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Letter to the Editor

Analytical results in a cellular automaton model of solar flare occurrence

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Abstract. Methods of the branching theory are applied to the Macpherson–MacKinnon (1997) cellular automation model for the occurrence of solar flares. The distribution of flare energies is shown to be a power law with the slope $\alpha = 3/2$ independent of dimensionality D of the model and in nice agreement with observations, arguing in favor of the model. An expression for the upper energy cut-off and a condition for the average flare size to become infinite are derived for an arbitrary D as well, providing an opportunity for comparing the analytical and numerical results.

Key words: Sun: corona – Sun: flares – Sun: magnetic fields

1. Introduction

It has been strongly argued on both theoretical (Somov 1991) and observational (Mandrini et al. 1993) grounds that energy release in solar flares is due to rapid magnetic reconnection in large-scale current sheets. At the same time observations clearly show that the energy release rate is highly variable, suggesting that flaring occurs as a sequence of small elementary energy releases—"nanoflares" (Parker 1988). This justifies the use of cellular automaton (CA) models to explore the statistical properties of flaring activity (Lu & Hamilton 1991; Galsgaard 1996; Georgoulis & Vlahos 1996; Litvinenko 1996). Of particular interest is the distribution of flare energies, which is observed to be a power law over several orders in magnitude of the total flare energy.

It is postulated in the CA approach that the elementary flaring events can self-organize in the sense that they can trigger activity in the neighboring regions, leading to an "avalanche" of nanoflares. Although this approach is based not on kinetic or MHD equations but rather on *ad hoc* triggering rules, it appears to be validated by computer MHD simulations of energy dissipation in the solar corona (e.g., Dmitruk & Gómez 1997), which demonstrate both a highly intermittent dissipation rate and robust power-law scalings of flare parameters.

It is of obvious interest to construct a CA model which would be simple enough to allow an analytical description and at the same time would lead to a realistic flare energy distribution. A very elegant CA model has been suggested by MacKinnon et al. (1996). Unlike the original Lu & Hamilton (1991) approach based on deterministic rules for the triggering and energy redistribution, the MacKinnon et al. (1996) model involves purely random activation of the neighboring sites, making possible an analytic description in one dimension. The results, however, conflicted with observations, and additional assumptions regarding the random triggering rules were necessary to remove the discrepancy. Alternatively, Macpherson & MacKinnon (1997), hereafter MM, improved the model by relaxing the assumptions concerning time scales of repeated flaring. Numerical results of the MM simulations were in agreement with observations already in the one-dimensional case ($D = 1$). Unfortunately, no exact analytical treatment appeared possible in the MM model for any D.

It is the goal of the present paper to show how some results of the branching theory can be used to obtain analytical asymptotic estimates for the MM model including its generalizations to dimensions higher than one. In the next section we briefly summarize the original MM approach. Next we present the estimates for the flare size distribution, which are valid asymptotically when the number of the elementary flaring events is large, and compare them with the available observational and numerical results. In the last section we discuss how the results of the MM model can be related to the flaring process on the Sun.

2. Description of the model and asymptotic results

MacKinnon et al. (1996) modeled the interaction of flaring elements on a one-dimensional spatial lattice evolving in discrete time steps. Each lattice site corresponds to a potential flaring element (nanoflare). The triggering rules are as follows. An inactive site can become active with probability p_1 if one of its neighbors is active at the previous time step. This site then will remain active for one time step. No repeated activation of any site is allowed and there is one active site at time $t = 0$. The rules above represent the idea that flaring at one site can trigger flaring at neighboring locations. The rules have some analogy to the reconnection process though the nature of flaring elements is left deliberately unspecified. The flare then is defined as a sequence of active sites in the lattice connected in space and time.

Assuming that the energy release is the same at each site, the total flare energy is proportional to the size of an event N that is the total number of active sites. Simple combinatoric arguments lead to the probability $P(N)$ of an event of size N:

$$P(N) = Np_1^{(N-1)}(1 - p_1)^2 \qquad (1)$$

(MacKinnon et al. 1996). This dependence on N does not agree with numerous observations that imply that $P(N) \sim N^{-1.5}$ for the solar flare energy distribution (e.g., Crosby et al. 1993; Shimizu 1995). To remove the discrepancy, MacKinnon et al. (1996) had to assume that the triggering probability p_1 is itself a random variable uniformly distributed between 0 and 1.

