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A regularization method for the extrapolation of the photospheric solar magnetic field

I. Linear force-free field

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Abstract. We present a method for reconstructing the magnetic field $\mathbf{B}$ above the photosphere $\{z = 0\}$ as the solution of the boundary value problem (BVP) for a bounded regular force-free magnetic field in $\Omega = \{z > 0\}$ from its boundary values supposed to be given on $\{z = 0\}$. We propose a way for regularizing the class of standard extrapolation methods which turns out to diverge quickly with height, because of the ill posedness of the BVP that prevents extrapolation at a reasonable height. Our method, which is a Projection Method, allows us to improve considerably the possible height of extrapolation that can be reached by non regularized or even approximate filtering method. In this paper, because of its linear nature the method is applied to the case of linear force-free field.

Key words: MHD – Sun: corona – Sun: magnetic fields

1. Introduction

A large variety of structures and phenomena, such as flares, Coronal Mass Ejections, prominences and coronal heating (Priest 1982) mainly dominated by the magnetic field. However it is not yet observationally accessible in the tenuous and hot plasma that fills the corona (see Sakurai 1989, Amari & Demoulin 1992, and references therein). One is thus led to consider the so-called *Reconstruction* problem of the coronal magnetic field: This consists of solving the equations of a model that assumes the coronal magnetic field to be in a force-free state as a Boundary Value Problem (BVP), in which the boundary conditions are the measured values of the magnetic field in the denser and cooler photosphere. Many problems have been encountered since the early attempts of Schmidt (1964). First one encounters the observational problem to get rid of the 180^o *ambiguity* that remains in the transverse component of the photospheric magnetic field (Amari et al. 1992, McClymont et al. 1997 and references therein). Other classes of problems range from the mathematical problems related to the nature of the boundary conditions leading to well-posed BVPs, to the numerical schemes that can then be used to solve them.

One may first make the simplest physical assumption which consists in assuming that the magnetic field is current-free. This only requires the longitudinal component of the photospheric field as a boundary condition. This case was first considered by Schmidt (1964), and has now led to an almost routine type reconstruction, used for observational purposes (Sakurai 1989), but also for building initial conditions for dynamical MHD numerical simulations (Amari et al. 1996, Mikic et al. 1996). The method of calculation is either a Green's function method or a Laplace solver method for the magnetic field or the vector potential. The mathematics of the various related BVPs (e.g., their well-(or ill-) posedness properties), are also known (Aly 1987).

However, the current free assumption is not relevant to many active regions, since the magnetic configuration is known to have stored free energy (i.e., its energy is above the minimum energy corresponding to the current free configuration with the same value of the normal component in the photosphere). This leads to the introduction of the so-called constant-α force-free hypothesis, which allows for the presence of electric currents in the corona. The magnetic field is computed from its longitudinal component (for a given value of α) by using either Fourier transform (Nakagawa et al. 1973, Alissandrakis 1981) or Green's function (Chiu & Hilton 1977, Semel 1988) techniques. Because of the non uniqueness of the solution this led some authors to consider the possibility of imposing more than one $\mathbf{B}$-component as boundary conditions. Hannakam et al. (1984) and Gary (1989) suggested that it is possible to impose two components of $\mathbf{B}$, while Kress (1989) proposed a least-squares approach (of which he presented a version valid for two-dimensional fields) in which the three $\mathbf{B}$-components are used, the solution being however only an approximate one. Some detailed aspects of these methods are discussed in Amari et al. 1997.

In this Paper, which is the first of a series, we would like to consider a particular class of methods called extrapolation methods that has been proposed for reconstructing the coronal magnetic field as a force-free field for its three component

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given on the boundary (Wu et al. 1985,1990, Cuperman et al. 1990a-b, Demoulin et al. 1991). These methods, associated to a simple BVP in which one use the three component given on the boundary $\{z = 0\}$ turns out to define an ill-posed mathematical problem, reducing to a Cauchy type problem a mathematical problem whose nature is essentially mixed (elliptic-hyperbolic), with no asymptotic condition at infinity taken into account in the BVP. The consequence is that these extrapolation methods turn out to diverge quickly with height.

The paper is organised as follows. In Sect. 2 we present the boundary value problem as well as some of its properties. In Sect. 3 we introduce the extrapolation method used to solve the BVP. Our regularization method is presented in Sect. 4 as well as some of the results obtained in Sect. 5, while Sect. 6 gathers some concluding remarks.

2. The problem

The set of equations that describe the equilibrium of the coronal magnetic field in some region (the half-space $\Omega = \{z > 0\}$) when the plasma pressure and gravitational forces are neglected, are the well known force-free equations (Priest 1982):

$$\nabla \times \mathbf{B} \;=\; \alpha(\mathbf{r})\mathbf{B} \quad \text{in} \quad \Omega \,, \tag{1}$$

$$\nabla \cdot \mathbf{B} \;=\; 0 \,, \quad in \quad \Omega \,, \tag{2}$$

in which $\alpha(\mathbf{r})$ as well as $\mathbf{B}$ are unknowns.

Depending on the set of boundary conditions on $\partial\Omega$ provided for this nonlinear system, different meaningful boundary value problems (BVP's) can be generated. Several aspects of the properties of these BVP's have been discussed elsewhere (see Grad & Rubin 1958, Aly 1988, Low & Lou 1991, Amari & Demoulin 1992, Amari et al. 1997), and are beyond the scope of this paper. We will restrict ourself to one BVP only.

2.1. An 'observational' boundary value problem: BVPO

Since the observations give the values $B_i, \ (i = x, y, z)$ of the three components of $\mathbf{B}$ on the photosphere, it seems natural to define the BVP (defined as BVPO in Amari & Demoulin (1992)) that consists of Eq. (1-2) and:

$$B_i|_{\partial\Omega} = f_i \;, (i = x, y, z), \tag{3}$$

as boundary conditions where f_i are three regular functions that represent the values of the three components of the magnetic field.

