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*Research Note*

# On beaming due to coherent inverse Compton scattering

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**Abstract.** We calculate the beaming of coherent emission by means of geometrical optics. When spherical elementary waves are phase-coupled in that way that constructive interference occurs along one axis, the intensity is lower for off-axis viewing angles. The spatial angle of constructive interference shrinks to a fraction of $4\pi$ when the length scale of the emission region is larger than the wavelength of the radiation. This leads to an amplification of the received radiation in addition to the relativistic beaming caused by the relativistic radial outflow which is thought to be important for the radio emission of pulsars. The effect will be calculated by numerical methods and can be approximated by a simple analytic expression.

**Key words:** stars: pulsars: general – radiation mechanisms: non-thermal

The main feature of a coherent process is that there are $N$ particles radiating *in phase* (e.g. Melrose 1991 and references therein). Therefore, the coherent intensity of the radiation is calculated as $N^2$ times the single particle amplitude. In the case of $N$ incoherently radiating particles the interference terms cancel and the total intensity is given by $N$ times the square of the single particle amplitude. Consequently the intensity of $N$ coherently radiating particles is $N$ times higher than in the incoherent case.

But, as will be shown, if the volume of coherently radiating particles (hereafter coherence volume) is about the same size or larger than the wavelength of the emitted radiation the phase coupling needed for the increased intensities can only be maintained in one direction, i.e. along one axis. Seen from off-axis angles there may even be destructive interference leading to zero intensity. In this contribution we calculate the angle at which the intensity drops to zero. Thus we get a reasonable estimate for the beaming due to coherent radiation.

Assuming that the reason for coherence is a free electron maser (FEM) process (i.e. inverse Compton scattering (ICS) of relativistic electrons and nonlinear electrostatic plasma waves (Asseo et al. 1990; Benford and Weatherall 1992; Asseo 1993)) we derive a formula for the beaming angle depending only on the density, the Lorentz factor of the electrons and the size of the coherence volume.

Consider a volume of $N$ particles which are able to radiate in phase. For simplicity we assume that the volume is a cube with one of its axes in the direction where fully constructive interference occurs. This seems reasonable since nonlinear electrostatic plasma waves appear pancake-shaped due to Lorentz contraction (along the magnetic field lines) and the electrons are moving perpendicularly to the extension of these plasma solitons.

Therefore it seems natural to expect that the direction of relative velocity between waves and particles describes the direction of phase coupling (this shall be chosen to be the z- axis being parallel to the vector (0,0,1)). The problem can be reduced to a two-dimensional one by orientating the coordinate system in such a way that the observer is located in the direction $(\sin\alpha, 0, \cos\alpha)$ and the y- axis can be disregarded in the calculations. If the coherence volume is $d^3$ (where $d$ is the length of a cube edge) we can choose the origin of the coordinate system to be a corner such that in the cube all three coordinates are normalized to the range of $[0..d]$. Throughout this paper we perform our calculations in the beam frame, i.e. a Lorentz factor belongs to the nonlinear wave structure approaching the electron beam (approximately equal to the Lorentz factor of the beam seen by an observer). The term *coherence volume* needs some clarification. It is the volume occupied by the coherently radiating particles which is the same as the volume of the considered soliton. In other words, we do not distinguish between the part of the soliton involved in the reaction and the volume of beam electrons radiating coherently except for the Lorentz contraction since the volume is observed in the beam frame. The soliton can be treated as the cause of coherence whereas the beam particles are coherently radiating when they interact with the soliton.

Next we calculate the phase shift of the wave from a radiating particle having the coordinates $(x, z)$ relative to a similar particle at the origin (0,0) when radiating at an angle of $\alpha$. (The y-coordinate is disregarded due to the choice of the coordinates). The phase difference can be expressed as

$$\phi(x, z) = k\,(\delta(x, z) + z) \tag{1}$$

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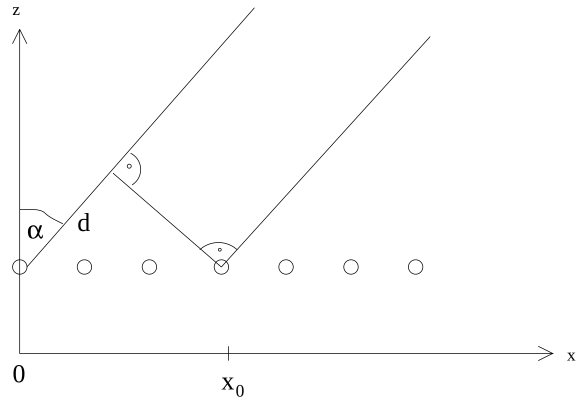


**Fig. 1.** Schematic picture showing how the path difference for two points is calculated having the same z- coordinate.