It is unrealistic not to allow repeated flaring for the lattice sites since flare energy release is often observed to occur locally for the duration of a solar flare. In fact the original CA flare model of Lu & Hamilton (1991) allowed repeated activation of the lattice sites. Intuitively, including repeated flaring during an ongoing event would increase the probability of large events and would make the probability distribution (1) less steep, possibly changing it into a power law. These considerations motivated MM to allow repeated activation. This means that any site remains "switched off" for one time step after being active, and can re-activate through triggering by neighbors at the next step (MM also considered more general models with an arbitrary "recharging time"). Numerical simulations with $p_1 = \text{const}$ gave $P(N) \sim N^{-\alpha}$ with $\alpha \approx 1.5$ already for the lattice dimension $D = 1$—remarkably close to the observed flare distribution.

Thus repeated activation indeed favors larger events so that a power-law flare energy distribution results. Another possibility to increase the probability of large events, as well as to make the model more realistic, would be to extend the treatment to dimensions $D > 1$ because this would increase the number of ways of getting an event of size N. Unfortunately, neither of these possibilities can be explored using the analytical approach of MacKinnon et al. (1996) that works only for $D = 1$. Nevertheless, we suggest below that useful asymptotic estimates are possible if the temporal development of the MM model is represented as a kind of branching process.

Consider a $(D + 1)$-dimensional lattice combining space and time steps. Connect all sites that have been active during an event (Fig. 1). Each node in the resulting tree-like structure represents an active site and branches from such a node represent the activation of its neighbors. We assume that the number of active sites at any given time is small compared with the total number of sites. "Small" here means that the situations when a site has more than one active neighbor can be ignored; hence no branch crossing can occur. Methods of the branching theory can be directly applied to the resulting tree (Harris 1963). Otherwise we would have to introduce probabilities $p_2, \ldots$ for a site with two or more active neighbors to become active. A similar simplified description of activation in a forest-fire model was adopted by Cristensen et al. (1993) who showed the description to be the better, the larger D is. In other words, as D increases, it becomes less likely that any given site will have more than one active neighbor out of the total of $2D$.

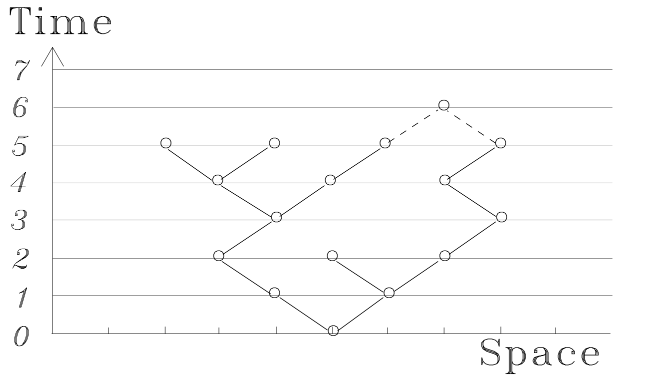

Fig. 1. A tree generated by a branching process. Events when a site is activated by more than one neighbor (dashed lines) are ignored.

Thus we expect the branching theory results to be in only qualitative agreement with the MM numerical results in the case $D = 1$, but suggest that the scalings below are more reliable for $D > 1$ and can be used as a guide for numerical simulations. The dimensionality D should reflect the complexity of a flaring region. For example, braiding and twisting of magnetic field lines would effectively increase D by creating multiple interacting regions of magnetic reconnection.

As flaring proceeds, each node in the tree branches off a number n of branches, where $0 \leq n \leq 2D$. The total number of nodes N in the tree corresponds to the size of the event. We now use the result due to Otter (1949) that the probability $P(N)$ for a tree in a branching process to have the total of N nodes is asymptotically

$$P(N) \sim N^{-3/2} \exp\left(-\frac{N}{N_0}\right), \quad 1 \ll N < \infty, \qquad (2)$$

where a cut-off size N_0 is a function of p_1 and D defined below. The same formula gives the distribution of flare sizes in the MM model for any D as long as the number of active sites at any given time is small. This is a generalization of the result obtained by MM numerically for $D = 1$. Therefore, the predicted power-law index $\alpha = 3/2$ is independent of dimensionality. This is an argument in favor of the MM model since the observed flare distribution does not show large temporal variations, whereas the magnetic field structure in the corona can change significantly in time implying that D changes as well. We stress that not all CA models are robust with respect to changes in D (see Wheatland & Sturrock 1996 and references therein). Because of combinatoric complications, it is not clear if scaling (2) could have been obtained by direct calculations.

