energy transport is due to and described by radiation diffusion, convection is disregarded. (ii) The angular velocity is constant over cylinders centered about the axis of rotation ($\frac{\partial\Omega}{\partial z} = 0$, "pseudo-barotrope").

Assumption (i) limits the validity of our discussion to the region below the "photosurface". Provided the second assumption is valid, a proportionality between the total energy flux and the gradient of the effective potential similar to Eq. (4) still holds, if convection contributes to the energy transport. However, then the diffusion coefficient is not necessarily constant over a surface of constant effective potential, which leads to a modification of von Zeipel's law of gravity darkening. Its precise form will depend on the details of the description of convection (in the presence of rotation), for which a reliable theory is not available. On the other hand, to derive the Eddington limit and the condition for critical rotation the energy transport equation is used only at the "photosurface", where in general convection is not present. Thus the results of Sect. 2 remain unaffected, even if convection is reponsible for energy transport in deeper layers of the star.

Rather than the first restriction the condition of a pseudo-barotrope is crucial being necessary for the existence of the effective potential (defined as the sum of gravitational and centrifugal potential), which many further arguments are based on. On the other hand, consideration of pseudo-barotropes may be justified as a dependence on z of Ω would lead to thermal instability (Fricke 1968, Goldreich & Schubert 1967). We note that the general derivation of the Eddington limit (Eq. 13) remains valid, even if $\frac{\partial\Omega}{\partial z} \neq 0$.

The Eddington limit for rotating pseudo-barotropes depends on the quantity f which has been estimated to vary between 1 for zero rotation and ≈ 0.6 for critical rotation. This estimate is based on evaluating the definition (7) for rigid rotation and assuming – similar to the approximation (5) – the gravitational potential to be spherically symmetric, i.e., we adopt a multipole expansion for the potential and keep only the monopole term. The latter is justified, if the mass contained in the stellar envelope is small compared to that of the stellar core (e.g., for evolved massive stars) and if the ratio of centrifugal and gravitational forces decreases from the surface to the center (e.g., for rigid rotation): Then the gravitational potential is determined by a spherical stellar core.

5. Conclusions

Taking into account radiative acceleration we have reexamined the condition for critical rotation. The latter is found to be identical with its classical counterpart, whereas the Eddington luminosity for rigidly rotating objects turns out to be reduced by up to ≈ 40 per cent by rotation in the critical case. The existence of an “Ω-limit” claimed in previous investigations is shown to be an artifact of disregarding the effect of gravity darkening. Thus, even if the bipolar structure of LBV nebulae suggests that rotation is a major ingredient in LBV eruptions, the conclusion that the Ω-limit is involved here (Langer 1997) cannot hold.

As a consequence of our study, rotation-dependent mass loss rates involving the Ω-limit which were adopted in recent stellar evolution calculations are unfounded. Estimating the line-driven mass loss rate on the basis of the scaling laws of the CAK theory we find its dependence on rotation to be rather weak. In particular, it never diverges due to the influence of rotation. Given these results, it seems unlikely that the dependence on rotation of stellar mass loss rates will have significant consequences for stellar evolution.

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