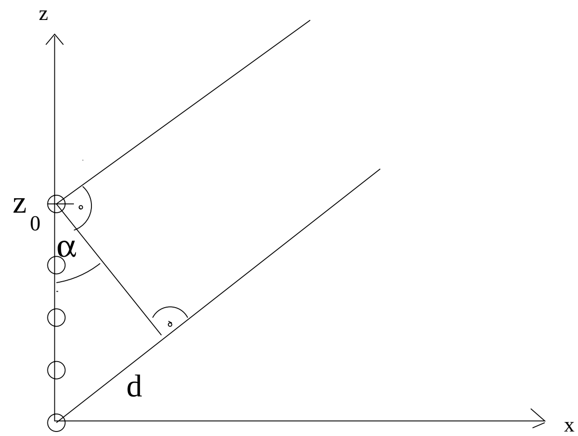


**Fig. 2.** The same as Fig. 1, but for two particles having the same x-coordinate.

where $k = 2\pi/\lambda$ is the wave number of the emitted radiation, $\delta(x, z)$ denotes the difference in pathlengths for radiation from the two particles (where the usual approximations are adopted for calculations of interference at optical grids). The second term inside the brackets, $z$, describes the *intrinsic* phase difference which is necessary for constructive interference in z-direction. Obviously a particle which is right ahead of another in the distance of $z$ must have a phase delay of $kz$ to interfere constructively.

$\delta_0$ is now calculated for two special cases:

1) z=0: It can be seen from Fig. 1 that $\delta_0 = x \sin\alpha \approx x\alpha$ for small values of $\alpha$.

2) x=0: In this case we find $\delta_0 = -z\cos\alpha$ (cf. Fig. 2) and thus $\phi(0, z) = z\,(1 - \cos\alpha) \approx z\alpha^2/2$ for small $\alpha$

We add these two phase differences to get the general result. This renders:

$$\phi(x, z) = \frac{2\pi}{\lambda}\left(x\alpha + \frac{1}{2}\,z\alpha^2\right) \tag{2}$$

For calculating the total intensity relative to the maximum value we integrate the amplitudes over one phase and over the coherence volume and take the square of the integral. In principle one has to sum up the complex amplitudes of waves from all the particles but replacing the sum by an integral does not affect the results heavily when $N \gg 1$. Using the integral has the advantage that particles need not to be arranged in a grid (as the electron temperature might be quite high) and the result will become more representative of the physics involved.

It is reasonable to assume that coherence is not provided instaneously at its maximum possible number $N$, but that the coherence volume will be zero initially and then expand in some way. Since we are mainly interested in the *maximum* possible effect, we do not assume a certain time dependence but just set a maximum size of the coherence cell. The coherence is caused by a plasma process, therefore the most natural assumption is to set the size to $d = c/\nu_{\rm pe}$. Here $\nu_{\rm pe} = \sqrt{(ne^2)/(m_{\rm e}\varepsilon_0\gamma)}/(2\pi)$ is the (frame invariant) plasma frequency (cf. Weatherall and Benford, 1991).

By re-scaling the integration interval to [0..1] (that means transforming the integration variables to dimensionless lengths in units of the coherence cell extension) we finally obtain for the relative phase averaged wave amplitudes in the direction of $\alpha$

$$\frac{A}{A_0} = \int_0^1\int_0^1 \cos\left[\frac{2\pi c}{\lambda\nu_{\rm pe}}\left(x\alpha + \frac{1}{2}z\alpha^2\right)\right]{\rm d}z\,{\rm d}x \tag{3}$$

where $A$ is the amplitude of the wave and $A_0 = A(\alpha = 0)$ the corresponding amplitude of waves emitted parallel to the particle propagation.

The double integral evaluates to

$$\frac{A}{A_0} = \frac{2}{M^2\alpha^3}\,(\sin a \sin b - \cos a \cos b + \cos a + \cos b - 1) \tag{4}$$

where $M = 2\pi\, c/(\lambda\nu_{\rm pe})$, $a = M\alpha$, $b = M\alpha^2/2$. The first zero of the above equation yields the angle of the minimum and we find the solution $a = \pi - b$ which leads to a simple quadratic equation for $\alpha$. The final result then reads

$$\alpha_{\rm min} = \sqrt{1 + \frac{2\pi}{M}} - 1 \approx \frac{\lambda\nu_{\rm pe}}{2c}\,. \tag{5}$$

The last approximation is valid as long as $M \gg 1$ which is always the case as we consider only plasma processes for radio emission, and thus can write $c/(\lambda\nu_{\rm pe}) = \nu_{\rm em}/\nu_{\rm pe} = \gamma^\varsigma$ where $\varsigma$ is an emission model dependent parameter which is at least 1.