Unlike the power-law slope, the upper cut-off N_0 of the distribution $P(N)$ is determined by both the triggering probability p_1 and the dimensionality D. The evaluation of N_0 may be useful for the purpose of comparing the asymptotic and numerical results. Let us introduce the probability generating function for the branching process:

$$f(s) \equiv \sum_{n=0}^{\infty} \wp_n s^n. \tag{3}$$

Here $\wp_n$ is the probability that a node has n branches, or equivalently, that a flaring element triggers n flaring elements at the next step:

$$\wp_n = \binom{2D}{n} p_1^n (1-p_1)^{2D-n}, \tag{4}$$

where $\binom{2D}{n}$ denotes the binomial coefficient. The sum is easily performed to give

$$f(s) = (1 - p_1 + p_1 s)^{2D}. \tag{5}$$

The average number of new branches from each node $m = \langle n \rangle$, called the branching ratio, is equal to

$$m = f'(1) = 2p_1 D. \tag{6}$$

The cut-off size N_0 of the distribution is

$$N_0 = \frac{1}{\ln w}, \tag{7}$$

where w is determined from

$$w = \frac{s_0}{f(s_0)} = \frac{1}{f'(s_0)} \tag{8}$$

(Otter 1949). For the generating function $f(s)$ above, we find

$$w = \frac{p_1^{-1} - 1}{2D - 1} \left[\frac{1 - (2D)^{-1}}{1 - p_1} \right]^{2D}. \tag{9}$$

Notice that $w > 1$ unless $m = 1$, giving rise to the exponential cut-off of the distribution. The formulas for α and N_0 should be tested against numerical results for $D > 1$ when they become available. Unfortunately, the MM results in the case $D = 1$ (as presented in their Fig. 4) do not have enough resulution for large N to test the predicted cut-off size N_0.

MM also computed the critical probability p_c such that for $p_1 = p_c$ the average event size

$$\langle N \rangle = \int N P(N) \, \mathrm{d}N \tag{10}$$

becomes infinite. Numerically, $p_c \approx 0.7$ for $D = 1$. The possibility of repeated flaring is the physical reason for p_c being less than 1. There appears to be no simple direct method of determining p_c in either one-dimensional case or for $D > 1$, whereas branching theory again provides a simple scaling. We note for clarity that the asymptotic Eq. (2) should not be used to find the moments of the distribution because $P(\infty) \neq 0$ for $m > 1$. The probability p_c itself, however, can be determined since it corresponds to the branching ratio $m = 1$ (Harris 1963). When $m \to 1$ the distribution becomes critical in the sense that the cut-off diverges ($w \to 1$ and $N_0 \to \infty$). This corresponds to the situation when each flaring element triggers on average exactly one new active element. Hence the critical probability is

$$p_c = \frac{1}{2D}. \tag{11}$$

This gives $p_c = 0.5$ for $D = 1$. As mentioned above, one would expect the branching theory results to be increasingly accurate with increasing D (Christensen et al. 1993).

3. Discussion

Cellular automaton models for flare occurrence exploit the idea that many small energy releases self-organize to produce a single large flare. Because these models are not based on a detailed physical mechanism for flare energy release, explanation of the flare distributions does not require the knowledge of much of the physics. There is a danger though that the CA approach could abandon the relevant physics altogether. Of particular concern is the dependence of the distribution power-law indices predicted by the CA models on their dimensionality D (Wheatland & Sturrock 1996). One would expect the latter to be determined by the complexity of the coronal magnetic field which can vary significantly, for example in the course of the solar cycle. Only small periodic changes in the distribution slope α have been reported (Bai 1993). Hence a viable CA approach should not predict a strong dependence of α on D.

The principal point made in this paper is that the Macpherson & MacKinnon (1997) CA model can be treated analytically as a branching process in an approximation which is appropriate for $D > 1$. One of the new results of this approach is the prediction of the power-law index of the flare energy distribution $\alpha = 3/2$ that is independent of either D or the triggering probability p_1. Moreover, this value of α

s in nice agreement with the value deduced from observations. We also derived expressions for the distribution cut-off N_0 and the critical probability $p_1 = p_c \sim D^{-1}$ corresponding to transition from a finite to the infinite average event size. We suggest using the analytical scalings as a guide to further numerical studies of this promising model.

The ultimate question, of course, is how the model relates to the physics of flares. The primary process in a solar flare is most likely a macroscopic instability followed by magnetic reconnection in a large-scale current sheet (Somov 1991; Mandrini et al. 1993). Nonlinear processes in the sheet, however, lead to its fragmentation and eventual formation of statistical distribution of micro-current sheets (Biskamp 1994). Therefore magnetic energy dissipation on a global scale can be consistent with the description in terms of stochastic magnetic energy release. This picture appears to be substantiated by MHD computer simulations of magnetic energy release in the corona (Dmitruk & Gómez 1997). Analytical estimates of the type presented in this paper could help to reconcile the CA and MHD models for the energy release in solar flares.

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