2.1.1. Well-posedness

To address the question of well-posedness of BVP0, in the sense of Hadamard (1932), one needs to consider the following points (Lavrent'ev et al. 1980):

- Existence: a solution of the problem should exist for all data in some closed subspace in a normed linear space of the type C^k, L_p, H_p^l, W_p^l and belongs to a space of the same type. The subspace is most often either the whole space or a part of it on which a finite collection of linear functionals vanishes.
- Uniqueness: the solution of the problem is unique in some analogous space.
- Continuity of the solution with respect to the boundary conditions: Any small variations of the data (boundary conditions) of the problem in the corresponding functional space, induce correspondingly small variations of the solution in the functional space to which the solution belongs.

Whether one or more of these three conditions is not fulfilled, so that this BVP is ill-posed, clearly depends on the boundary conditions too. Hadamard (1932) gives the following well known example of ill-posed BVP (Lavrent'ev et al. 1980):
Let $u = u(x, y)$ be the solution of the BVP:

$$\Delta u \;=\; 0, \;\; y > 0 \tag{4}$$

$$u(x, 0) \;=\; 0, \tag{5}$$

$$\frac{du((x, 0)}{dx} \;=\; k\sin(nx) \;\; for \;\; x \in [0, \pi]. \tag{6}$$

Clearly the solution exists and is unique: $u(x, y) = k/n \sin(nx) \sinh(ny)$. It is worth noting that the solution diverges at infinity for $y \to \infty$. Therefore the first two points in the definition of a well-posed BVP given above are satisfied. The main point is now that if one seeks a solution of the BVP above as a Cauchy BVP for the data given at $y = 0$, one gets that for any of the functional space listed above and any $\epsilon > 0, c > 0$ and $y > 0$ it is possible to choose k and n such that: $||\ k\sin(nx)\ || < \epsilon$ and $||\ k/n \sin(nx) \sinh(ny)\ || > c$, which shows that the last point in the definition of well-posedness above is not satisfied and therefore makes the BVP ill-posed.

In the solar extrapolation problem we have some data given by the observations (B_x, B_y, B_z) on $\{z = 0\}$, and whether we are going to use all the information or only a part (and use the remnants as compatibility conditions) to define a well-posed BVP are two separate issues.

i: Let us first consider BVP0 without imposing any added condition at infinity or elsewhere, and consider it as a Cauchy problem (Wu et al. 1985). As in the well known Hadamard example given above, this problem may admit of a solution independent of what happens in the far region at infinity. Again as in the Hadamard example the solution may diverge at infinity, and from the mathematical point of view any reconstruction method should be able to reproduce such a divergent solution. For example in the particular case $\alpha = 0$, using a scalar potential representation of the solution, and assuming periodic boundary conditions, the solution would be a superposition of elementary solutions tending to zero at infinity with z ($\exp(-kz)$) and diverging ($\exp(+kz)$). Note that although the requirements of existence and uniqueness would be satisfied there is no proof that the property of continuity with respect to the boundary conditons is fullifiled for this problem.

ii: Since most of the known models relevant to the magnetic structure of the corona (potential and force-free) reasonably assume that the magnetic field tends to zero at infinity, *we will hereafter assume that the solutions sought belong to the func-*

tional space of functions tending to zero at infinity.

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}| = 0\,. \tag{7}$$

This mathematical assumption (that does not seem to disagree with the observations even far from the sun) would mean that in the particular current-free example above, one would generally keep the $\exp(-kz)$ contribution only. The model in which the magnetic field is created by a dipole located under the photosphere is also one of the simplest models corresponding to this assumption. Now, on this hypothesis the far-field (boundary value) behaviour at infinity influences the local behaviour of the solution in any neighbourhood of the domain, since Eq. (1) has a partial elliptic structure component, the other one being hyperbolic (Grad & Rubin 1958, Amari & Demoulin 1992). Let us consider in the following for simplicity, the particular constant-α case (i.e α is supposed constant in Eq. (1)). The problem is then fully elliptic. To the question whether there exist a constant-α force-free field which tends to zero at infinity only for the tangential components (B_x, B_y), and α given on the boundary (note that if Bz is also given instead of α, then the compatibility condition resulting from Eq. (1) fixes α), Hannakam et al. (1984) claimed that the answer is yes, while Sakurai (1989) argued the contrary. We proved (Amari et al. 1997) a non-existence result. The argument can be summarized as follows. One makes a Fourier expansion and computes the solution, keeping the $c(k)$ usual arbitrariness in the solution. Then, assuming that the solution is continuous up to the boundary one finds that the tangential components computed from the solution inside the domain have no reason to match their given boundary value in general. This can also result from a Green function approach, as in the potential case for which the use of a Green function representation of the scalar potential Φ, ($\mathbf{B} = \nabla\Phi$), involving the value of the Φ and its derivative at the boundary, is subjected to the condition that the solution of the BVP exists. Therefore BVP0 is ill-posed if one seeks a solution that tends to zero at infinity, at least in the constant-α case. This shows ill-posedness in the sense of Hadamard, from a non-existence result, rather than non continuity of the solution with respect to the boundary conditions. Actually this last requirement is not even satisfied (as discussed in Low & Lou (1991)). They show that this continuity behaviour is violated in the case of a regular, smooth, potential solution in $\{z > 0\}$. Then an any arbitrary small deviations of the tangential components on the boundary $\{z = 0\}$ give rise to the appearance of singularities (divergence to infinity). In the less trivial case of a nonlinear force-free field, they show that the same behaviour is expected regarding absence of continuity with respect to boundary conditions. Our argument of non-existence of a solution (for constant-α) cannot be reproduced because it relies on the possible analytical computation of the solution. However, using the partial elliptic structure of the system of equations, Bineau (1972) proved that in a bounded domain, the problem is well posed if B_n is given on the boundary, but α given only on one part of the boundary (for example where $B_n > 0$.