This result can be compared to a more exact geometric model that we obtain by numerical integration over the phases of waves emitted from a soliton.

The soliton shape is chosen since a coherent plasma process requires efficient bunching of the radiating particles. Physically speaking this is the same as a high density modulation or a strong longitudinal plasma wave, which is well described by an almost non-dispersive longitudinal plasma soliton with a certain extension perpendicular to the magnetic field lines.

We assume a cylindrically symmetric shape like $\cosh\left(\frac{\rho}{R}\right)^{-2}$ (see Fig. 3) where $\rho$ denotes the distance from the centre of the soliton in units of its characteristic length (called "extension" of the soliton in the simplified discussion

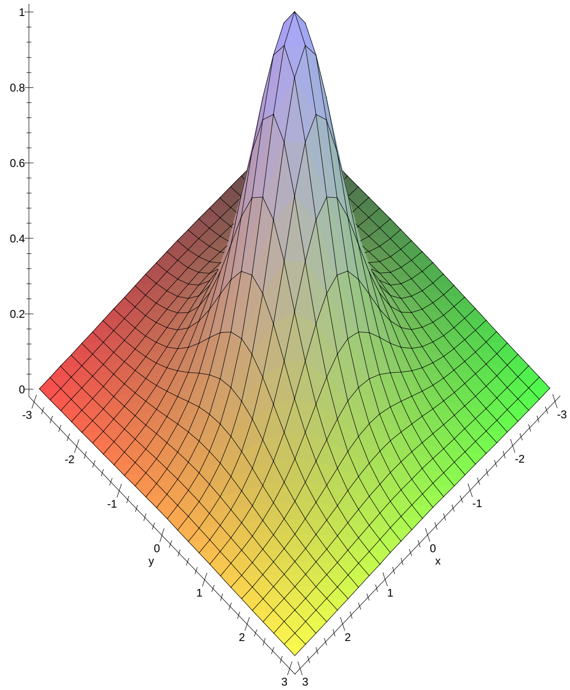


**Fig. 3.** Typical shape of a soliton. The maximum amplitude of the soliton is normalized to 1, and points in radial direction (z). The plane spanned by the line of sight and the direction of the outflowing particles as assumed to be the y=0 plane
. X- and y- axis are scaled in units of the characteristic length (see text).

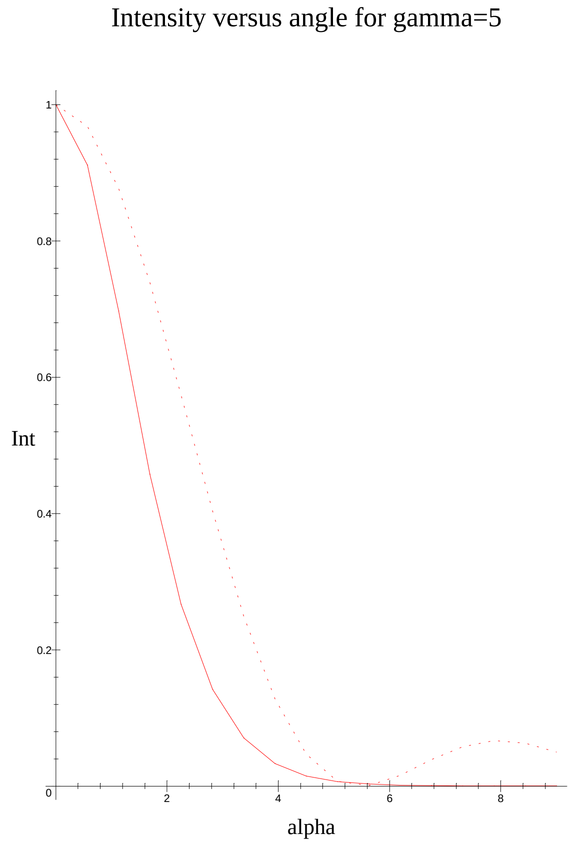


**Fig. 4.** Beaming pattern for the coherent emission. The exact result is shown by the fully drawn line whereas the dotted line shows the approximation. The angle between the line of sight and the direction of movement is given in degrees and the intensity is normalized to 1 for $\alpha = 0$

before) and $R$ as the characteristic length scale equal to $c/\nu_{\rm pe}$ or equal to $d$ used in the simple approximation above (e.g. Weatherall 1997).

The solitons developing from electrostatic plasma waves are negligibly thin in the direction of the field lines but have a large perpendicular extension. Let the observer be placed at a large distance $s$ and at an angle $\alpha$ with respect to the normal direction. Again we assume that the positive x- axis is the projection of the observer's line of sight. Then, $s$ being the distance and $\phi$ the meridional angle in the soliton plane we find the expression

$$r^2 = s^2 - 2s\rho\,\sin\phi\,\cos\alpha + \rho^2. \quad (6)$$

A spherical wave emitted from a point source can be written as

$$\Psi(r) = \frac{{\rm e}^{-{\rm i}kr}}{r} \quad (7)$$

Thus the superposition of all elementary waves from a soliton reads

$$A_{\rm sol}(s,\theta) = A_0 \int\limits_0^\infty \int\limits_0^{2\pi} \frac{{\rm e}^{-{\rm i}kr(s,\rho,\alpha,\phi)}}{r(s,\rho,\alpha,\phi)} \rho{\rm cosh}\left(\frac{\rho}{R}\right)^{-2}\,{\rm d}\phi\,{\rm d}\rho. \quad (8)$$

$k$ is the wave number with the same definition as above. This equation can be treated numerically. We find that the result is not altered significantly but only slightly modified by the different shape of the coherence cell. As can be seen in Fig. (4) the beaming is even stronger than found in the simple approximation. [1]

As mentioned above we assume a plasma process to be the origin of coherent emission. As an illustrative example we take the free electron maser mechanism which can (*cum grano salis*) be treated as inverse Compton scattering of plasma solitons and relativistic electrons. Then the incoming electron sees a plasma wave with a wavelength that is equal to that measured in the soliton frame (which is almost the same as the rest frame for very low-relativistic solitons) divided by $\gamma$ due to relativistic space contraction. Seen in the beam frame the electrostatic wave is simply reflected with the same frequency. In the rest frame the reflected wave has a frequency that is higher by another factor of $\gamma$ and is given by $\gamma^2\omega_{\rm pe}$ as can be verified by a Lorentz transformation from the beam frame to the observer's frame. A wave of this frequency cannot propagate as a plasma wave any more and will be decoupled from the plasma propagating as a free

[1] No great discrepancy arises even though we treated the "coherence cell" as a cubic box in the first case and in the numerical calculations we assumed the soliton to be pancake-shaped. But if the phase-coupled emitter extends over many wavelegths in all directions, $\lambda/d$ will dominate the interference effects and the symmetry of the emitter will be of secondary importance.

electromagnetic wave. Because we performed our calculations in the beam frame, only one $\gamma$ is required in the formula for the emitted frequency; that means $\varsigma = 1$ in the formula given above.

Thus we finally find the expression

$$\alpha_{\min} = \frac{1}{2\gamma} \tag{9}$$

From that equation we obtain that the beam becomes sharper when $\gamma$ is larger. Because we require the strongest coherence at *low* frequencies (and therefore naturally low Lorentz factors) it is natural to assume that we have non-stationary processes which start incoherently (at high Lorentz factors) and become more and more coherent when $\gamma$ decreases. Eq. (9) should only be valid for the lowest Lorentz factors when coherence is fully developed.

The angle of the minimum now provides an estimate for the spatial beaming angle due to coherence. This is given by

$$\Delta\Omega = \pi\alpha_{\min}^2 \tag{10}$$

which is true for small angles. This corresponds to a *relative beaming factor* of

$$\eta_1 = \frac{4\pi}{\Delta\Omega} = 16\gamma^2 \tag{11}$$

in the beam frame.

For a fixed observer in addition to coherent beaming there is, of course, beaming by a factor of $4\gamma^2/\pi$ (Rybicki and Lightman 1979) due to the relativistic motion of the solitons themselves. Thus an observer sees a relative beaming factor that reads

$$\eta = \frac{4\pi}{\Delta\Omega}\gamma^2 = \frac{64\gamma^4}{\pi} \tag{12}$$

It has been shown above that in coherent processes one can find a natural beaming effect that is caused by the non-isotropic emission pattern of a phase-coupled grid of point sources due to interference. This leads to higher fluxes of coherent radiation in special directions which can be important for estimations of bri ghtness temperatures or observed radiation ”power” (that means the power that would have to be emitted if the radiation was isotropic in the observers frame, see Manchester and Taylor 1977). The described effect might be a contributional cause of the very intense fluxes of radio photons in pulsars. Some detailled discussion of the flux problem will be subject of a later paper.

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