An important point that will not be discussed here and concerns the functional space in which a solution is sought, is the regularity of the solution since one may look for solution having possibly current sheets, (see the review of Amari 1991).

3. An extrapolation method: the vertical integration method

One can rewrite BVPO as

$$\frac{\partial[\mathbf{B},\alpha]}{\partial z} = \mathrm{M}[\alpha]\cdot[\mathbf{B}]\,, \tag{8}$$

$$B_i(x,y,0) = f_i(x,y),\; i = x,y,z\,, \tag{9}$$

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}| = 0\,. \tag{10}$$

where $\mathrm{M}[\alpha]$ is a matrix operator (whose explicit form it is not necessary to give at this stage) that contains derivatives with respect to x and y only (in the general nonlinear case, $\mathrm{M}[\alpha]$ would be a nonlinear matrix operator).

One can thus readily notice that the problem is equivalent to a Cauchy-type one in which the vertical coordinate z plays the role of time and the boundary conditions the role of initial conditions. The numerical method called the Vertical Integration Method (VIM) then consists in starting from the magnetograph data at the photospheric level $z = 0$, and extrapolating progressively the “initial conditions”. Originally this scheme was proposed by Wu et al. (1985),and then later used and improved by many authors. Most of its drawbacks have been already discussed (see the review of Amari et al. 1992, Démoulin 1997 and McClymont et al. 1997). Mainly it is extremely difficult to integrate up to large heights, small-length -scale solutions being rapidly amplified when z increases, and even growing exponentially, whatever filtering method is used (Demoulin et al. 1991). This blowing up of the solution with height was shown to exist in the non-constant and constant-α case and even in the potential case, on which some regularization procedures have been tested (Cuperman et al. 1989, Demoulin et al. 1991).

As discussed in the previous section, BVPO is ill-posed. Furthermore these formulations do not incorporate the asymptotic condition given by Eq. (7) at infinity, since only the information originating from the boundary is transmitted. Eventually there are no obvious boundary conditions to impose on the lateral faces of the numerical box when the method is implemented on a computer. Our starting point consists in addressing the following practical issue: Is it possible to improve the VIM in order to be able to catch at least the solutions that tend to zero at infinity. The idea then consists in, like for the analytical potential periodic solution discussed in Sect. 2, retaining the exponentially decreasing component of the solution. The previous formulations of the VIM fail to take into account the well-behaved asymptotic behaviour desired. In the next section we propose a regularization method whose central point is to incorporate the information contained in the asymptotic condition. In this paper we first confine ourself to the linear force-free case.

The so called constant-α force-free approximation, corresponds to the assumption $\alpha = constant$ in Eq. (1). Then the field $\mathbf{B}_\alpha$ satisfies

$$\begin{array}{ll} \nabla\times\mathbf{B}_\alpha = \alpha\mathbf{B}_\alpha & \text{in }\Omega\,,\\ \nabla\alpha = 0 & \text{in }\Omega\,. \end{array} \tag{11}$$

where we assume that $\mathbf{B}_\alpha$ is continuous up to the boundary $\partial\Omega$.

4. Regularization: the projection method

Let $\mathbf{B}_h$ stands for the discretized magnetic field:

$$\mathbf{B}_h(x,y,z) = \mathbf{B}(x_l, y_m, z) \tag{12}$$

for $x_l \le x \le x_{l+1}$ and $y_m \le y \le y_{m+1}$ and rewrite the 'Cauchy' problem after centered spatial discretization of the RHS of Eq. (10) as:

$$\begin{aligned}\frac{\partial B_{h,x}}{\partial z}(x,y,z) &= \alpha B_{h,y}(x,y,z) \\ &+\frac{1}{2\Delta x}(B_{h,z}(x+\Delta x,y,z) \\ &-B_{h,z}(x-\Delta x,y,z)),\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial B_{h,y}}{\partial z}(x,y,z) &= -\alpha B_{h,x}(x,y,z) \\ &+\frac{1}{2\Delta y}(B_{h,z}(x,y+\Delta y,z) \\ &-B_{h,z}(x,y-\Delta y,z)),\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial B_{h,z}}{\partial z}(x,y,z) &= -\frac{1}{2\Delta x}(B_{h,x}(x+\Delta x,y,z) \\ &-B_{h,x}(x-\Delta x,y,z)) \\ &-\frac{1}{2\Delta y}(B_{h,y}(x,y+\Delta y,z) \\ &-B_{h,y}(x,y-\Delta y,z)).\end{aligned} \tag{15}$$

Effecting a Fourier transform (Read & Simon; 1978) one gets

$$\frac{\partial \widehat{\mathbf{B}_\alpha h}}{\partial z}(\xi_1,\xi_2,z) = L(\xi_1,\xi_2)\widehat{\mathbf{B}_\alpha h}(\xi_1,\xi_2,z)\ , \tag{16}$$

where $L(\xi_1,\xi_2)$ stands for the matrix

$$\begin{bmatrix} 0 & \alpha & \frac{i}{\Delta x}\sin(\xi_1\Delta x) \\ -\alpha & 0 & \frac{i}{\Delta y}\sin(\xi_2\Delta y) \\ -\frac{i}{\Delta x}\sin(\xi_1\Delta x) & -\frac{i}{\Delta y}\sin(\xi_2\Delta y) & 0 \end{bmatrix}. \tag{17}$$

After some calculation one gets for the eigenvalues of $L(\xi_1,\xi_2)$ the set:$(0,\ \lambda$ et $-\lambda)$ with

$$\lambda^2 = \frac{1}{\Delta x^2}\sin^2(\xi_1\Delta x) + \frac{1}{\Delta y^2}\sin^2(\xi_2\Delta y) - \alpha^2, \tag{18}$$

and for the eigenvectors

$$\boldsymbol{v}_0 = \begin{pmatrix} \frac{i}{\Delta x}\sin(\xi_1\,\Delta x) \\ -\frac{i}{\Delta y}\sin(\xi_2\,\Delta y) \\ \alpha \end{pmatrix}, \tag{19}$$

$$\boldsymbol{v}_1 = \begin{pmatrix} \frac{\lambda}{\Delta x}\sin(\xi_1\,\Delta x) + \frac{\alpha}{\Delta y}\sin(\xi_2\,\Delta y) \\ \frac{\lambda}{\Delta y}\sin(\xi_2\,\Delta y) - \frac{\alpha}{\Delta x}\sin(\xi_1\,\Delta x) \\ -i(\frac{1}{\Delta x^2}\sin^2(\xi_1\,\Delta x) + \frac{1}{\Delta y^2}\sin^2(\xi_2\,\Delta y)) \end{pmatrix}, \tag{20}$$

$$\boldsymbol{v}_2 = \begin{pmatrix} \frac{\lambda}{\Delta x}\sin(\xi_1\,\Delta x) - \frac{\alpha}{\Delta y}\sin(\xi_2\,\Delta y) \\ \frac{\lambda}{\Delta y}\sin(\xi_2\,\Delta y) + \frac{\alpha}{\Delta x}\sin(\xi_1\,\Delta x) \\ i(\frac{1}{\Delta x^2}\sin^2(\xi_1\,\Delta x) + \frac{1}{\Delta y^2}\sin^2(\xi_2\,\Delta y)) \end{pmatrix}. \tag{21}$$

where i is the complex number such that $i^2 = -1$.

This linearly independent set of vectors forms a basis and the numerical solution can be expanded on this basis as

$$\widehat{\mathbf{B}_\alpha h} = \sum_{k=0}^{2} c_k \mathbf{v}_k e^{\lambda_k \cdot z} \tag{22}$$

One can clearly see the effect of the undesired component $\mathbf{v}_1$ which arises because of the lack of imposing the condition that the solution tends to zero at infinity in the VIM (Eq. (7)). Even if one starts at $\{z=0\}$ with the exact analytical values for $\widehat{\mathbf{B}_\alpha h}$ (and therefore for $\mathbf{B}_\alpha h$), which has no component on $\mathbf{v}_1$, then the smallest numerical errors (such as rounding-off errors or errors due to the discretization) will introduce into the numerical solution a component (though small) in $\mathbf{v}_1$ at the next vertical step, and this will unfortunately grow exponentially. This intrinsic pro- oerty is independent of the numerical scheme used to advance the LHS of the Cauchy problem (such as the Euler or Adams-Bashford schemes used in Cuperman et al. 1989 and Demoulin et al. 1991).

The key point of our method consists in constructing a regularization operator that is made of a projection matrix $\mathcal{P}$ such that:

$$\widehat{\mathbf{B}_\alpha h}^R = \mathcal{P}\cdot\widehat{\mathbf{B}_\alpha h}\ , \tag{23}$$

where $\widehat{\mathbf{B}_\alpha h}^R$ is the regularized form of $\widehat{\mathbf{B}_\alpha h}$. The two possible approaches of our Projection Method then depend on the choice of $\mathcal{P}$.

4.1. *Total projection*

In this case only the component in the space generated by $\mathbf{v}_2$ is kept. This choice corresponds (when λ is not a pure imaginary complex number) to the projection matrix

$$\mathcal{P}(\xi_1,\xi_2) = \frac{1}{2\lambda^2(u_1^2+v_1^2)}\mathcal{P}' \tag{24}$$

where $\mathcal{P}'$ is the matrix:

$$\mathcal{P}' = [\mathcal{P}'_1, \mathcal{P}'_2, \mathcal{P}'_3] \tag{25}$$

with

$$\mathcal{P}'_1 = \begin{bmatrix} \lambda^2 u_1^2 - \alpha^2 v_1^2 \\ (\lambda v_1 + \alpha u_1)(\lambda u_1 + \alpha v_1) \\ i\,(\lambda u_1 + \alpha v_1)(u_1^2+v_1^2) \end{bmatrix}, \tag{26}$$

$$\mathcal{P}'_2 = \begin{bmatrix} (\lambda u_1 - \alpha v_1)(\lambda v_1 - \alpha u_1) \\ \lambda^2 v_1^2 - \alpha^2 u_1^2 \\ i\,(\lambda v_1 - \alpha u_1)(u_1^2+v_1^2) \end{bmatrix}, \tag{27}$$

$$\mathcal{P}'_3 = \begin{bmatrix} -i\,(\lambda u_1 - \alpha v_1)(u_1^2+v_1^2) \\ -i\,(\lambda v_1 + \alpha u_1)(u_1^2+v_1^2) \\ (u_1^2+v_1^2)^2 \end{bmatrix}, \tag{28}$$

and where: $u_i = \frac{1}{\Delta x_i}\sin(\xi_i \Delta x_i),\ i = x,\ y.$

One can readily check that this choice corresponds to the total elimination of both increasing and oscillating modes.

4.2. *Partial projection*

In this case one chooses for $\mathcal{P}$, either the unit matrix when λ is a pure imaginary complex number, or when λ is a real number

$$\mathcal{P} = \mathcal{I} - \mathcal{P}_0 \quad (29)$$

where P_0 is the projection matrix onto $\mathbf{v}_1$, obtained by replacing λ by $-\lambda$ in (24)

It is worth noting that in the total projection version, only the normal component of $\mathbf{B}_\alpha$ on $\{z = 0\}$ is required.

4.3. *The projection method algorithm*

Implemented in practice the method consists in:

1. *Initializing* $\mathbf{B}_1$ *on* $\{z = 0\}$.
2. *Then for* $k = 1,\ \ N_z$*:*
3. *Computing* $\mathbf{B}_2$ *at* $\{z = (k+1)\Delta z\}$.
4. *Applying the regularization matrix* $\mathcal{R}$

   $$\mathcal{R} = \mathcal{F}^{-1} \circ \mathcal{P} \circ \mathcal{F} \quad (30)$$

   where $\mathcal{F}$ *stands for the discrete Fast Fourier Transform and* $\mathcal{F}^{-1}$ *for its inverse.*
5. *re-initializing* $\mathbf{B}_2 = \mathbf{B}_1$ *and go to step 2*

To get $\mathbf{B}_2$ from $\mathbf{B}_1$ we use the one level Euler scheme for $k = 1$, and the two levels Adams-Bashford scheme for $k \geq 2$ as in Cuperman et al. (1989) and Demoulin et al. (1991).

5. Results

5.1. *Numerical error*

Our regularization method is tested on a set of particular known analytical solutions. To perform the computation we take the numerical box $[0,\ L_x] \times [0,\ L_y] \times [0,\ L_z]$, discretized with the uniform mesh: $Nx\Delta x$, $Ny\Delta y$ et $Nz\Delta z$, where $(\Delta x, \Delta y, \Delta z)$ are the step sizes and (Nx, Ny, Nz) the corresponding node numbers.

Because of the errors due to the lack of boundary conditions on the lateral faces of the box, we avoid counting these errors by computing the error on a subdomain of the full box and use the following euclidian relative error:

$$e_A = \left(\frac{\sum_{i=N_{1x}}^{N_2 x} \sum_{j=N_{1y}}^{N_{2y}} ||A^h i, j - A^a_{i,j}||^2}{\sum_{i=N_{1x}}^{N_{2x}} \sum_{j=N_{1y}}^{N_{2y}} ||A^a_{i,j}||^2} \right)^{\frac{1}{2}}, \quad (31)$$

where A is a scalar or vectorial field ($\mathbf{B}$ or one of its components), A^h stands for its numerical approximation and A^a for its exact analytical value.

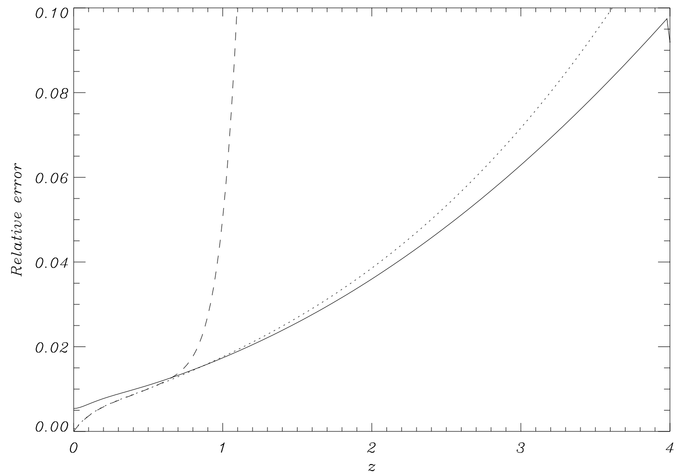

Fig. 1. Example #1, Relative error in the norm of the solution versus height, for a current free solution. Dashed: the VIM, without any regularization. Our regularization method is based upon a projection onto the space of regular solution. Dotted: Partial version of our Projection Method Solid: Total version of our Projection Method.

5.2. *Example 1: a potential magnetic field*

Let us take the field created by a dipole located below the surface $\{z = 0\}$ at $z = -L_0$ and given by (Jackson 1975)

$$B_x = 3\,B_0\,\frac{x_1\,z_1}{R^5}, \quad (32)$$

$$B_y = 3\,B_0\,\frac{y_1\,z_1}{R^5}, \quad (33)$$

$$B_z = B_0\,\frac{3\,z_1^2 - R^2}{R^5}, \quad (34)$$

where $z_1 = z + L_0$, $x_1 = x - \dfrac{L_x}{2}$, $y_1 = y - \dfrac{L_y}{2}$ et $R^2 = x_1^2 + y_1^2 + z_1^2$. Clearly this field is current free and thus corresponds to $\alpha = 0$. The horizontal size of the box is fixed with $L_x = L_y = 5\,L_0$. We choose: $N_x = N_y = 128$, and a vertical step size $\delta z = 0.02 L_0$. Fig. 1 shows that the V.I.M allows us to extrapolate up to a height of order L_0 with a global error of 0.06%. The Partial Projection of our Regularization Method allows us to reach a height of $2,8\,L_0$ with the same error and the best result is obtained with the Total Projection, reaching $3L_0$ with the same error.

5.3. *Example 2: a periodic constant-α force-free magnetic field*

We choose the specific exact solution:

$$B_x = -(\frac{4}{a\,b})^{\frac{1}{2}}\,\frac{\pi}{\lambda_{n,m}}\Big(\frac{n\,\gamma_{n,m}}{a}\cos(\frac{n\pi\,x}{a})\sin(\frac{m\pi\,y}{b}) - \frac{m\,\alpha}{b}\sin(\frac{n\pi\,x}{a})\cos(\frac{m\pi\,y}{b})\Big)e^{-\gamma_{n,m}\,z}$$

$$B_y = -(\frac{4}{a\,b})^{\frac{1}{2}}\,\frac{\pi}{\lambda_{n,m}}\Big(\frac{n\,\alpha}{a}\cos(\frac{n\pi\,x}{a})\sin(\frac{m\pi\,y}{b}) + \frac{m\,\gamma_{n,m}}{b}\sin(\frac{n\pi\,x}{a})\cos(\frac{m\pi\,y}{b})\Big)e^{-\gamma_{n,m}\,z}$$

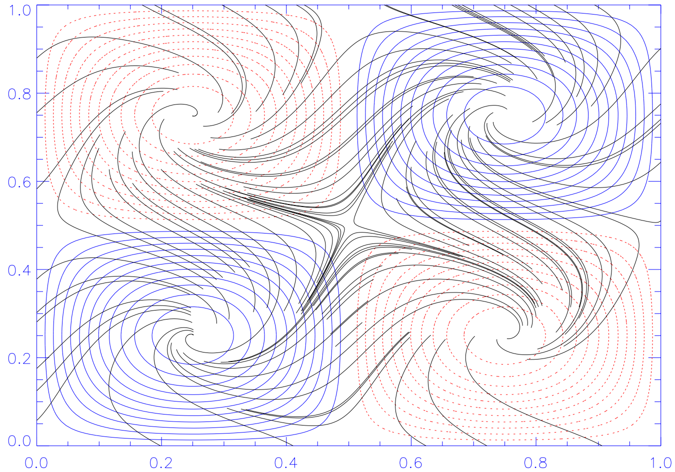

Fig. 2. Some field lines of a periodic constant-$alpha$ force-free configuration, traced above the contours of Bz. Note that the magnetic topology is complex, containing a separatrix.

$$B_z = -(\frac{4}{a\,b})^{\frac{1}{2}} \sin(\frac{n\pi\, x}{a}) \sin(\frac{m\pi\, y}{b}) e^{-\gamma_{n,m}\, z}$$

where $\lambda_{n,m} = \frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{a^2}$ and $\gamma_{m,n} = \sqrt{\lambda_{n,m} - \alpha^2}$.

We make the specific choice $n = m = 2$ and $a = b$ et $\alpha = \sqrt{\frac{\pi^2}{a^2} + \frac{4\,\pi^2}{b^2}} = \frac{\pi\sqrt{5}}{a}$. The magnetic topology of this configuration is shown in Fig. 2. It is a complex topology magnetic configuration since the presence of an X-type point shows the presence of a separatrix.

As for the previous example $N_x = N_y = 128$. The vertical integration step size is $\delta z = 0.66\ 10^{-2} \times L_0$. As shown in Fig. 3, the V.I.M gives an error of 0.05% at $0.4 \times L_0$. The Partial Projection regularization does not improve these results, while for the Total Projection the error is 0.05% at $z = 2 \times L_0$, wich is much better. These results show that the Projection Method can regularize the VIM even in the case of complex topology configuration.

5.4. Example 3: constant-α source fields

As a new test we choose the set of constant-α force-free magnetic fields created by magnetic charges located below the surface $\{z = 0\}$. This family of solutions is interesting since it is no longer periodic (unlike the previous example) and may model more adequately the topology of some active regions (Mandrini et al. 1995). Using the standard local spherical coordinate system $(r_i,\, \varphi_i,\, \theta_i)$:

$$\begin{cases} B_{r_i} &= Q_i \dfrac{cos(\alpha r_i)}{r_i^2}, \\ B_{\theta_i} &= Q_i \alpha tan(\dfrac{\theta_i}{2}) \dfrac{\sin(\alpha r_i)}{r_i}, \\ B_{\varphi_i} &= Q_i \alpha tan(\dfrac{\theta_i}{2}) \dfrac{\cos(\alpha r_i)}{r_i}, \end{cases} \quad (35)$$

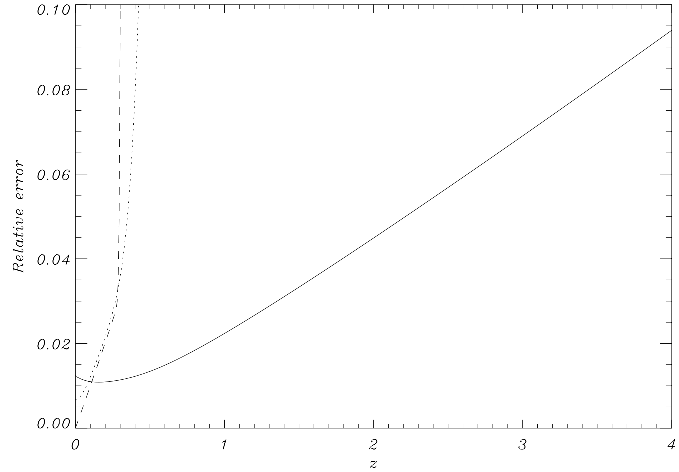

Fig. 3. Example #2 Relative error in the norm of the solution versus height, for a current free solution. Dashed: the VIM, without any regularization. Our regularization method is based upon a projection onto the space of regular solution. Dotted: Partial version of our Projection Method Solid: Total version of our Projection Method.

5.4.1. A bipolar configuration case

A set of two sources of opposite sign is taken located on the plane $\{z = -L_0\}$ at $(\frac{4}{9}L_x, \frac{1}{2}L_y)$ and $(\frac{6}{9}L_x, \frac{1}{2}L_y)$ with $L_x = L_y = 5 \times L_0$, while α takes the value $\frac{0.1}{L_0}$. The number of nodes is $N_x = N_y = 128$ and the vertical integration step is $\delta z = 0.02\, L_0$.

The V.I.M gives an error of 5% at $z = 1.3\, L_0$. This error increases then rapidily with z, even if the mesh is refined (Fig. 4). This case shows a different behaviour for the two versions of our Regularization Method. For the Total Projection method the error reaches the same value of 5% at a higher height of $z = 3\, L_0$, and the increases then quasi-parabolically, to reach 23% at $z = 8L_0$. A better result is obtained with the Partial Projection version since same value for the error (5%) is obtained even higher $(z = 3.8\, L_0)$.

5.4.2. A quadrupolar configuration case

Keeping the box size the same $(L_x = L_y = 5\, L_0)$, we take four charges $(Q_1, -2Q_1, 4Q_1, -3Q_1)$ located respectively at $5/12\ \ L_x, 5/12\ \ L_y, -L_0)$, $(7/12\ \ L_x, 5/12\ \ L_y, -L_0/2)$, $(5/12\ \ L_x, 7/12\ \ L_y, -L_0/2)$, $(7/12\ \ L_x, 7/12\ \ L_y, -L_0)$. Moreover we choose $\alpha = 0.1\, L_0^{-1}$, and, as for the previous case, $N_x = N_y = 128$ and $\delta z = 0.02\, L_0$.

The V.I.M gives an error of 5% at $z = 0.7\, L_0$ as shown in Fig. 5. Our Regularization Method allows us to reach the height of $3.1\, L_0$ with the same error with the Total Projection, while the Partial Projection gives the same error at only $2\, L_0$. Note that this case is also particularly interesting since it exhibits a good behaviour of our method in the case of a complex topology magnetic configuration, that contains separatrices surfaces, accross wich the field line connectivity is discontinuous.

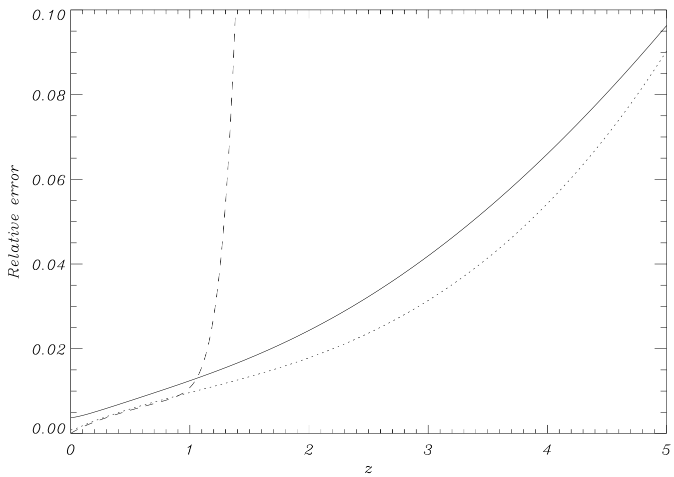

Fig. 4. Example #3, test #1, Relative error in the euclidian norm of the solution versus height for a constant-$alpha$ force-free magnetic field created by two charges located under the surface $\{z = 0\}$. For a description of the line symbols used see legend to Fig. 3 The Partial Projection gives slightly better results.

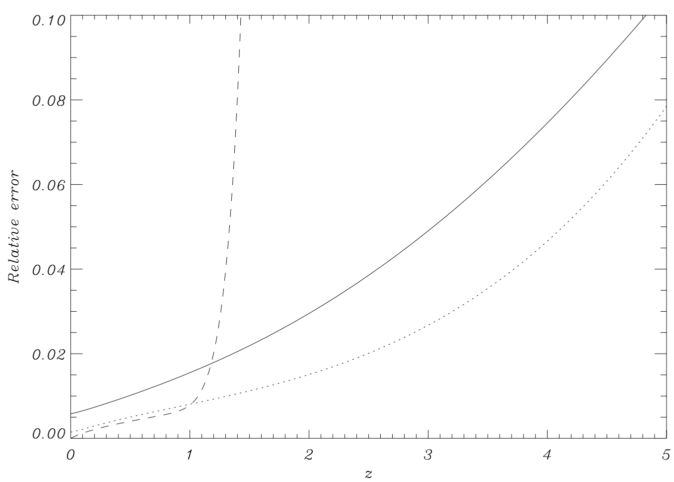

Fig. 5. Exemple #3, test #2. Relative error on the euclidian norm of the solution versus height for a complex topology constant-$alpha$ force-free magnetic field created by two magnetic charges located under the surface $\{z = 0\}$ For a description of the line symbols used see legend to Fig. 3 Now the Partial Projection is slightly better unlike for the complex topology periodic case whose results are shown in Fig. 3.

5.5. *Behaviour on a pathological case:*

As a last test we consider the case of the exact analytical solution of Low (1982), and given (in general for non constant α) by:

$$B_x = -\frac{B_0}{r}\cos\phi(r), \tag{36}$$

$$B_y = \frac{B_0 x_1 y_1}{r\rho^2}\cos\phi(r) - \frac{B_0 z_1}{\rho^2}\sin\phi(r), \tag{37}$$

$$B_z = \frac{B_0 x_1 z_1}{r\rho^2}\cos\phi(r) + \frac{B_0 y_1}{\rho^2}\sin\phi(r), \tag{38}$$

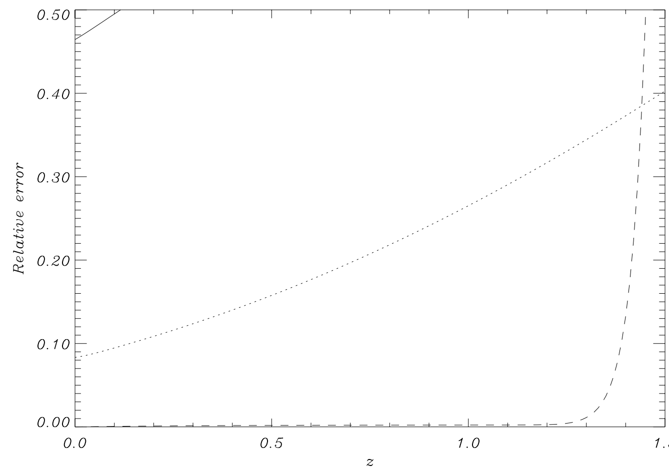

Fig. 6. Example #5. Relative error versus height for the constant-$alpha$ force-free magnetic solution of Low (1982). For a description of the line symbols used see legend to Fig. 3 The regularization method reveals a worse behaviour (although the errors increase at a smaller rate for the Partial Projection), than for the VIM because of the slow decreasing behaviour of the solution, even horizontally.

where $x_1 = x - \frac{L_x}{2}$, $y_1 = y - \frac{L_y}{2}$, $z_1 = z + L_0$, $\rho^2 = y_1^2 + z_1^2$, $r^2 = x_1^2 + \rho^2$. The so-called generating function $\phi(r)$ satisfies

$$\alpha(r) = -\phi'(r). \tag{39}$$

In the particular case of constant-α we are dealing with now, we choose

$$\phi(r) = -\alpha\, r + \phi_0. \tag{40}$$

The horizontal box size is $L_x = 6L_0$, $L_y = 6L_0$, and the mesh contains 128×128 nodes. The vertical integration step size is $\delta z = 0.01 \times L_0$. We choose $\alpha = 0$ as in Demoulin et al. (1991) Fig. 6 shows plots of the transverse magnetic field and contours of $B_T = (B_x^2 + B_y^2)^{\frac{1}{2}}$ at $z = 0$.

The error obtained with either the Partial Projection or Total Projection version of our Regularization Method (shown in Fig. 6) is not smaller than the one obtained with the V.I.M since they are 0.09% and 0.46% respectively ! already at the begining of the regularization process. This bad results can be understood as follows. The norm of $\mathbf{B}$ is given by

$$||\mathbf{B}|| = \frac{|B_0|}{\rho} \tag{41}$$

and is clearly independent of x. Therefore this solution does not belong to the functional space $L^2(\mathbf{R}^2)$ (function whose square is an integrable function in the plane -here $z = constant$). This solution is not a finite energy solution as one would expect from any physical solution. Therefore our regularization method which uses the Fourier transform cannot be applied successfully to this class of solutions (see Reed & Simon; 1978).

6. Conclusion

In this Paper, we have reported some new results we have obtained in devising a Regularization Method for the extrapolation of the 3D photospheric magnetic field for the class of extrapolation methods that is the Vertical Integration Method introduced by Wu (1985) and applied to regular bounded force-free magnetic fields. Most of the problems encountered in applying this method result from the impossibility of imposing or exploiting the asymptotic boundary condition. This is expected when a partially elliptic Boundary Value Problem (BVP) is considered as a Cauchy type BVP, and is therefore ill posed.

Our Regularization Method is based upon this boundary condition at infinity and consist in forcing the solution to belong to the functional space of functions tending to zero for $z \to \infty$. This can be performed by building a Regularization Operator which partly consists of a projection operator onto the right functional space.

We have successfully applied our method to a set of known analytical solutions having a simple or complex magnetic topology. Apart from a bad result obtained in the case of the pathological non finite energy (even in the horizontal plane) of Low (1982) for which the Fourier transform is in principle not permitted, our method always gives better results than the Vertical Integration Method, allowing an extrapolation to higher heights. It is also interesting to note that mesh refinement does not increase the error unlike in the other schemes. The effect of a still increasing error beyond some height is due to the underlying periodic assumption associated to the discrete Fast Fourier Transform that implies that, beyond some height of the order of the horizontal box size, the interaction with the neighboring repeated magnetic structure cannot be neglected. This analysis is compatible with a linear Green Function formulation that would give a simple estimation of this limit.

Clearly our approach rests on the *strong assumption that one seeks a solution tending to zero at infinity* and therefore should not reconver any diverging solution (as for the well known Hadamard solution example). Futhermore, since our projection procedure removes the divergent part of the solution, one could wonder how much of the original set of boundary information has been lost in the solution. Actually, for any exact force-free case, and in the framework of a solution tending to zero at infinity, one can go back continuously by infinitesimally small steps down to the boundary, and recover the boundary values up to the accuracy of the used numerical scheme. This is conceivable only if we assume that the data are compatible with a force-free solution as in the analytical examples presented in this Paper. In the case of a set of boundary data that would not be compatible with a force-free solution in the domain (as the presence of a heavy prominence), it is clear that this procedure like any force-free reconstruction would be invalid, and this even regardless the well-posedness property of the BVP. Whether our method or any other would lead to the presence of a singularity at the location of the prominence should be investigated (Low 1997, private communication) but no mathematical answer can be given yet. One could progressively investigate the effects of plasma pressure and gravity forces on the predictions of our force-free reconstruction procedure when applied to the particular set of exact non force-free solutions of Low (1991,1992) in which these forces can be smoothly introduced, in which the Bessel expansion now replace the periodic Fourier expansion one relevant to force-free configurations.

As for the current-free case, there are several limitations to the linear force-free assumption, as the total energy of the field in the unbounded domain is in general infinite, except for very particular solution (Aly 1992), and that the electric currents are uniformly distributed, while observations clearly show strong localized shear along the neutral line of many active region magnetic configurations (Hagyard 1988, Hofmann et al. 1991). It is therefore necessary to relax the constantα assumption and look for reconstruction methods with non linear force-free magnetic fields. Among the various attempts that have been developped (see review of Amari et al. 1997 and McClymont et al. 1997) the Vertical Integration Method has been also been tested and exhibites again the same kind of rapidly increasing error with heights Wu 1985, 1987, Cuperman et al. 1990, 1991, Demoulin et al. 1993), as for the linear case. We have started to generalize our Regularization Method to the non linear force-free extrapolation method and shall report about new results in a forthcoming paper.

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